185: Introduction to Complex Analysis

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

Our reality isn't about what's real, it's about what we pay attention to.

—Hank Green, [Gre20]

1.1 January 19

It is reportedly close enough to start.

1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using `bCourses` a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- The homework consists of readings (for each course day) and weekly problem sets. Late homework is never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is cumulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?

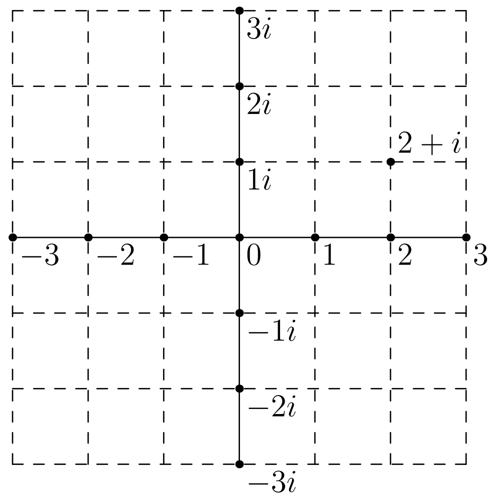
Idea 1.1. In complex analysis, we study functions $f : \mathbb{C} \to \mathbb{C}$, usually analytic to some extent.

There are two pieces here: we should study $\mathbb{C}$ in themselves, and then we will study the functions.

Definition 1.2 (Complex numbers)**.** The set of complex numbers $\mathbb{C}$ is $\{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Hopefully $\mathbb{R}$ is familiar from real analysis. As an aside, we see $\mathbb{R} \subseteq \mathbb{C}$ because $a = a + 0i \in \mathbb{C}$ for each $a \in \mathbb{R}$.

The complex numbers have an inherent geometry as a two-dimensional plane.



The point is that $\mathbb{C}$ looks like the real plane $\mathbb{R}^2$. More precisely, $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-vector space, where our basis is $\{1, i\}$.

We would like to understand $\mathbb{C}$ geometrically, "as a space." The first step here is to create a notion of size.

Definition 1.3 (Norm on $\mathbb{C}$)**.** We define the *norm map* $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $|z| := \sqrt{z\overline{z}}$. In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on $\mathbb{R}$: for $a \in \mathbb{R}$, we have $\sqrt{a^2} = |a|$.

Norm functions, as in the real case, give us a notion of distance.

Definition 1.4 (Metric on $\mathbb{C}$)**.** We define the *metric on* $\mathbb{C}$ to be $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$.

One can check that this is in fact a metric, but we will not do so here.

Remark 1.5. The distance in $\mathbb{C}$ is defined to match the distance in $\mathbb{R}^2$ under the basis $\{1, i\}$.

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned $\mathbb{C}$ into a topological space.

1.1.3 Complex Functions

There are lots of functions on $\mathbb{C}$, and lots of them are terrible. So we would like to focus on functions with some structure. We'll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone's favorite functions are holomorphic.

Example 1.6. Polynomials, $\exp$, $\sin$, and $\cos$ are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic* functions, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

Theorem 1.7. Holomorphic functions on $\mathbb{C}$ are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

Remark 1.8. It is very hard to spell Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

Definition 1.9 (Automorphisms of $\mathbb{C}$)**.** A function $f: \mathbb{C} \to \mathbb{C}$ is an *automorphism of* $\mathbb{C}$ if it is bijective and both f and f^{-1} are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

1.2 January 21

We're reviewing set theory today.

1.2.1 Set Theory Notation

We have the following definitions.

- $\varnothing$ means the empty set.
- $a \in X$ means that a is an element of the set X.
- $A \subseteq B$ means that A is a subset of B.
- $A \subsetneq B$ means that A is a proper subset of B.
- $A \cup B$ consists of the elements which are in at least one of A or B.
- $A \cap B$ consists of the elements which are in both A and B.
- $A \setminus B$ consists of the elements of A which are not in B.
- Two sets A and B are *disjoint* if and only if $A \cap B = \varnothing$.
- Given a set X, we define $\mathcal{P}(X)$ to be the set of all subsets of X.
- $|X| = \#X$ is the cardinality of X, or (roughly speaking) the number of elements of X.

As an example of unwinding notation, we have the following.

Proposition 1.10 (De Morgan's Laws)**.** Fix $\mathcal{S} \subseteq \mathcal{P}(X)$ a collection of subsets of a set X. Then

$$X \Bigg\backslash \bigcap_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} (X \setminus S) \qquad \text{and} \qquad X \setminus \bigcup_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} (X \setminus S).$$

Proof. We take these one at a time.

- Note $a \in X \setminus \bigcap \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcap \mathcal{S}$. However, $a \notin \bigcap \mathcal{S}$ is merely saying that a is not in all the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for one of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for some $S \in \mathcal{S}$, which is equivalent to $a \in \bigcup_{S \in \mathcal{S}} (X \setminus S)$.

- Note $a \in X \setminus \bigcup \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcup \mathcal{S}$. However, $a \notin \bigcup \mathcal{S}$ is merely saying that a is not in any of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for each of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for each $S \in \mathcal{S}$, which is equivalent to $a \in \bigcap_{S \in \mathcal{S}} (X \setminus S)$. ■

1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Importantly, $0 \in \mathbb{N}$.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}$ and $q \neq\}$ is the set of rationals.
- $\mathbb{R}$ is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq\}$ is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ is the positive real numbers.

- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$ is the nonpositive real numbers.
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is the set of nonzero complex numbers.

1.2.3 Relations

Let's review some set theory definitions.

Definition 1.11 (Cartesian product)**.** Given two sets A and B, we define the *Cartesian product* $A \times B$ to be the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$.

Definition 1.12 (Binary relation)**.** A *binary relation* on A is any subset $R \subseteq A^2 := A \times A$. We may sometimes notate $(x, y) \in R$ by xRy, read as "x is related to y."

Example 1.13. Equality is a binary relation on any set A; namely, it is the subset $\{(a, a) : a \in A\}$.

The best relations are equivalence relations.

Definition 1.14 (Equivalence relation)**.** An *equivalence relation* on A is a binary relation R satisfying the following three conditions.

- Reflexive: each $x \in A$ has $(x, x) \in R$.
- Symmetric: each $x, y \in A$ has $(x, y) \in R$ implies $(y, x) \in R$.
- Transitive: each $x, y, z \in A$ has $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Definition 1.15 (Equivalence class)**.** Fix A a set and $R \subseteq A^2$ an equivalence relation. Then, for given $x \in A$, we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of x.

The hope is that equivalence classes partition the set. What is a partition?

Definition 1.16 (Parition)**.** A *partition* of a set A is a collection of nonempty subsets $\mathcal{S} \subseteq \mathcal{P}(A)$ of A such that any two distinct $S_1, S_2 \in \mathcal{S}$ are disjoint while $A = \bigcup_{S \in \mathcal{S}} S$.

And now let's manifest our hope.

Lemma 1.17. Equivalence relations are in one-to-one correspondence with partitions of A.

Proof. Given an equivalence relation R, we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that $R \mapsto \mathcal{S}(R)$ is our needed bijection. We have the following checks.

- Well-defined: observe that $\mathcal{S}(R)$ does partition A: if we have $[x]_R, [y]_R \in \mathcal{S}$, then $[x]_R \cap [y]_R \neq \varnothing$ implies there is some z with $(x,z) \in R$ and $(z,y) \in R$, so $x \in [y]_R$ and then $[x]_R \subseteq [y]_R$ follows. So by symmetry, $[y]_R \subseteq [x]_R$ as well, so we finish the disjointness check.

  Further, we see that
  $$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x]_R \subseteq A$$
  because $x \in [x]_R$, so indeed the equivalence classes cover A.

- Injective: suppose R_1 and R_2 have $\mathcal{S}(R_1) = \mathcal{S}(R_2)$. We show that $R_1 \subseteq R_2$, and $R_2 \subseteq R_1$ will follow by symmetry, finishing.

  We notice that, for any $\mathcal{S}$ partitioning A, being a partition, will have exactly one subset which contains x. But for $\mathcal{S}(R)$ for an equivalence relation R, we see $x \in [x]_R \in \mathcal{S}(R)$, so this equivalence class must be the one.

  So because $[x]_{R_1}$ and $[x]_{R_2}$ are the only subsets of $\mathcal{S}(R_1)$ and $\mathcal{S}(R_2)$ containing x (respectively), we must have $[x]_{R_1} = [x]_{R_2}$. Thus, $(x,y) \in R_1$ implies $y \in [x]_{R_1} = [x]_{R_2}$ implies $(x,y) \in R_2$.

- Surjective: suppose that $\mathcal{S}$ is a partition of A. As noted above, each $x \in A$ is a member of exactly one set $S \in \mathcal{S}$, which we call $[x]$. Then we define $R \subseteq A^2$ by $(x,y) \in R$ if and only if $y \in [x]$. One can check that this is an equivalence relation, which we will not do here in detail.[1]

  The point is that
  $$[x]_R = \{y : (x,y) \in R\} = \{y : y \in [x]\} = [x],$$
  so $\mathcal{S}(R) = \mathcal{S}$. So our mapping is surjective. ■

We continue our discussion.

Definition 1.18 (Quotient set)**.** Given an equivalence relation $R \subseteq A^2$, we define the *quotient set* A/R is the set of equivalence classes of R. In other words,
$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar" or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Definition 1.19 (Representatives)**.** Given an equivalence relation $R \subseteq A^2$, we say that $C \subseteq A$ is a *set of representatives of R-equivalence classes of A* if and only if C consists of exactly one element from each equivalence class in A/R.

1.2.4 Functions

To finish off, we discuss functions.

Definition 1.20 (Functions)**.** A *function* $f \colon X \to Y$ is a relation $f \subseteq X \times Y$ satisfying the following.

- For each $x \in X$, there is some $y \in Y$ such that $(x,y) \in f$. Intuitively, each $x \in X$ goes somewhere.
- For each $x \in X$ and given some $y_1, y_2 \in Y$ such that $(x,y_1),(x,y_2) \in f$, then $y_1 = y_2$. Intuitively, each $x \in X$ goes to at most one place.

We will write $f(x) = y$ as notational sugar for $(x,y) \in f$. Note this equality is legal because the value y with $(x,y) \in f$ is uniquely given.

[1] Note $x \in [x]$ by definition of $[x]$. If $y \in [x]$, then note $y \in [y]$ as well, so $[x] = [y]$ is forced by uniqueness, so $x \in [y]$. If $y \in [x]$ and $z \in [y]$, then again by uniqueness $[x] = [y] = [z]$, so $z \in [x]$ follows.

We would like to create new functions from old. Here are two ways to do this.

Definition 1.21 (Restriction)**.** Given a function $f\colon X \to Y$ and a subset $A \subseteq X$, we define

$$f|_A := \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function $f|_A\colon A \to Y$.

We will not check that $f|_A$ is actually a function; it is, roughly speaking inherited from f.

Definition 1.22. Given two functions $f\colon X \to Y$ and $g\colon Y \to Z$, we define the *composition* of f and g to be some function $g \circ f\colon X \to Z$ defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Definition 1.23 (Image)**.** Given a function $f\colon X \to Y$, we define the *image* of f to be

$$\operatorname{im} f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x)y\}.$$

Namely, $\operatorname{im} f$ consists of all elements hit by someone in X hit by f.

Definition 1.24 (Fiber, pre-image)**.** Given a function $f\colon X \to Y$ and some $y \in Y$, we define the *fiber* of f over y to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset $A \subseteq X$ to be

$$f^{-1}(A) := \{x \in A : f(x) \in A\} = \bigcup_{a \in A} \{x \in A : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Definition 1.25 (Inj-, sur-, bijective)**.** Fix a function $f\colon X \to Y$. We have the following.

- Then f is *injective* or *one-to-one* if and only if, given $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Then f is *surjective* or *onto* if and only if $\operatorname{im} f = Y$. In other words, for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Then f is *bijective* if and only if it is both injective and surjective.

Here is an example.

Definition 1.26 (Identity)**.** For a given set X, the function $\operatorname{id}_X\colon X \to X$ defined by $\operatorname{id}_X(x) := x$ is called the *identity function*.

For completeness, here are the checks that id_X is bijective.

- Injective: given $x_1, x_2 \in X$, we see $\operatorname{id}_X(x_1) = \operatorname{id}_X(x_2)$ implies $x_1 = \operatorname{id}_X(x_1) = \operatorname{id}_X(x_2) = x_2$.
- Surjective: given $x \in X$, we see that $x \in \operatorname{im} \operatorname{id}_X$ because $x = \operatorname{id}_X$.

We leave with some lemmas, to be proven once in one's life.

Lemma 1.27. Fix finite sets X and Y such that $\#X = \#Y$. Then a function $f\colon X \to Y$ is bijective if and only if it is injective or surjective.

Proof. Certainly if f is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on $\#X = \#Y$. If $\#X = \#Y = 0$, then $X = Y = \varnothing$, and all functions $f\colon \varnothing \to \varnothing$ are vacuously bijective: for injective, note that any $x_1, x_2 \in \varnothing$ have $x_1 = x_2$; for surjective, note that any $x \in \varnothing$ has $f(x) = x$.

Otherwise, $\#X = \#Y > 0$. We have two cases.

- Take f injective; we show f is surjective. In this case, $\#X > 0$, so choose some $a \in X$. Note that $x \in X$ with $x \neq a$ will have $f(x) \neq f(a)$ by injectivity, so we may define the restriction
$$f|_{X\setminus\{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}.$$
Observe that $f|_{X\setminus\{a\}}$ is injective because f is: if $x_1, x_2 \in X \setminus \{a\}$ have
$$f(x_1) = f|_{X\setminus\{a\}}(x_1) = f|_{X\setminus\{a\}}(x_2) = f(x_2),$$
then $x_1 = x_2$ follows.
Now, $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$, so by induction $f|_{X\setminus\{a\}}$ will be bijective because it is injective. In particular, f by way of $f|_{X\setminus\{a\}}$ fully hits $Y \setminus \{f(a)\}$ in its image, so because $f(a) \in \operatorname{im} f$ as well, we conclude $\operatorname{im} f = Y$. So f is surjective.

- Take f surjective; we show f is injective. Define a function $g\colon Y \to X$ as follows: for each $y \in Y$, the surjectivity of f promises some $x \in X$ such that $f(x) = y$, so choose any such x and define $g(y) := x$.[2] Observe that $f(g(y)) = y$ by construction.
Now, we notice that g is injective: if $y_1, y_2 \in Y$ have $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. So the previous case tells us that g is in fact bijective.
So now choose any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. The surjectivity of f promises some $y_1, y_2 \in Y$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$, so we see that
$$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$
proving our injectivity. ■

Lemma 1.28. Fix $f\colon X \to Y$ a bijective function. Then there is a unique function $g\colon Y \to X$ such that $f \circ g = \operatorname{id}_Y$ and $g \circ f = \operatorname{id}_X$.

Proof. We show existence and uniqueness separately.

- We show existence. Note that, because $f\colon X \to Y$ is surjective, each $y \in Y$ has some $x \in X$ such that $f(x) = y$. In fact, this $x \in X$ is uniquely defined because $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so we may define $g(y)$ as the value x for which $f(x) = y$.
By construction, $f(g(y)) = y$, so $f \circ g = \operatorname{id}_Y$. Additionally, we note that, given any $x \in X$, the value x_0 for which $f(x) = f(x_0)$ is $x = x_0$ by the injectivity, so $g(f(x)) = x$. Thus, $g \circ f = \operatorname{id}_X$, as claimed.

- We show uniqueness. Suppose that we have two functions $g_1, g_2\colon Y \to X$ which satisfy
$$f \circ g_1 = f \circ g_2 = \operatorname{id}_Y \qquad \text{and} \qquad g_1 \circ f = g_2 \circ f = \operatorname{id}_X .$$
Then we see that
$$g_1 = g_1 \circ \operatorname{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \operatorname{id}_X \circ g_2 = g_2,$$
where we have used the fact that function composition associates. This finishes. ■

[2] Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.

THEME 2

COMPLEX NUMBERS AND THEIR TOPOLOGY

This somewhat laborious proof could have been avoided if one had defined a complex analytic structure

—Jean-Pierre Serre, [Ser12]

2.1 January 24

Good morning everyone.

2.1.1 Algebraic Structure

Today we are reviewing the complex numbers (reportedly, "some basics"). Or at least it is hopefully mostly review. Here is our main character this semester.

Definition 2.1 (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here i is some symbol such that $i^2 = -1$ formally.

In particular, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

The complex numbers also have some algebraic structure.

Definition 2.2 (Plus and times in $\mathbb{C}$)**.** Given complex numbers $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$, we define

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i,$$

and

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,$$

defined essentially by direct expansion, upon recalling $i^2 = -1$.

Here is the corresponding algebraic structure.

Proposition 2.3. The set $\mathbb{C}$ with the above operations is a two-dimensional $\mathbb{R}$-vector space with basis $\{1, i\}$.

Proof. The elements $\{1, i\}$ span $\mathbb{C}$ because all complex numbers in $\mathbb{C}$ can be written as $a + bi = a \cdot 1 + b \cdot i$ by definition.

To see that these elements are linearly independent, suppose $a + bi = 0$. If $b = 0$, then $a = 0$ follows, and we are done. Otherwise, take $b \neq 0$, but then we see $(-a/b) = i$, so

$$(-a/b)^2 = -1 < 0,$$

which does not make sense for real numbers. This finishes. ■

Proposition 2.4. The set $\mathbb{C}$ with the above operations is a field.

Proof. We have the following checks.

- The element $0 + 0i$ is our additive identity. Indeed, one can check that $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i) = a + bi$.
- The element $1 + 0i$ is our multiplicative identity. Indeed, one can check that $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$.
- Commutativity of addition and multiplication follow from by expansion.
- The distributive laws can again be checked by expansion.
- The additive inverse of $a + bi$ is $(-a) + (-b)i$.
- The multiplicative inverse of $a + bi$ can be found by wishing really hard and writing

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

Then one can check this works. ■

Sometimes we would like to extract our coefficients from our basis.

Definition 2.5 (Real, imaginary parts)**.** Given $z := a + bi \in \mathbb{C}$, we define the operations

$$\operatorname{Re} z := a \qquad \text{and} \qquad \operatorname{Im} z := b.$$

Importantly, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$.

Because we are merely doing basis extraction, it makes sense that these operations will preserve some (additive) structure.

Proposition 2.6. Fix $z = a + bi$ and $w = c + di$. Then the following.

(a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$.

(b) $\operatorname{Im}(z + w) = \operatorname{Im} z + \operatorname{Im} w$.

Proof. We proceed by direct expansion. Observe

$$\operatorname{Re}(z + w) = \operatorname{Re}((a + c) + (b + d)i) = a + c = \operatorname{Re} z + \operatorname{Re} w,$$

and

$$\operatorname{Im}(z + w) = \operatorname{Im}((a + c) + (b + d)i) = b + d = \operatorname{Im} z + \operatorname{Im} w.$$

This finishes. ■

It also turns out that the complex numbers have a very special transformation.

Definition 2.7 (Conjugate). Given $z := a+bi \in \mathbb{C}$, we define the *complex conjugate* to be $\overline{z} := a-bi \in \mathbb{C}$.

We promised conjugation would be special, so here are some special things.

Proposition 2.8. Fix $z = a + bi \in \mathbb{C}$. Then the following.

(a) $z + \overline{z} = 2\operatorname{Re} z$.

(b) $z - \overline{z} = 2i\operatorname{Im} z$.

(c) $\overline{\overline{z}} = z$.

Proof. We take these one at a time.

(a) Write $a + bi + \overline{a+bi} = a + bi + a - bi = 2a$.

(b) Write $a + bi - \overline{a+bi} = a + bi - (a - bi) = 2bi$.

(c) Write $\overline{\overline{a+bi}} = \overline{a-bi} = a + bi$. ■

In fact, more is true.

Proposition 2.9. Fix $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$. Then the following.

(a) $\overline{z+w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.

Proof. We take these one at a time.

- Write
$$\overline{z+w} = (a+c) - (b+d)i = (a-bi) + (c-di) = \overline{z} + \overline{w}.$$
- Write
$$\begin{aligned}\overline{z} \cdot \overline{w} &= (a-bi)(c-di) \\ &= (ac-bd) - (ad+bc)i \\ &= \overline{(ac-bd) + (ad+bc)i} \\ &= \overline{zw}.\end{aligned}$$

This finishes. ■

2.1.2 Defining Distance

Complex conjugation actually gives rise to a notion of size.

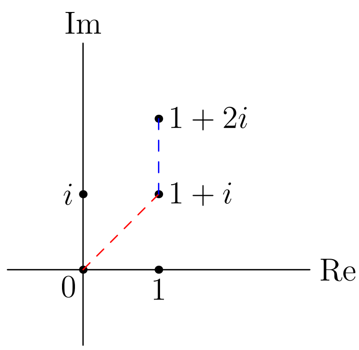
Definition 2.10 (Norm on $\mathbb{C}$). Given $z := a + bi$, we define the *norm function on* $\mathbb{C}$ by
$$|z| := \sqrt{a^2+b^2}.$$

Size actually gives distance.

Definition 2.11 (Distance on $\mathbb{C}$)**.** Given complex numbers $z = a + bi$ and $w = c + di$, we define the *distance* between z and w to be

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}.$$

Here are some examples.



One can ask what is the distance between $0+0i$ and $1+1i$, and we can compute directly that this is $\sqrt{1+1} = \sqrt{2}$. Similarly, the distance between $1+2i$ and $1+i$ is $|(1+2i)-(1+i)| = |i| = 1$. It should agree with our geometric intuition.

We mentioned complex conjugation is involved here, so we have the following lemma.

Lemma 2.12. Fix $z, w \in \mathbb{C}$. The following are true.

(a) $|z|^2 = z\overline{z}$.

(b) $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

(c) $|z| = |\overline{z}| = |-z|$.

(d) $|z| = 0$ if and only if $z = 0$.

(e) $|zw| = |z| \cdot |w|$.

Proof. We take these one at a time. Set $z = a + bi$.

(a) We have

$$|z|^2 = a^2 + b^2 = (a+bi)(a-bi) = z\overline{z}.$$

Here we have used subtraction of two squares, which one can see when writing $a^2 + b^2 = a^2 - (ib)^2$.

(b) We have $a^2 \leq a^2 + b^2$ and $b^2 \leq a^2 + b^2$ by the Trivial inequality, so

$$|\operatorname{Re} z| = |a| \leq \sqrt{a^2 + b^2} = |z|,$$

and similarly,

$$|\operatorname{Im} z| = |b| \leq \sqrt{a^2 + b^2} = |z|.$$

(c) Note

$$|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|,$$

and

$$|-z| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(d) From (b), we know that $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$, but $|z| = 0$ then forces $\operatorname{Re} z = \operatorname{Im} z = 0$, so $z = 0$.

(e) From (a), we can write $|zw|^2 = zw \cdot \overline{zw}$, which will expand out into

$$z \cdot w \cdot \overline{z} \cdot \overline{w}.$$

We can collect this into $z\overline{z} \cdot w\overline{w} = |z|^2|w|^2$. Thus, by (a) again, $|zw|^2 = |z|^2|w|^2$. But because all norms must be nonnegative real numbers, we may take square roots to conclude $|zw| = |z| \cdot |w|$. ■

Remark 2.13. Norms are actually more general constructions. For example, the requirement $|zw| = |z| \cdot |w|$ makes $|\cdot|$ into a "multiplicative" norm.

To finish off, we actually show that our distance function is good: we show the triangle inequality.

Lemma 2.14 (Triangle inequality)**.** For every $x, y, z \in \mathbb{C}$, we claim

$$|z - x| \le |z - y| + |y - z|.$$

This claim should be familiar from real analysis. Intuitively, it means that travelling between z and x cannot be made into a shorter trip by taking a detour to some other point y first.

Proof. Let $a := z - y$ and $b := y - z$ so that $a + b = z - x$. Thus, we are showing that

$$|a + b| \overset{?}{\le} |a| + |b|,$$

which is nicer because it only has two letters. For this, because everything is a nonnegative real numbers, it suffices to show the square of this requirement; i.e., we show

$$(|a| + |b|)^2 - |a + b|^2 \overset{?}{\ge} 0.$$

Fully expanding, it suffices to show

$$|a|^2 + |b|^2 + 2|a| \cdot |b| - |a + b|^2 \overset{?}{\ge} 0.$$

Expanding out $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$, we are showing

$$a\overline{a} + b\overline{b} + 2|a| \cdot |b| - (a + b)(\overline{a} + \overline{b}) \overset{?}{\ge} 0.$$

This is nice because the expansion of the rightmost term will induce some cancellation: it expands into $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b}$, so we are left with showing

$$2|a| \cdot |b| - (a\overline{b} + b\overline{a}) \overset{?}{\ge} 0.$$

Note that $\overline{a}b = \overline{a\overline{b}}$, so we can collect the final term as $2\operatorname{Re}(a\overline{b})$. Similarly, we can write $|a| \cdot |b| = |a| \cdot |\overline{b}| = |a\overline{b}|$, so we are showing

$$2|a\overline{b}| - 2\operatorname{Re}(a\overline{b}) \ge 0,$$

which is true because the real part does exceed the norm. This finishes. ■

2.2 January 26

In-person class should start on Monday. Homework #2 will be released on Friday.

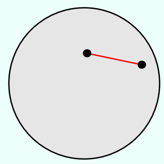
2.2.1 Geometry on $\mathbb{C}$

So let's try to build a topology on $\mathbb{C}$ today. We pick up the following definition.

Definition 2.15 (Convex)**.** A subset $X \subseteq \mathbb{C}$ is *convex* if and only if, for every $z, w \in X$ and $t \in [0,1]$, we have that $w + t(z-w) \in X$.

Intuitively, "convex" means that X contains the line segment of any two points in X.

Example 2.16. The disk is convex: any line with endpoints in the circle lives in the circle.

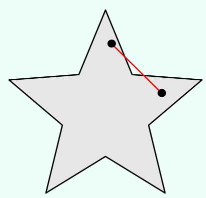


More explicitly, given $z, w \in B(z_0, r)$ for $r > 0$, we see that any $t \in [0,1]$ will have

$$|w + t(z-w) - z_0| = |(1-t)(w-z_0) + t(t-z_0)| \le (1-t)|w-z_0| + (1-t)|z-z_0| = (1-t)r + tr = r,$$

so $w + t(z-w) \in B(z_0, r)$. Replacing the $<$ with $\le$ shows that $\overline{B(z_0, r)}$ is convex.

Non-Example 2.17. The star-shape is not convex: the given line goes outside the star.



To define our open sets, we will define balls first.

Definition 2.18 (Open ball)**.** Given some $z_0 \in \mathbb{C}$, then *open ball* centered at z_0 with radius $r > 0$ is

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Observe $z_0 \in B(z_0, r)$.

Open balls let us define all sorts of properties.

Definition 2.19 (Isolated)**.** Fix $X \subseteq \mathbb{C}$. A point $z \in X$ is isolated in X if and only if there exists $r > 0$ such that

$$B(z, r) \cap X = \{z\}.$$

Definition 2.20 (Discrete)**.** A subset $X \subseteq \mathbb{C}$ is *discrete* if and only if every point is isolated.

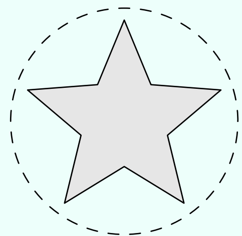
Example 2.21. Any finite subset of $X \subseteq \mathbb{C}$ is discrete. Namely, any point $z \in X$ can take

$$r = \frac{1}{2} \min_{w \in X \setminus \{z\}} |z - x|.$$

Example 2.22. The subset $\mathbb{Z} \subseteq \mathbb{C}$ is isolated. Namely, take $r = \frac{1}{2}$ for any given point.

Definition 2.23 (Bounded)**.** A subset $X \subseteq \mathbb{C}$ is *bounded* if and only if there is an M such that $X \subseteq B(0, M)$.

Example 2.24. The star from earlier fits into a large circle and is therefore bounded.



And here is our fundamental definition for our topology.

Definition 2.25 (Open)**.** A subset $X \subseteq \mathbb{C}$ is *open* if and only if, for each $z \in X$, there exists $r > 0$ such that $B(z, r) \subseteq X$.

Remark 2.26 (Nir)**.** We should probably show that open balls are open; let $B(z, r)$ be an open ball. Well, for any $w \in B(z, r)$, set $r_w := r - |z - w|$, which is positive because $w \in B(z, r)$ requires $|z - w| < r$. Now, $w' \in B(w, r_w)$ implies that $|w - w'| < r - |z - w|$, so by the triangle inequality,

$$|z - w'| \le |z - w| + |w - w'| < r,$$

so $w' \in B(z, r)$ follows. So indeed, each $w \in B(z, r)$ has $B(w, r_w) \subseteq B(z, r)$.

Open lets us define closed.

Definition 2.27 (Closed)**.** A subset $X \subseteq \mathbb{C}$ is *closed* if and only if $\mathbb{C} \setminus X$ is open.

Warning 2.28. Sets are not doors: a set can be both open and closed.

2.2.2 Unions and Intersections

Here are some basic properties of our topology.

Lemma 2.29. The subsets $\varnothing$ and $\mathbb{C}$ are open and closed in $\mathbb{C}$.

Proof. It suffices to show that $\varnothing$ and $\mathbb{C}$ are both open, by definition of closed. That $\varnothing$ is open holds vacuously because one cannot find any $z \in \varnothing$ anyways. That $\mathbb{C}$ is open holds because open balls are subsets of $\mathbb{C}$, so any $z \in \mathbb{C}$ can take $r = 1$ so that

$$B(z, r) \subseteq \mathbb{C}.$$

So we are done. ■

Lemma 2.30. Fixing some $z \in \mathbb{C}$, the set $\{z\}$ is closed.

Proof. We show that $U := \mathbb{C} \setminus \{z\}$ is open. Well, fix any $w \in U$, and because $w \neq z$, we note $|z - w| > 0$, so we set $r := \frac{1}{2}|z - w|$. It follows that

$$z \notin B(w, r)$$

because $|z - w| > r$. But this is equivalent to $B(w, r) \subseteq \mathbb{C} \setminus \{x\} = U$, so we are done. ■

We would like to make new open and closed subsets from old ones. Here is one way to do so.

Lemma 2.31. The following are true.

(a) Arbitrary union: if $\mathcal{U}$ is any collection of open subsets of $\mathbb{C}$, then the union $\bigcup_{U \in \mathcal{U}} U$ is also open.

(b) Arbitrary intersection: if $\mathcal{V}$ is any collection of closed subsets of $\mathbb{C}$, then intersection $\bigcap_{V \in \mathcal{V}} V$ is also closed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcup_{U \in \mathcal{U}} U$. We need to show there is some $r > 0$ such that

$$B(z, r) \stackrel{?}{\subseteq} \bigcup_{U \in \mathcal{U}} U.$$

Well, we know there must be some $U_z \in \mathcal{U}$ such that $z \in U_z$ by definition of the union. But now U_z is open, and therefore we are promised an $r > 0$ such that

$$B(z, r) \subseteq U_z \subseteq \bigcup_{U \in \mathcal{U}} U,$$

so we are done.

(b) Fix $\mathcal{V}$ a collection of closed subsets of $\mathbb{C}$. We want to show that

$$\mathbb{C} \Bigg\backslash \bigcap_{V \in \mathcal{V}} V$$

is open, which by de Morgan's law is equivalent to

$$\bigcup_{V \in \mathcal{V}} (\mathbb{C} \setminus V)$$

being open. However, each $V \in \mathcal{V}$ is closed, so $\mathbb{C} \setminus V$ will be open, so we are done by (a). ■

Lemma 2.32. The following are true.

(a) Finite intersection: if $\{U_k\}_{k=1}^n$ is a finite collection of open subsets of $\mathbb{C}$, then the intersection $\bigcap_{k=1}^n U_k$ is also open.

(b) Finite union: if $\{V_k\}_{k=1}^n$ is a finite collection of closed subsets of $\mathbb{C}$, then $\bigcup_{k=1}^n V_k$ is also closed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcap_{k=1}^n U_k$ so that we need to find $r > 0$ such that

$$B(z,r) \overset{\subseteq}{\bigcup}_{k=1}^{n} U_k.$$

Well, $z \in U_k$ for each k, and each U_k is open, so there is an $r_k > 0$ such that $B(z, r_k) \subseteq U_k$. Thus, we set $r := \min_k\{r_k\}$; because there are only finitely many r_k, we are assured that $r > 0$. Now, we observe that

$$B(z,r) \subseteq B(z,r_k) \subseteq U_k.$$

(Explicitly, $|w - z| < r$ implies $|w - z| < r_k$ because $r \le r_k$.) Thus, it follows that

$$B(z,r) \subseteq \bigcap_{k=1}^{n} U_k,$$

as desired.

(b) We use de Morgan's laws. We want to show that

$$\mathbb{C} \Bigg\backslash \bigcup_{k=1}^{n} V_k$$

is open, which by de Morgan's laws is the same thing as showing that

$$\bigcap_{k=1}^{n} (\mathbb{C} \setminus V_k)$$

is open. However, each $\mathbb{C} \setminus V_k$ is open by hypothesis on the V_k, so the full intersection is open by (a). This finishes. ■

Remark 2.33. The finiteness is in fact necessary. For example,

$$\bigcap_{n \in \mathbb{N}} B(0, 1/n) = \{0\}.$$

Then one can check that each open ball is open while singletons in $\mathbb{C}$ are not.

2.2.3 Interior, Closure

Let's see more definitions.

Definition 2.34 (Interior)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *interior* X° of X to be the union of all open sets contained in X (which will be open by Lemma 2.31).

Remark 2.35. In fact, X° is the largest open subset of X, for any open subset $U_0 \subseteq \mathbb{C}$ contained in X will have

$$U_0 \subseteq \bigcup_{\text{open } U \subseteq X} U = X^\circ.$$

It follows X is open if and only if $X = X^\circ$: if $X = X^\circ$, then X is open because X° is open; if X is open, then X is the largest open subset of $\mathbb{C}$ contained in X, so $X = X^\circ$.

Definition 2.36 (Closure)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *closure* $\overline{X}$ of X to be the intersection of all closed sets containing X (which will be closed by Lemma 2.31).

Remark 2.37. In fact, X° is the smallest closed set containing X, for any closed subset $V_0 \subseteq \mathbb{C}$ containing X will have

$$V_0 \supseteq \bigcap_{\text{open } V \supseteq X} V = \overline{X}.$$

It follows X is closed if and only if $X = \overline{X}$: if $X = \overline{X}$, then X is open because $\overline{X}$ is closed; if X is closed, then X is the smallest closed subset of $\mathbb{C}$ containing X, so $X = \overline{X}$.

By the above definitions, it is not too hard to see that $X^\circ \subseteq X \subseteq \overline{X}$.

The interior and closure also let us define the boundary.

Definition 2.38 (Frontier, boundary)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *frontier* or *boundary* ∂X of X to be $\overline{X} \setminus X^\circ$.

2.2.4 Connectivity

Definition 2.39 (Disconnected)**.** A subset $X \subseteq \mathbb{C}$ is *disconnected* if and only if there exists nonempty disjoint open subsets U_1 and U_2 such that $X \subseteq U_1 \cup U_2$ and $X \cap U_1, X_\cap U_2 \neq \varnothing$. (In other words, the subspace of $X \subseteq \mathbb{C}$ is (topologically) disconnected.) In this case, we say that U_1 and U_2 *disconnect* X.

Lastly, we say X is *connected* if and only if it is not disconnected.

Example 2.40. The set $\varnothing$ is connected because it is impossible for $U \cap \varnothing \neq \varnothing$ for any open set U of $\mathbb{C}$.

Example 2.41. Any singleton $\{z\}$ is connected. In fact, one cannot decompose $\{x\}$ into two disjoint sets at all, much less into disjoint sets of the form $U \cap \{x\}$ with U open.

Example 2.42. Any open ball $B(z, r)$ is connected. This is surprisingly annoying to check. We will show this shortly by showing that $B(z, r)$ is path-connected.

Example 2.43. The set $\{1, 2\}$ is disconnected by $U_1 = B(1, 1/2)$ and $U_2 = B(2, 1/2)$.

Connectivity plays nicely with the rest of our definitions as well.

Lemma 2.44. A given subset $X \subseteq \mathbb{C}$ is connected if and only if the only subsets of X which are both open and closed (in the subspace topology) are $\varnothing$ and X.

Proof. We take the directions independently. For the forwards direction, take X connected, and suppose that $U \subseteq X$ is open and closed. In the subspace topology, we get that $X \setminus U$ will also be open, and then the subsets U and $X \setminus U$ are both open, disjoint and have

$$X = U \cup (X \setminus U).$$

Thus, we require $U = \varnothing$ or $X \setminus U = \varnothing$, so $U \in \{\varnothing, X\}$.

We leave the reverse direction as an exercise. Suppose that X is disconnected, and we show that there is a nonempty proper closed and open subset of X. Well, because X is disconnected, we have disjoint open sets U_1 and U_2 of $\mathbb{C}$ such that $X \cap U_1, X \cap U_2 \neq \varnothing$ and $X \subseteq U_1 \cup U_2$. It follows that

$$X = (U_1 \cap X) \cup (U_2 \cap X). \tag{$*$}$$

However, now consider the open subset $U := U_1 \cap X$ of X. We note that $(U_1 \cap X) \cap (U_2 \cap X) = \varnothing$, so by ($*$) we see that $U_1 \cap X = X \setminus (U_2 \cap X)$, so $U_1 \cap X$ is closed as well.

To finish, we note that $U \neq \varnothing$ is nonempty, and its complement is $X \setminus U = U_2 \cap X$ is also nonempty, so $U \neq X$ is proper. Thus, U is a proper nonempty closed and open subset of X. This finishes. ■

Remark 2.45 (Nir)**.** It is actually important that the open subsets in the above lemma are in the subspace topology and are not required to be $\mathbb{C}$-open. For example, $X = \{1, 2\}$ is disconnected, but it has no nonempty $\mathbb{C}$-open subsets to witness this.

Lemma 2.46. Fix $\mathcal{S}$ a collection of connected subsets of $\mathbb{C}$. If $\bigcap_{S \in \mathcal{S}} S$ is nonempty, then $\bigcup_{S \in \mathcal{S}} S$ will be connected.

Proof. Suppose $\bigcup_{S \in \mathcal{S}} S$ is contained in the disjoint open subsets U_1 and U_2 of $\mathbb{C}$; we claim $U_1 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing$ or $U_2 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing$, which will finish.

Pick up some

$$z \in \bigcap_{S \in \mathcal{S}} S,$$

which exists because the intersection is nonempty. Without loss of generality, we may assume that $z \in U_1$.

Now, $z \in S$ for each $S \in \mathcal{S}$, so we see $U_1 \cap S \neq \varnothing$, so because $(U_1 \cap S) \cup (U_2 \cap S) = S$, we see that $U_2 \cap S = \varnothing$ by hypothesis on S's connectivity. Thus, taking the union over the $U_2 \cap S = \varnothing$,

$$U_2 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing,$$

which finishes the proof. ■

Remark 2.47. The condition with nonempty intersection is necessary: $\{0\}$ and $\{1\}$ are connected, but $\{0\} \cup \{1\}$ is not.

2.3 January 28

Hopefully we'll be in-person on Monday. Homework 2 will be released later today, due next Friday.

2.3.1 Sequences

Today we're talking about sequences, building towards a theory of sequences and series. Next week we will begin studying holomorphic functions and actually doing complex analysis.

Anyways, here is a series of definitions.

Definition 2.48 (Sequence)**.** A *sequence* of complex numbers is a function $f \colon \mathbb{N} \to \mathbb{C}$. Often we will notate this by $\{z_n\}_{n \in \mathbb{N}}$ where $z_n := f(n)$.

By convention, all of our sequences will be sequences of complex numbers unless otherwise stated.

Definition 2.49 (Subsequence)**.** A sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a *subsequence* of a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ if and only if there is some strictly increasing function $g\colon \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{g(n)}$.

Definition 2.50 (Bounded)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is *bounded* if and only if there exists a positive real number $M > 0$ such that

$$\{z_n\}_{n\in\mathbb{N}} \subseteq B(0, M).$$

In other words, $|z_n| < M$ for each $n \in \mathbb{N}$.

We are in particular interested in convergence in analysis.

Definition 2.51 (Converges)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *converges* to some $z \in \mathbb{C}$ if and only if, for each $\varepsilon > 0$, there exists some N such that $n > N$ implies

$$|z - z_n| < \varepsilon.$$

We will notate this by $z_n \to z$ or $\lim_{n\to\infty} z_n = z$.

Note that the definition of the limit above says that

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} |z_n - z| = 0.$$

Intuitively, the distance between the z_n and the z has to "narrow in" on z.

We would like some real-analytic tools for our complex analysis. Here is a convergence lemma.

Lemma 2.52. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence. Then, letting $z_n := x_n + y_n i$, we have that $z_n \to z$ where $z = x + yi$ if and only if $x_n \to x$ and $y_n \to y$.

Proof. This is essentially by definition of the metric on $\mathbb{C}$. We take the directions one at a time.

- Suppose that $z_n \to z$ in $\mathbb{C}$. Then we claim that $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z_n$ in $\mathbb{R}$. Indeed, for any $\varepsilon > 0$, there is N such that

  $$n > N \implies |z - z_n| < \varepsilon.$$

  But now we see that $|\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| \le \sqrt{(\operatorname{Re} z_n - \operatorname{Re} z)^2 + (\operatorname{Im} z_n - \operatorname{Im} z)^2}$, so it follows

  $$n > N \implies |\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| < \varepsilon,$$

  finishing.

- Suppose that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to y$. We claim that $z_n \to x + yi$. Indeed, for any $\varepsilon > 0$, there exists N_x such that

  $$n > N_x \implies |\operatorname{Re} z_n - x| < \varepsilon/2$$

  and N_y such that

  $$n > N_y \implies |\operatorname{Im} z_n - y| < \varepsilon/2.$$

  It follows that

  $$n > \max\{N_x, N_y\} \implies |z_n - (x + yi)| = \sqrt{|\operatorname{Re} z_n - x|^2 + |\operatorname{Im} z_n - y|^2} \le \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2} < \varepsilon.$$

  This finishes. ■

Essentially, this means that checking convergence of complex numbers is the same as checking real and imaginary parts individually, so we can turn convergence questions into ones from real analysis.

We also have the following basic properties about convergence.

Proposition 2.53. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a convergent sequence. The following are true.

(a) $\{z_n\}_{n\in\mathbb{N}}$ is bounded.

(b) The limit of $\{z_n\}_{n\in\mathbb{N}}$ is unique.

(c) Every subsequence of $\{z_n\}_{n\in\mathbb{N}}$ converges to z.

Proof. We take the claims one at a time. Let $z \in \mathbb{C}$ be so that $z_n \to z$.

(a) Fix $\varepsilon = 1$ so that there exists N so that $n > N$ implies $|z_n - z| < 1$. Now set

$$M := \max(\{|z_n| + 1 : n \le N\} \cup \{|z| + 1\}).$$

We claim that $|z_n| < M$ for each $n \in \mathbb{N}$. We have two cases.

- If $n \le N$, then $|z_n| < |z_n| + 1 \le M$.
- Otherwise, $n > N$ so that
$$|z_n| \le |z_n - z| + |z| < |z| + 1 \le M,$$
so we are done.

(b) Suppose that $z_n \to z'$ for some $z' \in \mathbb{C}$, and we show $z = z'$. Indeed, if $z = z'$, then we are done, so suppose that $z \ne z'$ so that $|z - z'| \ne 0$. Then we set $\varepsilon := \frac{1}{2}|z - z'| > 0$, and we are promised some N, N' such that
$$n > N \implies |z - z_n| < \frac{\varepsilon}{2} \quad \text{and} \quad n > N' \implies |z' - z_n| < \frac{\varepsilon}{2}.$$
In particular, we see that, for $n > \max\{N, N'\}$, we have
$$|z - z'| \le |z - z_n| + |z_n - z'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{1}{2}|z - z'|.$$
But because $0 \le |z - z'|$, we see that this forces $|z - z'| = 0$, so $z = z'$ follows. (Technically we have hit contradiction, but we do not need to use this.)

(c) Note that subsequences can be characterized by choosing a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ so that we want to show $z_{f(n)} \to z$. Indeed, for any $\varepsilon > 0$, we are promised some N so that
$$n > N \implies |z - z_n| < \varepsilon.$$
Now, for each $n \in \mathbb{N}$, we have[1] $f(n) \ge n$, so we see that
$$n > N \implies f(n) > N \implies |z - z_{f(n)}| < \varepsilon,$$
which finishes. ■

Sequences themselves have an arithmetic.

Proposition 2.54. Fix $\{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ sequences such that $z_n \to z$ and $w_n \to w$. Then the following hold.

(a) $z_n + w_n \to z + w$.

(b) $z_n w_n \to zw$.

(c) If $w \ne 0$ and $w_n \ne 0$ for each $n \in \mathbb{N}$, then $\frac{1}{w_n} \to \frac{1}{w}$.

[1] We can show this by induction on n, for $f(0) \ge 0$ and $f(n+1) > f(n)$ forces $f(n+1) \ge f(n) + 1$.

Proof. We take these one at a time, essentially borrowing the proof from metric spaces.

(a) Fix some $\varepsilon > 0$. We can find some N_z such that

$$n > N_z \implies |z - z_n| < \varepsilon/2$$

and some N_w such that

$$n > N_w \implies |w - w_n| < \varepsilon/2.$$

Now, taking $N := \max\{N_z, N_w\}$ so that the triangle inequality gives

$$n > N \implies |(z + w) - (z_n + w_n)| \le |z - z_n| + |w - w_n| < \varepsilon,$$

which finishes.

(b) We have to use the fact that the sequences are bounded here. Because $w_n \to w$, the sequence is bounded, so there is an M so that $|w_n| < M$ for each $n \in \mathbb{N}$. Now, the key inequality is that

$$|z_n w_n - zw| \le |z_n w_n - zw_n| + |zw_n - zw| \le M|z_n - z| + |z| \cdot |w_n - w|. \qquad (*)$$

So with this in mind, fix any $\varepsilon > 0$, and we see that we are promised N_z such that

$$n > N_z \implies |z_n - z| < \frac{\varepsilon}{2M}$$

and some N_w such that

$$n > N_w \implies |w_n - w| < \frac{\varepsilon}{2|z|}$$

so that $(*)$ implies

$$n > \max\{N_x, N_w\} \implies |z_n w_n - zw| < \varepsilon,$$

finishing.

(c) We begin with some motivating arithmetic. Observe that

$$\left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|ww_n|}.$$

We can upper-bound the numerator without tears, so we see the main difficulty is lower-bounding the denominator. Well, because $w \ne 0$, we can set $\varepsilon = |w|/2$ so that there exists N_0 such that

$$n > N_0 \implies |w_n - w| < |w|/2.$$

In particular, it follows that $|w_n| \ge |w| - |w - w_n| = |w|/2$ for $n > N_0$.

With this in mind, fix any $\varepsilon > 0$. Then we are promised some N_1 such that

$$n > N_1 \implies |w_n - w| < |w|^2\varepsilon/2$$

so that we see

$$n > \max\{N_0, N_1\} \implies \left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|w| \cdot |w_n|} \le \frac{|w|^2\varepsilon/2}{|w| \cdot |w|/2} = \varepsilon,$$

finishing. ■

2.3.2 Limit Points

Here is our main character.

Definition 2.55 (Limit point). Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is a *limit point* if and only if there exists some sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$.

Definition 2.56 (Accumulation point). Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is an *accumulation point* if and only if there exists some sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X \setminus \{z\}$ such that $z_n \to z$.

Essentially accumulation points do not allow isolated points while limit points do.

The above essentially gives a more directly topological definition of "closed set." It also gives us a more directly topological definition of the closure.

Lemma 2.57. Fix $X \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. The following are equivalent.

(a) We have that $z \in \overline{X}$.

(b) For all $\varepsilon > 0$, we have $B(z, \varepsilon) \cap X \neq \varnothing$.

(c) There exists $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$.

Proof. We show our directions one at a time.

- We show (a) implies (b). Suppose $z \in \overline{X}$, and for the sake of contradiction suppose we have $\varepsilon > 0$ such that $B(z, \varepsilon) \cap X = \varnothing$. In particular, $z \notin X$.

  Now, $z \in \overline{X}$ implies that z is contained in every closed set containing X by definition of $\overline{X}$. But because $B(z, \varepsilon)$ is open and is disjoint from X, we see

  $$z \in \overline{X} \subseteq \mathbb{C} \setminus B(z, \varepsilon),$$

  which is a contradiction.

- We show (b) implies (c). For each $n \in \mathbb{N}$, we know that $B(z, 1/n) \cap X \neq \varnothing$, so we find some $z_n \in B(z, 1/n)$. Now, for any $\varepsilon > 0$, choose $N := 1/\varepsilon$ so that

  $$n > N \implies |z_n - z| < \frac{1}{n} < \frac{1}{N} = \varepsilon,$$

  so indeed $z_n \to z$.

- We show (b) implies (a). We proceed by contraposition. Suppose that $z \notin \overline{X}$. It follows that $z \in \mathbb{C} \setminus \overline{X}$, which is open, so there exists an $r > 0$ such that

  $$B(z, r) \subseteq \mathbb{C} \setminus \overline{X} \subseteq \mathbb{C} \setminus X.$$

  It follows that $B(z, r) \cap X = \varnothing$.

- We show (c) implies (b). Suppose $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ has $z_n \to z$ for some $z \in \mathbb{C}$. For any $\varepsilon > 0$, there exists N such that

  $$n > N \implies |z_n - z| < \varepsilon,$$

  so in particular, choosing any $n := \lceil N \rceil + 1$ has $z_n \in B(z, \varepsilon) \cap X$, so $B(z, \varepsilon) \cap X \neq \varnothing$. ■

The above discussion can give us a more directly topological definition of "closed."

Lemma 2.58. A subset $X \subseteq \mathbb{C}$ is closed in $\mathbb{C}$ if and only if X contains all of its limit points.

Proof. By the previous lemma, we see that $z \in \overline{X}$ if and only if z is a limit point of X, so $\overline{X}$ is the set of limit points of X. Now, X is closed if and only if $X = \overline{X}$, so X is closed if and only if all limit points of X are in fact points of X. (Note that all points of X are automatically limit points essentially because $X \subseteq \overline{X}$ for free.) ■

While we're here, we can pick up a nice topological result.

Lemma 2.59. Fix $X \subseteq \mathbb{C}$ a connected subset. Then $\overline{X}$ is also connected.

Proof. This argument is purely topological. We proceed by contraposition: suppose $\overline{X}$ is disconnected by $U_1, U_2 \subseteq \mathbb{C}$. We claim that U_1, U_2 disconnect X. Well, we already know that $A \subseteq \overline{A} \subseteq U_1 \cup U_2$, and we already know that U_1 and U_2 are disjoint.

We claim that, for $U \subseteq \mathbb{C}$ an open subset, if $U \cap \overline{X} \neq \varnothing$, then $U \cap X \neq \varnothing$ as well. Indeed, we proceed by contraposition: if $U \cap X = \varnothing$, then $X \subseteq \mathbb{C} \setminus U$, but $\mathbb{C} \setminus U$ is closed, so

$$\overline{X} \subseteq \mathbb{C} \setminus U,$$

so $\overline{X} \cap U = \varnothing$.

Thus, it follows from $U_1 \cap \overline{X}, U_2 \cap \overline{X} \neq \varnothing$ that $U_1 \cap X, U_2 \cap X \neq \varnothing$. This finishes the proof that U_1 and U_2 disconnect X. Indeed, ■

2.3.3 Cauchy Sequences

Just like in real analysis, we will be interested in Cauchy sequences.

Definition 2.60 (Cauchy sequence)**.** A sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is a *Cauchy sequence* if and only if, for each $\varepsilon > 0$, there exists an N such that

$$n, m > N \implies |z_n - z_m| < \varepsilon.$$

We have the following results on Cauchy sequences.

Proposition 2.61. Fix $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ a sequence. If $\{z_n\}_{n \in \mathbb{N}}$ converges, it is Cauchy.

Proof. This proof uses no special properties of $\mathbb{C}$. If $z_n \to z$, then for a given $\varepsilon > 0$, there exists N such that

$$n > N \implies |z_n - z| < \varepsilon/2.$$

It follows that

$$n, m > N \implies |z_n - z_m| < |z_n - z| + |z_m - z| < \varepsilon,$$

finishing. ■

Proposition 2.62. Every Cauchy sequence in $\mathbb{C}$ converges.

Proof. If $\{z_n\}_{n \in \mathbb{N}}$ is Cauchy, then we claim $\{\operatorname{Re} z_n\}_{n \in \mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n \in \mathbb{N}}$ are Cauchy sequences. Indeed, for any $\varepsilon > 0$, there exists N so that

$$n, m > N \implies |z_n - z_m| < \varepsilon,$$

but then $|\operatorname{Re} z_n - \operatorname{Re} z_m| < |z_n - z_m|$ and $|\operatorname{Im} z_n - \operatorname{Im} z_m| < |z_n - z_m|$, so the same N witnesses that $\{\operatorname{Re} z_n\}_{n \in \mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n \in \mathbb{N}}$ are Cauchy in $\mathbb{R}$.

Now, Cauchy sequences in $\mathbb{R}$ converge, so there are reals $x, y \in \mathbb{R}$ such that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to w$. It follows that $z_n \to x + yi$, finishing. ■

2.3.4 A Little More Topology

We close with one more topological definition.

Definition 2.63 (Sequentially compact)**.** A subset $X \subseteq \mathbb{C}$ is *sequentially compact* if and only if every $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ has a convergent subsequence which converges in X.

Remark 2.64. This happens to be equivalent to X is compact because $\mathbb{C} \cong \mathbb{R}^2$ satisfies the fact that all compact sets are closed and bounded.

Example 2.65. Every finite set is compact.

And here is a last definition.

Definition 2.66 (Tends to infinity)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *tends to infinity* (notated $z_n \to \infty$) if and only if each $M > 0$ has some $N \in \mathbb{N}$ such that

$$n > N \implies |z_n| > M.$$

Essentially the points of $\{z_n\}_{n\in\mathbb{N}}$ wander infinitely away.

2.4 January 31

So we are lecturing in-person today. Good morning everyone.

Quote 2.67. If I don't fall off the stage, I will consider it a major accomplishment.

Homework 2 is due Friday, the 4th of February. Office hours will occur at the usual times, but they will now occur in-person at Evans 749.

2.4.1 Series

Today we're mostly talking about series, and on Friday we'll talk about continuous functions.

Definition 2.68 (Series)**.** An *infinite series* over $\mathbb{C}$ is an infinite sum

$$S := \sum_{n=1}^{\infty} z_n$$

where $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a sequence of complex numbers.

With respect to series, we really want to know when various series converge so that the series is well-defined.

Definition 2.69 (Converge, diverge)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

As usual, we start with some basic examples.

Exercise 2.70. Fix some $z \in \mathbb{C}$ with $|z| < 1$, we define $z_n := z^n$. Then we have

$$S = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Proof. Fix some partial sum

$$S_N := \sum_{k=0}^{N} z^k = 1 + z + z^2 + \cdots + z^N.$$

Multiplying by z, we see that

$$zS_n = z + z^2 + \cdots + z^N + z^{N+1}.$$

It follows that

$$S_N - zS_N = 1 - z^{N+1}.$$

Because $|z| < 1$, we have $z \neq 1$, so we may write

$$S_N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}.$$

However, we may note that as $N \to \infty$, the bad term z^{N+1} will have size

$$\left|z^{N+1}\right| = |z|^{N+1},$$

which goes to 0 (because $|z| < 1$).[2] It follows that

$$\lim_{N\to\infty} S_N = \frac{1}{1-z},$$

which is what we wanted. ■

Anyways, here are some basic lemmas.

Lemma 2.71 (Divergence test). Suppose that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum z_n$ converges. Then $z_n \to 0$ as $n \to \infty$.

Proof. Let S_n be the nth partial sum so that we are given $S_n \to L$ for some $L \in \mathbb{C}$. But now we see that

$$z_{n+1} = \left(\sum_{k=0}^{N+1} z_k\right) - \left(\sum_{k=0}^{N} z_k\right) = S_{n+1} - S_n.$$

Using limit laws, we see that

$$\lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = L - L = 0.$$

Shifting the indices back gives $z_n \to 0$ as $N \to \infty$. ■

Here is an important example of a divergent series.

[2] This is surprisingly annoying to rigorize with an ε-δ proof, so we won't do so here. The interested can try to use induction to manually bound $|z|^n$ by $\frac{c}{n}$ for some c.

Exercise 2.72. We claim that

$$S = \sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge.

Proof. We will show that the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is not Cauchy, which will show that the series diverges. Well, observe that

$$S_{2^{n+1}} - S_{2^n} = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}$$

after cancelling out all of our terms. However, each term in the sum is at least $\frac{1}{2^{n+1}}$, so we bound

$$S_{2^{n+1}} - S_{2^n} \geq \frac{1}{2^{n+1}} \left(2^{n+1} - 2^n\right) = \frac{1}{2}.$$

We now show that the partial sums are not Cauchy. Fix ε. Supposing for the sake of contradiction that the sequence is Cauchy, there exists N so that $n, m > N$ has

$$|S_n - S_m| < \frac{1}{2}.$$

However, we can find some power of 2 named 2^r which exceeds N, in which case we find $2^{r+1}, 2^r > N$ and

$$|S_{2^{r+1}} - S_{2^r}| \geq \frac{1}{2},$$

which is our contradiction. ■

Remark 2.73. Because a sequence will converge if and only if its real and imaginary parts do, we can turn a convergence test into a real-number test by taking the real and imaginary parts of the sum.

2.4.2 The Comparison Test

Recall the comparison test in $\mathbb{R}$.

Theorem 2.74 (Comparison test)**.** Fix $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ sequences of real numbers. Further, suppose that we there exists a positive constant $c > 0$ such that $0 \leq x_n \leq cy_n$. Then the following hold.

- If $\sum y_n$ converges, then $\sum x_n$ converges as well.
- If $\sum x_n$ diverges, then $\sum y_n$ diverges as well.

Proof. We appeal to real analysis. The interested can see Theorem 2.1.21 in Eterović. The main point is to use the Monotone sequence theorem. ■

Here is an example.

Exercise 2.75. Fix $s > 1$ an integer. Then the series

$$S = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges.

Proof. Because s is an integer, we have $s \geq 2$. Namely, $\frac{1}{k^s} \leq \frac{1}{k^2}$, so by the comparison test it suffices to just show the convergence of

$$S' := \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

For this, we apply some trickery. In particular, for $k > 1$, we bound

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

In particular,

$$S' = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

Thus, by the comparison test, it suffices to show the convergence of

$$T := \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

But the nth partial sum will telescope, giving

$$T_n := \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 - \frac{1}{n},$$

so $T_n \to 1$ as $n \to \infty$, and $T = 1$. It follows that S' is upper-bounded by $1 + T \leq 2$. ■

2.4.3 Absolute Convergence

The following kind of convergence is nontrivially stronger, but that makes it better.

Definition 2.76 (Absolute convergence)**.** Fix a sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ of complex numbers. Then the sum $S := \sum z_n$ converges *absolutely* if and only if the series

$$\sum_{n=0}^{\infty} |z_n|$$

also converges. In other words, the partial sums of the above series converges.

We have the following quick lemma to justify naming this "convergence."

Lemma 2.77. If a series converges absolutely, then the series also converges.

Proof. The idea is to use the triangle inequality. Fix our series

$$S := \sum_{n=0}^{\infty} z_n$$

for which

$$T := \sum_{n=0}^{\infty} |z_n|$$

converges. Let S_n be the nth partial sum of S and T_n the n the partial sum of T.

Our goal is to show that $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy. Observe $\{T_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers because $|z| \geq 0$ always. To start off our arithmetic, we note that, for $n, m \in \mathbb{N}$ with $n > mn$, we have

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} z_k\right|,$$

which by the triangle inequality can be bounded by

$$|S_n - S_m| \leq \sum_{k=m+1}^{n} |z_k| = T_m - T_n.$$

But now we can use the fact that $\{T_n\}_{n\in\mathbb{N}}$ must be Cauchy to finish: for any $\varepsilon > 0$, there exists some N such that $n > m > N$ implies $T_m - T_n < \varepsilon$. But then this same N promises $n > m > N$ implies

$$|S_n - S_m| < T_m - T_n < \varepsilon,$$

which is what we wanted. ■

Here is a surprise tool that will help us later.

Lemma 2.78. Fix a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of nonzero complex numbers. Further, suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ tends to infinity (i.e., $|a_n| \to \infty$ as $n \to \infty$), then for any positive real number $r \in \mathbb{R}^+$, the series

$$\sum_{k=0}^{\infty} \left(\frac{r}{|a_k|}\right)^k$$

converges.

Proof. We need the a_n to be nonzero in order to allow division, so the real puzzle is to determine how to use the fact $|a_n| \to \infty$. Well, there exists some N such that $n > N$ has

$$|a_n| > 2r.$$

But then $\frac{r}{|a_n|} < \frac{1}{2}$ for each $n > N$, so we can use the comparison test as follows: observe that

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

will converge, and there will exist some $c > 1$ so that

$$\frac{r}{|a_k|} < \frac{c}{2^k}$$

for $0 \leq k \leq N$; and then for $n > N$, we get the above inequality anyways as discussed earlier (observe we took $c > 1$). ■

Quote 2.79. I can't break math on the first day of class. I can do it later on.

Lemma 2.80. Suppose that we have two series $S := \sum_{k\in\mathbb{N}} z_k$ and $T := \sum_{k\in\mathbb{N}} w_k$ are both absolutely convergent. Then the sum

$$P := \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j\right)$$

is absolutely convergent as well. In fact, P will converge to ST.

Proof. We sketch the result, and the remaining details are in Eterović. As usual, consider the partial sums

$$A_n := \sum_{k=0}^{n} |z_k| \qquad \text{and} \qquad B_n = \sum_{k=0}^{n} |w_k|,$$

both of which will converge as $n \to \infty$. Brazenly multiplying these together, we see that

$$A_nB_n = \sum_{i,j=0}^{n} |z_iw_j| = \sum_{k=0}^{n} \sum_{\substack{i+j=k \\ 0\le i,j\le n}} |z_iw_j| + \sum_{k>n} \sum_{\substack{i+j=k \\ 0\le i,j\le n}} |z_iw_j|.$$

In the first sum, observe that any time $i+j=k$, we will automatically have $i,j \le k \le n$. It follows that

$$A_nB_n = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_iw_j \right) + \underbrace{\sum_{\substack{i+j>n \\ 0\le i,j\le n}} |z_iw_j|}_{R_n}.$$

The key claim is that $R_n \to 0$. The main idea is that $i+j>n$ implies that $i \ge n/2$ or $j \ge n/2$, so we can write

$$|R_n| \le \sum_{i=0}^{n} \sum_{j=n/2}^{n} |z_iw_j| + \sum_{i=n/2}^{n} \sum_{j=0}^{n} |z_iw_j| = \left(\sum_{i=0}^{n} |z_i| \right) \left(\sum_{j=n/2}^{n} |w_j| \right) + \left(\sum_{i=n/2}^{n} |z_i| \right) \left(\sum_{j=0}^{n} |w_j| \right).$$

Now, fix any $\varepsilon > 0$, and we show there exists X so that $n > X$ has $|R_n| < \varepsilon$. Note $A := \sum |z_k|$ and $B := \sum |w_k|$ both converge and hence have Cauchy partial sums. Because the partial sums are increasing, we bound

$$|R_n| \le A \left(\sum_{j=n/2}^{n} |w_j| \right) + B \left(\sum_{i=n/2}^{n} |z_i| \right)$$

So there exists N such that $n > m > N$ has

$$\sum_{i=m+1}^{n} |z_i| < \frac{\varepsilon}{2B}$$

Similarly there exists M so that $n > m > M$ has

$$\sum_{j=m+1}^{n} |w_j| < \frac{\varepsilon}{2A},$$

from which it follows that $n > n/2 > \max\{N, M\}$ will have

$$|R_n| \le A \cdot \frac{\varepsilon}{2A} + B \cdot \frac{\varepsilon}{2B} = \varepsilon,$$

which finishes.

Now, because $R_n \to 0$, we see

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right) = \lim_{n\to\infty} A_nB_n - \lim_{n\to\infty} R_n,$$

which does indeed converge, so indeed the series

$$\sum_{k=0}^{\infty} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right)$$

will converge. By the comparison test (using the triangle inequality), it follows that

$$P = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j \right)$$

will also absolutely converge.

To show that P converges to ST, we observe that the difference of the nth partial sum is

$$P_n - S_n T_n = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) - \sum_{i,j=0}^{n} z_i w_j = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) - \sum_{k=0}^{n} \left(\sum_{i+j=k}^{n} z_i w_j \right) + \sum_{\substack{0 \le i,j \le n \\ i+j<n}} z_i w_j,$$

so

$$P_n - S_n T_n = \sum_{\substack{0 \le i,j \le n \\ i+j<n}} z_i w_j.$$

But by the triangle inequality, we see $|P_n - S_n T_n| \le R_n$, so $P_n - S_n T_n \to 0$ as $n \to \infty$. It follows P_n and $S_n T_n$ have the same limit as $n \to \infty$ (which exists because S_n and T_n have a limit). So indeed, $P = ST$. ■

2.5 February 2

Good morning everyone. Here is some house-keeping.

- Homework #2 is due on Friday at 11:59, on GradeScope. The assignment has just been added.
- There are office hours to talk about the homework. Please come if you have questions.

2.5.1 Summation Review

Today we finish our discussion of series, for now. We quickly recall the definitions.

Definition 2.68 (Series)**.** An *infinite series* over $\mathbb{C}$ is an infinite sum

$$S := \sum_{n=1}^{\infty} z_n$$

where $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is a sequence of complex numbers.

Definition 2.69 (Converge, diverge)**.** Fix a sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

Today we are building towards proving Dirichlet's convergence theorem. We pick up the following lemmas.

Lemma 2.81. Fix sequences $\{z_{k,\ell}\}_{k,\ell\in\mathbb{N}}$ a collection of complex numbers satisfying the following conditions.

- Fixing any k, the sum $\sum_{\ell=0}^{\infty}|z_{k,\ell}|$ converges.
- The sum $\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|$ converges.

Then the following are true.

(a) Fix any ℓ, the sum $\sum_{k=0}^{\infty}|z_{k,\ell}|$ converges; i.e., the terms in the left sum below are well-defined.

(b) We have that
$$\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}z_{k,\ell}=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}z_{k,\ell},$$
and both sums converge.

Intuitively, the first condition is requiring that the series horizontally does not grow too fast. The second condition is requiring an absolute convergence condition. Then the first conclusion says we can sum vertically instead, and the second conclusion says that we can move around the order of summation.

Proof. We will sketch this proof; we prove (a) and (b) more or less simultaneously. To turn the infinite double sum into something we can consider finite partial sums of, we set, for each natural N,
$$S_n:=\sum_{k=0}^{n}\sum_{\ell=0}^{n}|z_{k,\ell}|.$$
The main claim is that
$$\lim_{n\to\infty}S_n\stackrel{?}{=}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|.$$
Indeed, fix any $\varepsilon>0$. Because the latter sum converges, there exists some natural A such that
$$\sum_{k>A}\sum_{\ell=0}^{\infty}|z_{k,\ell}|<\frac{\varepsilon}{2}.$$
Further, there exists some natural B_k such that
$$\sum_{\ell>B_k}|z_{k,\ell}|<\frac{\varepsilon}{2A}$$
for each $k\in\mathbb{N}$. Take $B:=\max_{0\le k<A}B_k$. Now, we set $N:=\max\{A,B\}$. To start off our inequalities, we note that
$$S_n=\sum_{k=0}^{n}\sum_{\ell=0}^{n}|z_{k,\ell}|\le\sum_{k=0}^{n}\sum_{\ell=0}^{\infty}|z_{k,\ell}|\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|,$$
so we know the sign of our difference. In particular, for any $n>N$, we see that
$$S_n=\sum_{k=0}^{N}\sum_{\ell=0}^{N}|z_{k,\ell}|\ge\sum_{k=0}^{K}\sum_{\ell=0}^{L}|z_{k,\ell}|.$$
Thus,
$$0\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|-S_n\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|-\sum_{k=0}^{K}\sum_{\ell=0}^{L}|z_{k,\ell}|=\sum_{k>K}\sum_{\ell=0}^{\infty}|z_{k,\ell}|+\sum_{k=0}^{K}\sum_{\ell>L}|z_{k,\ell}|$$
after some cancellation. But we can upper-bound the last quantity by $\frac{\varepsilon}{2}+K\cdot\frac{\varepsilon}{2K}=\varepsilon$, so we are done.

The main point of the above lemma is that we are told each $\varepsilon > 0$ has some N so that $n > N$ implies

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| - S_n = \sum_{\substack{(k,\ell)\in\mathbb{Z}^2 \\ k>n \text{ or } \ell>n}} |z_{k,\ell}| < \varepsilon.$$

We now take the two parts in sequence.

(a) Fix an index ℓ'; we show absolute convergence by showing that the partial sums of $\sum_{k=0}^{\infty}|z_{k,\ell'}|$ are Cauchy. Indeed, fix some $\varepsilon > 0$, and we know there exists N so that each $n > N$ has

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| - S_n < \varepsilon.$$

Now, we see that any $n > m > N$ will have

$$\sum_{k=m+1}^{n} |z_{k,\ell'}| \le \sum_{k=m+1}^{N}\sum_{\ell=0}^{\infty}|z_{k,\ell}| \le \sum_{k=N+1}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| + \sum_{k=0}^{N}\sum_{\ell=N+1}^{\infty}|z_{k,\ell}| < \varepsilon,$$

so we are done.

(b) As above, fix some $\varepsilon > 0$, and we are promised N so that

$$\sum_{\substack{(k,\ell)\in\mathbb{Z}^2 \\ k>N \text{ or } \ell>N}} |z_{k,\ell}| < \varepsilon/2.$$

Observe, for $K, L > N$, we have by the triangle inequality that

$$\left|\sum_{\ell=0}^{L}\sum_{k=0}^{K} z_{k,\ell} - S_N\right| < \varepsilon/2.$$

This bounds holds for any K, so we can send K arbitrarily large; that inner sum will converge, so in fact we can send K to ∞ without ill effect. (Formally, the inner term is an increasing sequence bounded above, so it will converge as $K \to \infty$.) This gives

$$\left|\sum_{\ell=0}^{L}\sum_{k=0}^{\infty} z_{k,\ell} - S_N\right| \le \varepsilon/2.$$

Again, the inner term is an increasing sequence as $L \to \infty$ but still bounded above as $\varepsilon/2$, so the inner sum will converge as $L \to \infty$ and still give the inequality

$$\left|\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} z_{k,\ell} - S_N\right| < \varepsilon.$$

Now as we send $\varepsilon \to 0$, we see that $\lim_{N\to\infty} S_N = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} z_{k,\ell}$, which finishes. ■

2.5.2 Dirichlet Test

We now go directly for the Dirichlet test for convergence.

Lemma 2.82 (Summation by parts). Fix sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ sequences of complex numbers. Then we define

$$B_n := \sum_{k=0}^{N} b_n,$$

and $B_{-1} = 0$ to be the empty sum. It follows that, for any $n, m \in \mathbb{N}$ with $n > m$,

$$\sum_{k=m}^{n} a_k b_k = a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1} B_k(a_k - a_{k+1}).$$

Proof. This comes down to a direct computation. The main point is that $b_k = B_k - B_{k-1}$, which even works with $k = 0$. Indeed,

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} a_k (B_k - B_{k-1}) \\ &= \sum_{k=m}^{n} a_k B_k - \sum_{k=m}^{n} a_k B_{k-1} \\ &\stackrel{*}{=} a_n B_n + \sum_{k=m}^{n-1} a_k B_k - a_m B_{m-1} - \sum_{k=m}^{n} a_{k+1} B_k \\ &= a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1} B_k (a_k - a_{k+1}),\end{aligned}$$

which is what we wanted. The important step to pay attention to is the rearrangement we did in $\stackrel{*}{=}$ in order to message the sums together. ■

And here is our theorem.

Theorem 2.83 (Dirichlet's test)**.** Fix $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a sequence of real numbers and $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence of complex numbers satisfying the following conditions.

- The sequence $\{a_n\}_{n\in\mathbb{N}}$ is decreasing.
- We have $a_n \to 0$ as $n \to \infty$.
- Bounded partial sums: there exists a positive real number M such that

$$\left| \sum_{k=0}^{n} b_k \right| < M$$

  for each $n \in \mathbb{N}$.

Then we claim that

$$\sum_{k=0}^{\infty} a_k b_k$$

converges.

Proof. As usual, fix our partial sums

$$S_n := \sum_{k=0}^{n} a_k b_k \qquad \text{and} \qquad B_n := \sum_{k=0}^{n} b_k.$$

We are given that the B_k are bounded, so we are going to want to use Lemma 2.82, which tells us that

$$S_n = a_n B_n + \sum_{k=0}^{n-1} B_k (a_k - a_{k+1}).$$

We examine the convergence of these terms individually.

- For the sum, we will show that it absolutely converges. We are given that the partial sums B_n are bounded by M, so we note $|B_k(a_k - a_{k+1})| < M|a_k - a_{k+1}|$, so it suffices to show that

$$M \sum_{k=0}^{n-1} |a_k - a_{k+1}|$$

converges as $n \to \infty$. It would be great if this would telescope, and indeed it does! Because the a_k are decreasing,

$$\sum_{k=0}^{\infty} |a_k - a_{k+1}| = \sum_{k=0}^{\infty} (a_k - a_{k+1}) = a_0 - a_{n+1}.$$

Because $a_n \to 0$ as $n \to \infty$, we see that this sum will converge to a_0. It follows that

$$\sum_{k=0}^{\infty} |B_k(a_k - a_{k+1})|$$

will converge by the Comparison test, so

$$\sum_{k=0}^{\infty} B_k(a_k - a_{k+1})$$

converges by absolute convergence.

- Note that the B_n are bounded in norm by M, so $|a_nB_n| \le M|a_n|$, but $|a_n| \to 0$ as $n \to \infty$, so $|a_nB_n| \to 0$. ■

Eterović has lots of different convergence tests in his notes, but we don't care about most of them. Here is one that we do care about.

Theorem 2.84 (Integral test)**.** Fix a decreasing function $f : [1, \infty) \to \mathbb{R}^+$ and for which

$$\int_k^{k+1} f(x)\, dx$$

always exists. Then the sequence of integrals $I_n := \int_1^n f(x)\, dx$ converges if and only if the summation

$$\sum_{k=1}^{\infty} f(k)$$

converges.

Proof. We omit this proof; it's a reasonably standard real-analytic test. ■

2.6 February 4

Today we are talking about continuity.

Warning 2.85. The first half of this lecture was transcribed from Professor Morrow's notes because I had to miss class for a job interview

2.6.1 Limits

Before defining continuity, we have the following definitions.

Definition 2.86 (Limit)**.** Fix $f\colon X \to \mathbb{C}$ a function and $z_0 \in \overline{X}$. Then we say *the limit of $f(z)$ as z approaches z_0 equals w*, denoted

$$\lim_{z\to z_0} f(z) = w,$$

if and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - w| < \varepsilon$$

for $z \in X$.

This is the standard ε-δ definition.

We also pick up the following convention as a surprise tool that may help us later.

Definition 2.87 (Infinite limits)**.** Fix $f\colon X \to \mathbb{C}$ a function. Then we say *the limit of $f(z)$ as z tends to infinity equals w*, denoted

$$\lim_{z\to\infty} f(z) = w,$$

if and only if, for each $\varepsilon > 0$, there exists $N > 0$ such that

$$|z| > N \implies |f(z) - w| < \varepsilon$$

for $z \in X$.

As in real analysis, the ε-δ definition of a limit can be translated to a statement about sequences.

Proposition 2.88. Fix $\alpha \in \overline{X}$. Then $\lim_{z\to\alpha} f(z) = w$ if and only if, for each $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to \alpha$ as $n \to \infty$, we have $f(z_n) \to w$ as $n \to \infty$.

Proof. In the forwards direction, fix $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to \alpha$, and we show that $f(z_n) \to w$. Well, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon,$$

where $z \in X$. But for this $\delta > 0$, there exists N such that

$$n > N \implies |z_n - \alpha| < \delta \implies |f(z_n) - f(\alpha)| < \varepsilon.$$

So indeed, $f(z_n) \to f(\alpha)$.

In the reverse direction, suppose that $f(z)$ does not approach w as $z \to \alpha$. Then, there exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$, there is $z \in X$ such that $|z - \alpha| < \delta$ while $|f(z) - w| > \varepsilon_0$. Well, for any $n \in \mathbb{N}$, taking $\delta = 1/(n+1)$, we are promised $z_n \in X$ such that

$$|z_n - \alpha| < \frac{1}{n+1} qquad \text{and} \qquad |f(z_n) - w| > \varepsilon_0.$$

So to finish, we claim that $z_n \to \alpha$ as $n \to \infty$, but $f(z_n)$ does not approach w as $n \to \infty$.

- For any $\varepsilon > 0$, we note that $N := 1/\varepsilon$ has $n > N$ implies

$$|z_n - \alpha| < \frac{1}{n+1} < \frac{1}{N+1} < \frac{1}{N} = \varepsilon,$$

so indeed $z_n \to \alpha$ as $N \to \infty$.

- We note that $\varepsilon_0 > 0$ satisfies that

$$|f(z_n) - w| > \varepsilon_0$$

for any $n \in \mathbb{N}$, so no N will have $n > N$ implies $|f(z_n) - w| < \varepsilon_0$. Thus, $f(z_n)$ does not approach w as $n \to \infty$.

The sequence $\{z_n\}_{n\in\mathbb{N}}$ now completes the proof by showing the reverse direction by contraposition. ■

While we're here, we pick up the following definitions.

Definition 2.89 (Bounded). A function $f\colon X \to \mathbb{C}$ is *bounded* if there exists $R > 0$ such that $\operatorname{im} f \subseteq B(0, R)$.

Definition 2.90 (Bounded near). Fix a nonempty open subset $\Omega \subseteq \mathbb{C}$ and $z_0 \in \Omega$. Then $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ is *bounded near* z_0 if and only if

$$\lim_{z\to z_0} (z - z_0)f(z) = 0.$$

2.6.2 Continuity

And here is our central definition for today.

Definition 2.91 (Continuous). A function $f\colon X \to \mathbb{C}$ is *continuous at* $z_0 \in X$ if and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon,$$

where $z \in X$. Further, f is *continuous on* X if and only if f is continuous at each $z_0 \in X$.

We have the following lemma of equivalent definitions.

Lemma 2.92. Suppose that $f\colon X \to \mathbb{C}$.

(a) Then f is continuous at w if and only if every sequence $\{z_n\} \subseteq X$ such that $z_n \to z$ implies $f(z_n) \to f(z)$.

(b) We have that f is continuous on X if and only if every open set $U \subseteq \mathbb{C}$ has $f^{-1}(U)$ open in X.

(c) We have that f is continuous on X if and only if each closed set $V \subseteq X$ has $f^{-1}(V)$ closed in X.

(d) Lastly, we have that f is continuous at if and only if, for each $\varepsilon > 0$ and $z \in \mathbb{C}$, we have that $f^{-1}(B(z, \varepsilon))$ is open in X.

Proof. We take the parts one at a time.

(a) We could use Proposition 2.88, but we will just do this by hand. For the forwards direction, suppose that $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ converges to some w. Then let $\varepsilon > 0$. By assumption, there exists some $\delta > 0$ such that

$$|z - w| < \delta \implies |f(x) - f(w)| < \varepsilon.$$

It follows from $z_n \to w$ that there exists some N such that

$$n > N \implies |z_n - w| < \delta \implies |f(z_n) - f(z)| < \varepsilon,$$

so it follows that $f(z_n) \to f(z)$.

In the reverse direction, take f not continuous at w, so there exists $\varepsilon > 0$ so that for all $n \in \mathbb{N}$, there exists some chosen z_n with

$$|z_n - w| < \delta \implies |f(z_n) - f(w)| > \varepsilon.$$

But as $z_n \to w$, we see that $f(z_n)$ does not approach $f(w)$, so we are done.

(b) In the forwards direction, suppose that $U \subseteq \mathbb{C}$ is open, and we show that $f^{-1}(U)$ is open in X. Well, suppose that $z \in f^{-1}(U)$, and we will find $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(U)$.

Well, $f(z) \in U$, so there exists $\varepsilon > 0$ such that $B(f(z), \varepsilon) \subseteq U$. Thus, continuity of f requires some $\delta > 0$ such that
$$|w - z| < \delta \implies |f(w) - f(z)| < \varepsilon,$$
which implies $f(w) \in B(f(z), \varepsilon) \subseteq U$ implies $w \in f^{-1}(U)$. So indeed, $B(z, \delta) \subseteq f^{-1}(U)$.

In the reverse direction, suppose that each open $U \subseteq \mathbb{C}$ has $f^{-1}(U)$ is open. Now fix any $z \in X$ and $\varepsilon > 0$. The set $B(f(z), \varepsilon)$ is open, so
$$f^{-1}(B(f(z), \varepsilon))$$
is open. But $z \in f^{-1}(B(f(z), \varepsilon))$, so we can find $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(B(f(z), \varepsilon))$. Thus, $|w - z| < \delta$ implies $w \in f^{-1}(B(f(z)), \varepsilon)$ implies $f(w) \in B(f(z), \varepsilon)$ implies $|f(w) - f(z)| < \varepsilon$, finishing.

(c) In the forwards direction, suppose f is continuous so that $U \subseteq \mathbb{C}$ open implies $f^{-1}(U) \subseteq X$ is open. But then, if $V \subseteq \mathbb{C}$ is closed, then $\mathbb{C} \setminus V$ is open, so[3]
$$f^{-1}(\mathbb{C} \setminus V) = f^{-1}(\mathbb{C}) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$$
is open, so $f^{-1}(V)$ is closed.

In the backwards direction, suppose f^{-1} preserves closed sets. Then, if $U \subseteq \mathbb{C}$ is open, $\mathbb{C} \setminus U$ is closed, so
$$f^{-1}(\mathbb{C} \setminus U) = f^{-1}(\mathbb{C}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$$
is closed, so $f^{-1}(U)$ is open. Thus, f^{-1} preserves open sets, so f must be continuous.

(d) In the forwards direction, fix $\varepsilon > 0$ and $z \in \mathbb{C}$, so $B(z, \varepsilon)$ is open, so $f^{-1}(B(z, \varepsilon))$ is open in X, finishing.

In the other direction fix $\varepsilon > 0$ and $z \in \mathbb{C}$ to consider $B(f(z), \varepsilon) \subseteq U$. Thus, continuity of f requires some $\delta > 0$ such that
$$|w - z| < \delta \implies |f(w) - f(z)| < \varepsilon,$$
which implies $f(w) \in B(f(z), \varepsilon) \subseteq U$ implies $w \in f^{-1}(U)$. So indeed, $B(z, \delta) \subseteq f^{-1}(U)$.

In the reverse direction, fix U open, and we show that $f^{-1}(U)$ is open. Well, each $z \in U$ has some ε_z such that $B(z, \varepsilon_z) \subseteq U$. But $f^{-1}(B(z, \varepsilon_z))$ is open by hypothesis, so
$$f^{-1}(U) = f^{-1}\left(\bigcup_{z \in U} B(z, \varepsilon_z)\right) = \bigcup_{z \in U} f^{-1}(B(z, \varepsilon_z))$$
is an arbitrary union of open sets and hence open. ■

And here are some special examples.

Example 2.93. Fix some $z_0 \in \mathbb{C}$. Then $f(z) := |z - z_0|$ is continuous on $\mathbb{C}$. Indeed, fix any $w \in \mathbb{C}$. Then for any $\varepsilon > 0$, we set $\delta := \varepsilon$ so that $|z - w| < \delta$ implies
$$|f(z) - f(w)| = \big||z - z_0| - |w - z_0|\big| \leq |z - w| < \delta = \varepsilon.$$

Example 2.94. The functions Re and Im is continuous. Indeed, fix any $w \in \mathbb{C}$. Then, for any $\varepsilon > 0$, take $\delta := \varepsilon$ so that $|z - w| < \delta$ implies
$$|\operatorname{Re} z - \operatorname{Re} w| = |\operatorname{Re}(z - w)| \leq |z - w| < \delta = \varepsilon,$$
and similarly,
$$|\operatorname{Im} z - \operatorname{Im} w| = |\operatorname{Im}(z - w)| \leq |z - w| < \delta = \varepsilon,$$

Continuous functions also have some arithmetic.

[3] To see $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, note that $x \in f^{-1}(A \setminus B)$ if and only if $f(x) \in A \setminus B$ if and only if $f(x) \in A$ but $f(x) \notin B$ if and only if $x \in f^{-1}(A)$ but $x \notin f^{-1}(B)$.

Proposition 2.95. Fix $f, g\colon X \to \mathbb{C}$ to functions continuous at $z_0 \in X$. Then $f+g$, $f\cdot g$ are both continuous at $z_0 \in X$, and f/g is continuous at z_0 provided $g(z_0) \neq 0$.

Proof. The point is to appeal to the corresponding results on convergence of sequences. In particular, we use the idea that f is continuous at z_0 if and only if each sequence $z_n \to z_0$ in X has $f(z_n) \to f(z_0)$. We omit the details because they are essentially the same as in a real analysis class. ■

Corollary 2.96. Every polynomial in one variable is a continuous function $X \to \mathbb{C}$ for any $X \subseteq \mathbb{C}$.

Proof. Note that $x \mapsto x$ is continuous, so by induction $x \mapsto x^n$ is continuous for each $n \in \mathbb{N}$. Taking a $\mathbb{C}$-linear combination gives arbitrary polynomials. ■

Here is another sort of arithmetic.

Lemma 2.97. The composition of two continuous functions is continuous.

Proof. Omitted. ■

2.6.3 Connectedness

We want to build towards a particular type of continuous function.

Proposition 2.98. Fix $X \subseteq \mathbb{C}$ a connected subset. Then a continuous function $f\colon X \to \mathbb{C}$ has connected image $f(X)$.

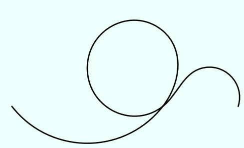
Proof. The main point is to use the topological characterization of continuity. In particular, suppose that $f(X)$ is disconnected, and we show that X is disconnected. In particular, suppose that U_1 and U_2 disconnect $f(X)$, and we have that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ disconnect X. We will not run all the checks here; the main point is that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open because f is continuous. ■

Definition 2.99 (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ where $a < b$ are real numbers.

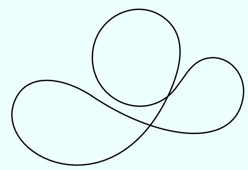
Definition 2.100. We say that a path γ is *closed* if and only if $\gamma(a) = \gamma(b)$. We say that γ is *simple* if and only if $\gamma\colon (a,b) \to \mathbb{C}$ is injective.

Remark 2.101. The point of restricting γ to the open interval at the end so that closed, simple paths are allowed to exist.

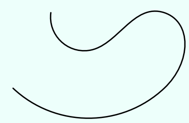
Example 2.102. Here is a path.



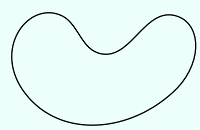
Example 2.103. Here is a closed path.



Example 2.104. Here is a simple path.



Example 2.105. Here is a closed, simple path, also called a loop.

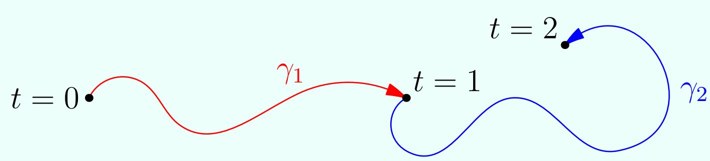


Definition 2.106 (Concatenation)**.** Fix $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d]$ paths in $\mathbb{C}$ such that $\gamma_1(b) = \gamma_2(c)$. Then we define the *concatenation* of γ_1 and γ_2 to be

$$(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a,b], \\ \gamma_2(t-b+c) & t \in [b, d-c+b]. \end{cases}$$

The main point is that we are doing one path after the other.

Example 2.107. The following shows an example concatenation of $\gamma_1 * \gamma_2$, where $\gamma_1, \gamma_2\colon [0,1] \to \mathbb{C}$.



The entire path is $\gamma_1 * \gamma_2$.

Paths give us the following notion.

Definition 2.108 (Path-connected)**.** A subset $X \subseteq \mathbb{C}$ is *path connected* if and only if, for any two $x_0, x_1 \in X$, there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Lemma 2.109. The open ball $B(z,r)$ and closed ball $\overline{B(z,r)}$ are both path-connected.

Proof. The point is that $B(z,r)$ and $\overline{B(z,r)}$ are both convex, so the path

$$\gamma(t) := z_0 + t(z_1 - z_0)$$

will work. ■

Here is the basic result.

Proposition 2.110. A space X is path-connected implies that X is connected. If X is open and connected, then X is path-connected.

Proof. We will show this next class. ■

2.7 February 7

Good morning everyone. A few announcements.

- Homework #3 is due on Friday.
- There will be no in-person class on Wednesday or Friday.
- Office hours this week are today (1:00PM–2:30PM) and tomorrow (2:00PM–3:30PM).

2.7.1 Connectedness

Today we're going to talk more about continuous functions.

Last time we ended with the following proposition.

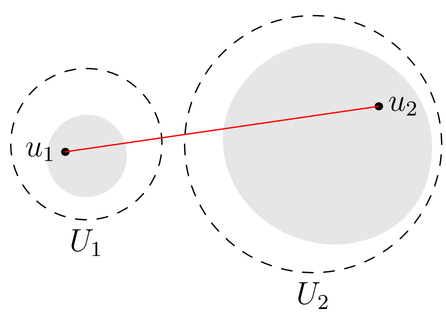
Proposition 2.110. A space X is path-connected implies that X is connected. If X is open and connected, then X is path-connected.

Proof. We do these separately.

- Suppose that $X = U_1 \sqcup U_2$ is disconnected, and we show that X is not path-connected. Namely, we have $U_1, U_2 \subseteq X$ open subsets (in X) which are disjoint and nonempty. Because U_1 and U_2 are nonempty, find $x_1 \in U_1$ and $x_2 \in U_2$.

  However, we claim there is no continuous path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Indeed, the image of $\gamma([0,1])$ must be connected, but then we can disconnect $\gamma([0,1])$ by U_1 and U_2: $\gamma([0,1]) \subseteq U_1 \cup U_2$ and $x_\bullet \in U_\bullet \cap \gamma([0,1])$ and $U_1 \cap U_2 = \varnothing$.

  At a high level, here is the image that a disconnected X cannot have a path between any two pair points: there is no possible red path below which stays in the gray region.

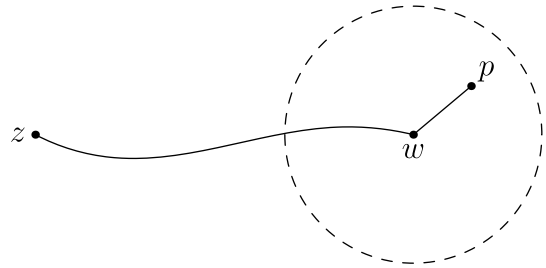


- Suppose we have a point $z \in X$, and we set

$$C(z) \coloneqq \{w \in X : \text{there is a path from } z \text{ to } w\}.$$

  We claim that $C(z)$ is closed and open in X, which will force $C(z) = X$ because X is connected and $C(z)$ is nonempty ($z \in C(z)$ by the trivial path $\gamma\colon t \mapsto z$).

  We start by showing $C(z)$ is open: because X is open, there exists $r > 0$ such that $B(w,r) \subseteq X$. But with $w \in C(z)$, there will be a path between any point in $p \in B(w,r)$ and w, so there is a path from z to w to p. Here is the image.



Now we show that $C(z)$ is closed. Suppose that $w \in X \setminus C(z)$, and we have to show that there is an open ball around w in $X \setminus C(z)$. To see this, fix an open ball $B(w,r) \subseteq X$ for $r > 0$, but now there can be no path from z to anywhere in $B(w,r)$, for then we could just run the above argument again to show that $w \in C(z)$. ■

Remark 2.111. The proof for the second part merely needs X to be locally path-connected, not a metric space.

Corollary 2.112. We have that $\mathbb{C}$ is path-connected and therefore connected.

Proof. Given any two points $z, w \in \mathbb{C}$, we choose the path $\gamma\colon [0,1] \to \mathbb{C}$ by

$$\gamma(t) = tz + (1-t)w.$$

Indeed, $\gamma(0) = w$ and $\gamma(1) = z$, and γ is somewhat clearly continuous by, say, checking component-wise. ■

2.7.2 Compactness

Let's do compactness better this time.

Lemma 2.113. Fix $X \subseteq \mathbb{C}$ (sequentially) compact. Then X is both closed and bounded.

Proof. We start by showing X is closed. For this, we show that X contains all of its limit points.

Well, suppose that $z \in X$ is a limit point so that we have a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$. But by the (sequential) compactness of X, this sequence has a convergent subsequence $\{z_{\sigma n}\}_{n\in\mathbb{N}}$ which does converge in X. But any subsequence will converge to the same limit (!), so $z_{\sigma n} \to z$ as well, so $z \in X$ is forced.

We now show that X is bounded. We proceed by contraposition: if X is unbounded, then for any $n \in \mathbb{N}$, then we can find some $z_n \in X \setminus B(0,n)$. But then we can check that $\{z_n\}_{n\in\mathbb{N}}$ has no convergent subsequence, essentially because it tends off to infinity. ■

Our goal for the rest of class is to prove the following two results.

Proposition 2.114. A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if it is closed and bounded.

Theorem 2.115 (Heine–Borel)**.** A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if every open cover of X has a finite subcover.

On the homework, we showed the backward direction of Theorem 2.115.

Remark 2.116. Our hope is to have lots of equivalent characterizations of compactness so that we can have easier proofs of statements about compact sets.

To start off, here are some lemmas we will need.

Lemma 2.117. Fix $X \subseteq \mathbb{C}$ (sequentially) compact. For any $\varepsilon > 0$, there exist only finitely many points $z_1, \ldots, z_n \in X$ such that

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

Proof. The point is to build some inductive argument: one fixes an $\varepsilon > 0$ and then continues choosing random points out of X until we cover X. Indeed, if the process does not terminate, then the sequence we generate has no convergent subsequence.

Rigorously, if X is empty, then just choose no points at all and be done. Otherwise, we can find some $z_1 \in X$. Inductively, suppose we have a sequence $\{z_1, \ldots, z_m\}$. If

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon),$$

then we are done. Otherwise, we can find $z_{m+1} \in X \setminus \bigcup_{k=1}^{n} B(z_k, \varepsilon)$.

If the above inductive process terminates, then we get the result. Otherwise, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$z_{n+1} \in X \Bigg\backslash \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

We claim that $\{z_n\}_{n\in\mathbb{N}}$ has subsequence converging in X. Indeed, suppose for the sake of contradiction that $z_{\sigma n} \to z$ for some strictly increasing σ and $z \in X$. Then there exists N such that $n > N$ implies

$$|z_{\sigma n} - z| < \varepsilon/2.$$

But then, finding some $n+1, n > N$, we have

$$|z_{\sigma(n+1)} - z_{\sigma n}| < |z_{\sigma(n+1)} - z| + |z_{\sigma n} - z| < \varepsilon,$$

so

$$z_{\sigma(n+1)} \in \bigcup_{k=1}^{\sigma(n+1)-1} B(z_k, \varepsilon),$$

which is our contradiction to the construction of $z_\bullet$. ■

Lemma 2.118. Fix $X \subseteq \mathbb{C}$ (sequentially) compact with some open cover $\mathcal{U}$ of X. Then there is an $\varepsilon > 0$ such that, for every $z \in X$, there is $U \in \mathcal{U}$ such that $B(z, \varepsilon) \subseteq U$.

Proof. Suppose that, for all $\varepsilon > 0$, there exists some $z \in X$ such that no $U \in \mathcal{U}$ has $B(z, \varepsilon) \subseteq U$. We construct a sequence in X with no subsequence converging in X. Indeed, for any $n \in \mathbb{N}$, we find $z_n \in X$ such that no $U \in \mathcal{U}$ has $B(z_n, 1/n) \subseteq U$. We claim that $\{z_n\}_{n\in\mathbb{N}}$ has no subsequence converging in X.

Indeed, suppose that we have $z \in X$ and strictly increasing $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that $z_{\sigma n} \to z$. We will then be able to find some z_n such that $B(z_n, 1/n) \subseteq U$ for some $U \in \mathcal{U}$, which will be a contradiction. Indeed, $z \in X$, and $\mathcal{U}$ covers z, so there is some $U \in \mathcal{U}$ with $z \in U$. In fact, U is open, so there is an $\varepsilon > 0$ such that

$$B(z, \varepsilon) \subseteq U.$$

Now, there is N such that for $n > N$, we can guarantee that $|z - z_n| < \varepsilon/2$. Further, for $n > 2/\varepsilon$, we have $1/n < \varepsilon/2$. So $n > \max\{N, 2/\varepsilon\}$ will have $\sigma n > \max\{N, 2/\varepsilon\}$, implying

$$|w - z_n| < 1/n < \varepsilon/2 \implies |w - z| < |w - z_n| + |z - z_n| = \varepsilon \implies w \in B(z, \varepsilon) \subseteq U,$$

so $B(z_n, 1/n) \subseteq U$. This contradiction finishes. ■

This is saying that there is a universal ε that we can find for our open cover.

Lemma 2.119. Fix X a bounded set. Then, for any $\varepsilon > 0$, there exist finitely many points $z_1, \ldots, z_n$ such that

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

Proof. The point is to reduce this to the case of $[-M, M]^2$ which can cover X because X is bounded, and then we can create the cover for X by hand. ■

Now let's attack one of our equivalent conditions for compactness.

Proposition 2.114. A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if it is closed and bounded.

Proof. The forwards direction we have already done.

In the backwards direction, suppose that $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ is some sequence. Our main goal is to construct a convergent subsequence. Because X is bounded, we can choose $w_{1,1}, \ldots, w_{1,\ell_1}$ such that

$$X \subseteq \bigcup_{k=1}^{\ell_1} B(w_{1,k}, 1/2).$$

Now, because $\{z_n\}_{n\in\mathbb{N}}$ is infinite, there must be some index $w_1 := w_{1,k_1}$ such that

$$L_1 = \{n \in \mathbb{N} : z_n \in B(w_{1,k_1}, 1/2)\}$$

is infinite. The important point is that $\{z_n\}_{n\in L_1}$ has formed a subsequence which lives inside a ball of radius $1/2$. We can continue this process: again using our bounded condition, we can find some $w_{2,1}, \ldots, w_{2,\ell_2} \in B(w_{1,k_1}, 1/2)$ such that

$$B(w_{1,k_1}, 1/2) \subseteq \bigcup_{k=1}^{\ell_2} B(w_{2,k}, 1/4).$$

Then we can choose L_2 from here by choosing one of the $w_{2,k}$ with infinitely many indices. Continuing this process forces our sequence to converge.

To more explicitly appeal to choice, we note that we can always find some sequence $\{w_{k,i}\} \subseteq X$ such that

$$X \subseteq \bigcup_{k=1}^{\ell_n} B(w_{k,i}, 1/2^k),$$

but L_{i-1} is infinite, so there is a specific $w_k := w_{k,i}$ such that

$$L_i := \{n \in L_{i-1} : |z_n - w_k| < 2^{-k}\}$$

is infinite. To actually construct our sequence from these infinite subsets, we define a choice function over our indices: define $\varphi \colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n+1)$ is the smallest number exceeding $\varphi(n)$ with $\varphi(n+1) \in L_{n+1}$. Then we know that

$$|z_{\varphi(n)} - w_k| < 2^{-n}$$

for each $1 \le k \le n$. Thus, for $n \ge m > N$, we have

$$|z_{\varphi(n)} - z_{\varphi(m)}| \le |z_{\varphi(n)} - w_m| + |z_{\varphi(n)} - w_m| < 2 \cdot 2^{-m} < 2^{-N+1},$$

so for any $\varepsilon > 0$, we can choose $N := 1 - \log_2 \varepsilon$ sufficiently large so that $n, m > N$ implies

$$|z_{\varphi(n)} - z_{\varphi(m)}| < 2^{-N+1} = \varepsilon.$$

It follows that the subsequence defined by φ is Cauchy and hence converges. But because X is closed, any convergent sequence in X will be in X, so our sequence in X has a convergent subsequence. ■

2.8 February 9

2.8.1 More Compactness

To wrap up from last class, we show the following.

Theorem 2.115 (Heine–Borel)**.** A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if every open cover of X has a finite subcover.

Proof. The direction that sequentially compact implies closed and bounded was done on the homework.

We focus on the other direction. Fix $\mathcal{U}$ an open cover of X. By Lemma 2.118, we know there exists $\varepsilon > 0$ such that, for each $z \in X$, there is some $U \in \mathcal{U}$ such that $B(z, \varepsilon) \subseteq U$. But in fact, with this $\varepsilon > 0$, Lemma 2.119 tells us that there exists finitely many points $z_1, \ldots, z_\ell$ such that

$$X \subseteq \bigcup_{k=1}^{\ell} B(z_k, \varepsilon).$$

But now, finding U_k such that $B(z_k, \varepsilon) \subseteq U_k$ (possible by construction of ε), we see that $\{U_k\}_{k=1}^{\ell}$ will be our finite subcover. ■

Remark 2.120. The conclusion of the above theorem is the usual notion of compactness, so I will stop writing "(sequentially)" whenever I say "compact."

Let's see a use for our definitions of compactness.

Corollary 2.121. Let $X \subseteq \mathbb{C}$ be a compact space and $f \colon X \to \mathbb{C}$ continuous. Then $f(X)$ is compact.

Proof. Give $f(X)$ some open cover $\mathcal{U}$. Because f is continuous, we see that

$$\left\{f^{-1}(U)\right\}_{U \in \mathcal{U}}$$

is an open cover for X. But X is compact, so we can find some finite subcover $\{U_k\}_{k=1}^{n} \subseteq \mathcal{U}$ so that $\left\{f^{-1}(U_k)\right\}_{k=1}^{n}$ covers X. But then the $\{U_k\}_{k=1}^{n}$ will cover X by taking the union over our open subcover. ■

2.8.2 Uniform Continuity

The point of uniform convergence is to make fewer choices in our notion of continuity.

Definition 2.122 (Uniform continuity)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset. Then a function $f \colon X \to \mathbb{C}$ is *uniformly continuous* if and only if, for each $\varepsilon > 0$, there exists a single $\delta > 0$ so that $z, w \in X$ have

$$|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Importantly, this definition has δ not depend on either z nor w, where continuity would allow δ to depend on one of them.

Example 2.123. The functions $\mathrm{id}_{\mathbb{C}}$ and $z \mapsto \overline{z}$ are both uniformly continuous on $\mathbb{C}$. Letting f be either of these functions, we see that, for any $\varepsilon > 0$, we may take $|z - w| < \varepsilon$ to imply

$$|f(z) - f(w)| = |z - w| < \varepsilon.$$

Here is a nice result.

Proposition 2.124. Fix X a nonempty, compact subset. Then any continuous function $f\colon X \to \mathbb{C}$ is uniformly continuous.

Proof. The point is to let $\delta \to 0$ until we can fit some prescribed ε bound. Choose $\delta = 1/n$ as n varies over positive integers, and we imagine fixing sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ such that

$$|z_n - w_n| < 1/n.$$

Now we use the sequential compactness of X, which forces $\{z_n\}_{n=1}^\infty$ to have a convergent subsequence, so we conjure $\alpha \in X$ and a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_{\sigma n} \to \alpha$ as $n \to \infty$.

We now claim that $w_{\sigma n} \to \alpha$ as well. In particular, for any $\delta > 0$, there is some N_1 so that $n > N_1$ implies

$$|z_{\sigma n} - \alpha| < \delta/2.$$

Choosing N to be larger than N_1 and $2/\delta$, we see that $n > N$ will have

$$|w_{\sigma n} - \alpha| \le |z_{\sigma n} - w_{\sigma n}| + |z_{\sigma n} - \alpha| < \frac{1}{\sigma n} + \frac{\delta}{2} \le \frac{1}{n} + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so indeed $w_{\sigma n} \to \alpha$ as $n \to \infty$.

Only now we suppose for the sake of contradiction we have some $\varepsilon > 0$ such that any $\delta > 0$ has some z and w such that $|z - w| < \delta$ actually has $|f(z) - f(w)| \ge \varepsilon$. Taking $\delta := 1/n$, we are promised some sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ so that

$$|z_n - w_n| < \delta \qquad \text{and} \qquad |f(z_n) - f(w_n)| \ge \varepsilon.$$

Using the above machinery, we see that $z_{\sigma n} \to \alpha$ and $w_{\sigma n} \to \alpha$ force $f(z_{\sigma n}) \to f(\alpha)$ and $f(w_{\sigma n}) \to f(\alpha)$ by continuity of f, but the sequences $f(z_{\sigma n})$ and $f(w_{\sigma n})$ are supposed to be ε far apart! Explicitly, we can find sufficiently large N_1 and N_2 such that

$$\begin{aligned} n > N_1 &\implies |f(z_{\sigma n}) - \alpha| < \varepsilon/4, \\ n > N_2 &\implies |f(w_{\sigma n}) - \alpha| < \varepsilon/4, \end{aligned}$$

which by the triangle inequality means that any $n > \max\{N_1, N_2\}$ will give

$$|f(z_{\sigma n}) - f(w_{\sigma n})| \le |f(z_{\sigma n}) - \alpha| + |f(w_{\sigma n}) - \alpha| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,$$

which is a contradiction to the construction of $z_{\sigma n}$ and $w_{\sigma n}$. ■

2.8.3 Uniform Convergence

We next talk about uniform convergence for functions. Here is our starter pack.

Definition 2.125 (Sequence of functions)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is a function $\varphi\colon \mathbb{N} \to (X \to \mathbb{C})$. Namely, for each $n \in \mathbb{N}$, we are given a function $\varphi(n)\colon X \to \mathbb{C}$.

Definition 2.126 (Pointwise convergence). Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}$ *converges to some* $g\colon X \to \mathbb{C}$ *pointwise* if and only if, for each $z \in X$, we have $f_n(z) \to g(z)$ as $n \to \infty$. We write this as $f_n \to g$.

This is called pointwise convergence because we are only checking convergence at each individual point $z \in X$, without caring about the larger structure of the function. This will cause problems later but not now.

Definition 2.127 (Bounded). We say that a function $f\colon X \to \mathbb{C}$ is *bounded* if and only if $f(X) \subseteq \mathbb{C}$ is bounded. In other words, there is some $M > 0$ so that $f(X) \subseteq B(0, M)$.

Definition 2.128 (Uniform convergence). Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}$ *converges to some* $g\colon X \to \mathbb{C}$ *pointwise* if and only if, for each $\varepsilon > 0$, there is some N so that

$$n > N \implies |f_n(z) - g(z)| < \varepsilon$$

for each $z \in X$.

The uniformity here is that the value of N is no longer allowed to depend on z. Here is an alternate definition.

Proposition 2.129. Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$ some function. Then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to g if and only if

$$\lim_{n\to\infty} \sup_{z\in X}\{|f_n(z) - g(z)|\} = 0.$$

Proof. We take the directions independently.

- In the forward direction, we know that there is an N_1 so that $n > N_1$ implies each $z \in X$ has

  $$|f_n(z) - g(z)| < 1.$$

  In particular, for $n > N_1$, the set $\{|f_n(z) - g(z)| : z \in X\}$ is bounded above by 1, so the supremum will exist; set $y_n := \sup\{|f_n(z) - g(z)| : z \in X\}$ so that we want to show $y_n \to 0$ as $n \to \infty$.

  More generally, for any $\varepsilon > 0$, there exists some N so that $n > N_0$ implies

  $$|f_n(z) - g(z)| < \varepsilon/2.$$

  So $n > \max\{N_0, N_1\}$, we will have that $y_n = \sup\{|f_n(z) - g(z)| : z \in X\}$ both exists and has $y_n \le \varepsilon/2 < \varepsilon$. So we do get $y_n \to 0$ as $n \to \infty$.

- In the reverse direction, set $y_n := \sup\{|f_n(x) - g(x)| : x \in X\}$ so that $y_n \to 0$ as $n \to \infty$. Namely, for each $\varepsilon > 0$, there exists some N so that $n > N$ has $y_n < \varepsilon$. In particular, we see $n > N$ has

  $$|f_n(x_0) - g(x_0)| \le \sup\{|f_n(x) - g(x)| : x \in X\} = y_n < \varepsilon$$

  for each $x_0 \in X$. So indeed, f_n converges to g uniformly. ■

2.8.4 Distances Between Functions

Later in life it will be nice to view functions as forming a metric under $d(f, g) := \sup\{|f(x) - g(x)|\}$. However, this supremum need not only exist; here is one condition in which it does.

Lemma 2.130. Fix $f, g\colon X \to \mathbb{C}$ bounded functions. Then $\sup\{|f(x) - g(x)| : x \in X\}$ exists.

Proof. Because f is bounded, there exists M_f so that each $x \in X$ has $|f(x)| < M_f$. Similarly, because g is bounded, there exists M_g so that each $x \in X$ has $|g(x)| < M_g$. It follows that, for each $x \in X$,

$$|f(x) - g(x)| \le |f(x)| + |g(x)| \le M_f + M_g,$$

so the set $\{|f(x) - g(x)| : x \in X\}$ is bounded above and in particular has a supremum. ■

Proposition 2.131. Fix $f, g, h\colon X \to \mathbb{C}$ all bounded functions. Then

$$\sup_{x\in X}\{|f(x) - h(x)|\} \le \sup_{x\in X}\{|f(x) - g(x)|\} + \sup_{x\in X}\{|g(x) - h(x)|\}.$$

Note that all the suprema above exist by Lemma 2.130

Proof. The point is to reduce to the typical triangle inequality. Indeed, for any $x \in X$, we see that

$$|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)|.$$

Thus,

$$\begin{aligned}\sup_{x\in X}\{|f(x) - h(x)|\} &\le \sup_{x\in X}\{|f(x) - g(x)| + |g(x) - h(x)|\}\\ &\le \sup_{x\in X}\{|f(x) - g(x)|\} + \sup_{x\in X}\{|g(x) - h(x)|\},\end{aligned}$$

which is what we wanted. We have used the fact that $\sup(A + B) \le \sup A + \sup B$ for $A, B \subseteq \mathbb{R}$, which is not hard to show: if $a + b \in A + B$, then $a \le \sup A$ and $b \le \sup B$, so $a + b \le \sup A + \sup B$; thus, $\sup(A + B) \le \sup A + \sup B$.[4] ■

Remark 2.132 (Nir). Viewing Lemma 2.130 as providing a distance metric on the space of bounded functions $X \to \mathbb{C}$, the above proposition proves the triangle inequality for this metric. The other checks as follows; fix two bounded functions $f, g\colon X \to \mathbb{C}$.

- Note that $\sup\{|f(x) - g(x)| : x \in X\} = 0$ if and only if $|f(x) - g(x)| = 0$ for all $x \in X$ if and only if $f = g$.
- Note that $|f(x) - g(x)| = |g(x) - f(x)|$ for each $x \in X$, so $\{|f(x) - g(x)| : x \in X\} = \{|g(x) - f(x)| : x \in X\}$, so they also have equal suprema.

We can also build a Cauchy criterion for uniform convergence.

Proposition 2.133. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}_{n\in\mathbb{N}}$ converges to some function uniformly if and only if the quantity $\sup_{x\in X}\{f_n(x) - f_m(x)\}$ exists and, for any $\varepsilon > 0$, there exists some N so that $n, m > N$ implies

$$\sup_{x\in X}\{|f_n(x) - f_m(x)|\} < \varepsilon$$

for any $x \in X$.

[4] In fact, $\sup A + \sup B \le \sup(A + B)$ as well. We show $\sup A \le \sup(A + B) - \sup B$. Fixing $a \in A$, we need $a \le \sup(A + B) - \sup B$, so we show $\sup B \le \sup(A + B) - a$. Fixing $b \in B$, we need $b \le \sup(A + B) - a$, which is clear.

We note that the hypothesis that the supremum exists can be removed if the functions are presupposed to be bounded.

Proof. We again take the directions independently.

- Suppose that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges to a function g uniformly. Then, for any $\varepsilon > 0$, we are promised some N so that $n > N$ will have

  $$|f_n(x) - g(x)| < \varepsilon/4$$

  for any $x \in X$. In particular, for any $n, m > N$, we see

  $$|f_n(x) - f_m(x)| < |f_n(x) - g(x)| + |f_m(x) - g(x)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

  so

  $$\sup_{x\in X}\{|f_n(x) - f_m(x)|\} \le \frac{\varepsilon}{2} < \varepsilon,$$

  which is what we wanted.

- There are two steps.
  - We begin by constructing g. Well, for each $x \in X$, we note that any $\varepsilon > 0$ will have some N so that $n, m > N$ implies

    $$|f_n(x) - f_m(x)| \le \sup_{x\in X}\{|f_n(x) - f_m(x)|\} < \varepsilon,$$

    so the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges in $\mathbb{C}$. We define $g(x)$ to be the limit of $f_n(x)$ as $n \to \infty$.
  - Next we show the uniform convergence. Fix some $\varepsilon > 0$. Then we are promised some N so that $n, m > N$ has

    $$\sup_{x\in X}\{|f_n(x) - f_m(x)|\} < \varepsilon.$$

    In particular, for any $x \in X$

    $$f_n(x) - g(x) = \lim_{m\to\infty}(f_n(x) - f_m(x)),$$

    so because $z \mapsto |z|$ is continuous, any $n > N$ will have

    $$|f_n(x) - g(x)| = \lim_{m\to\infty}|f_n(x) - f_m(x)| \le \lim_{m\to\infty}\sup_{z\in X}\{|f_n(x) - f_m(x)|\} < \lim_{m\to\infty}\varepsilon = \varepsilon,$$

    where the last inequality holds by taking m sufficiently large (that is, $m > N$). So we have been provided uniform convergence. ■

Remark 2.134 (Nir)**.** In the language of metric topology, the above proposition asserts that the space of (bounded) functions is metrically complete. For this, one must technically show that $\{f_n\}_{n\in\mathbb{N}}$ being bounded implies that the convergent g is bounded, but this is not hard: there is N so that $n > N$ has $|f_n(x) - g(x)| < 1$ for each $x \in X$.

Remark 2.135. In lecture, Professor Morrow asserted that we require these functions to be bounded. I do not think this is the case; indeed, the above proof never uses this hypothesis.

We close with one result which shows that uniform continuity is nice.

Proposition 2.136. Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ all continuous at some $x \in X$. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to some function $g\colon X \to \mathbb{C}$, then g is also continuous.

Proof. The idea is to well-approximate g by f_ns. Fix any $\varepsilon > 0$. By the uniform convergence, there will be some N so that

$$|f_n(z) - g(z)| < \varepsilon/3$$

for any $n > N$ and $z \in X$; fix some $n > N$. Because f_n is continuous, we are promised some $\delta > 0$ (allowed to vary with our chosen $x \in X$) so that

$$|z - x| < \delta \implies |f_n(z) - f_n(x)| < \varepsilon/3$$

for any $z \in X$. Well, if $|z - x| < \delta$, then the triangle inequality gives

$$|g(z) - g(x)| \le |g(z) - f_n(z)| + |f_n(z) - f_n(x)| + |f_n(x) - g(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which is what we needed. ■

Remark 2.137 (Nir)**.** In fact, if the $\{f_n\}_{n\in\mathbb{N}}$ are uniformly continuous, then g will also be uniformly continuous. The argument is similar.

THEME 3

DIFFERENTIATION

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.

—Charles Hermite

3.1 February 11

The wheel of time marches on. Today, we actually start talking about complex analysis.

3.1.1 Differentiability

We are going to talk about holomorphic functions.

Convention 3.1. We set Ω to be some open subset of $\mathbb{C}$.

This gives the following definition.

Definition 3.2 (Differentiable)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ a function. Then f is *complex differentiable at* $z_0 \in \Omega$ *with derivative* $\alpha \in \mathbb{C}$ if and only if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \alpha.$$

We write this as $f'(z_0) = \alpha$.

If f' is itself a differentiable function, then f would be "twice" differentiable, and we denote this function by f''. In general, if f can be differentiated n times, we denote the corresponding function by $f^{(n)}$.

Warning 3.3. In the definition of complex differentiability, we are taking the limit with $h \in \mathbb{C}$, not $h \in \mathbb{R}$. This will make complex differentiability significantly more structured.

Differentiability gives rise to the following definition.

Definition 3.4 (Holomorphic, entire)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ a function. Then f is *holomorphic on* Ω if and only if f is complex differentiable at each $z_0 \in \mathbb{C}$. If $\Omega = \mathbb{C}$, then we say f is *entire*.

Here is a small usual lemma.

Lemma 3.5. Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ a function. Then if f is differentiable at $z_0 \in \Omega$, then f is continuous at $z_0 \in \mathbb{C}$.

Proof. We compute that

$$\begin{aligned}\lim_{z\to z_0}\big(f(z)-f(z_0)\big) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0)\\ &\overset{*}{=} \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\cdot\lim_{z\to z_0}(z-z_0)\\ &= f'(z_0)\cdot 0\\ &= 0.\end{aligned}$$

It follows by rearrangement that $\lim_{z\to z_0} f(z) = f(z_0)$, which is what we wanted. Notably, $\overset{*}{=}$ sets $h := z - z_0$. ∎

3.1.2 Basic Properties

As usual, differentiable functions have an arithmetic.

Proposition 3.6. Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f, g\colon \Omega \to \mathbb{C}$ functions differentiable at $z_0 \in \mathbb{C}$.

(a) We have that $(af+bg)'(z_0) = af'(z_0) + bg'(z_0)$, where $a, b \in \mathbb{C}$.

(b) We have that $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

(c) If $g'(z_0) \neq 0$, then
$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. We copy the proofs from real analysis.

(a) We check that
$$\begin{aligned}\lim_{h\to 0}\frac{(af+bg)(z_0+h)+(af+bg)(z_0)}{h} &= a\cdot\lim_{h\to 0}\frac{f(z_0+h)+f(z_0)}{h} + b\cdot\lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h}\\ &= a\cdot f'(z_0) + b\cdot g'(z_0),\end{aligned}$$
which is what we wanted.

(b) The key idea is to add and subtract $f(z_0)g(z_0+h)$. Indeed, we see
$$\begin{aligned}\lim_{h\to 0}\frac{(fg)(z_0+h)-(fg)(z_0)}{h} &= \lim_{h\to 0}\frac{f(z_0+h)g(z_0+h)-f(z_0)g(z_0+h)}{h}\\ &\quad+\lim_{h\to 0}\frac{f(z_0)g(z_0+h)-f(z_0)g(z_0)}{h}\\ &= \left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right)\left(\lim_{h\to 0} g(z_0+h)\right)\\ &\quad+ f(z_0)\left(\lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h}\right)\\ &= f'(z_0)g(z_0) + f(z_0)g'(z_0),\end{aligned}$$

which is what we wanted.

(c) This will follow from applying the product rule to $f \cdot \frac{1}{g}$, where we can compute the derivative of $\frac{1}{g}$ by the chain rule. We refer to Eterović's notes for the details. ■

Remark 3.7 (Nir)**.** Technically part (c) will require us to compute the derivative of $f(z) := \frac{1}{z}$ for $z \neq 0$ to finish the proof. Well, for any $z \in \mathbb{C} \setminus \{0\}$, we see that

$$\frac{f(z+h)-f(z)}{h} = \frac{\frac{1}{z+h}-\frac{1}{z}}{h} = \frac{z-(z+h)}{hz(z+h)} = -\frac{1}{z(z+h)}.$$

Taking $h \to 0$ reveals that the derivative is in fact $f'(z) = -\frac{1}{z^2}$.

Let's give some examples of entire functions.

Exercise 3.8. Fix n some positive integer. We show that the function $f(z) := z^n$ is entire with derivative $f'(z) := nz^{n-1}$.

Proof. We employ the usual proof involving the binomial theorem. Note that

$$f(z+h) = (z+h)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k,$$

so

$$\frac{f(z+h)-f(z)}{h} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k} h^{k-1},$$

where notably the $k=0$ term was killed by the $-f(z)$. Thus,

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k} \left(\lim_{h\to 0} h^{k-1}\right),$$

but all terms except $k=1$ will now vanish as $h \to 0$, so we are left with nz^{n-1} as our limit. ■

Remark 3.9 (Nir)**.** One could also show this by induction, using the product rule.

Corollary 3.10. Any polynomial function is entire.

Proof. Polynomials are (finite) linear combinations of the monomials $f_n(z) := z^n$, so this follows from combining the above two results. ■

3.1.3 Advanced Properties

We also have a notion of L'Hôpital's rule.

Proposition 3.11 (L'Hôpital's rule)**.** Fix $\Omega \subseteq \mathbb{C}$ an open subset with $f, g \colon \Omega \to \mathbb{C}$ holomorphic functions. Then, given $z_0 \in \Omega$ with $f(z_0) = g(z_0) = 0$ while $g'(z_0) \neq 0$, we have that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof. Note that, because $f(z_0) = g(z_0) = 0$, we see that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)}{z - z_0} \qquad \text{and} \qquad g'(z_0) = \lim_{z\to z_0} \frac{g(z)}{z - z_0}.$$

Dividing, we see that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \lim_{z\to z_0} \frac{f'(z_0)}{g'(z_0)} = \frac{f'(z_0)}{g'(z_0)},$$

which is what we wanted. ■

Remark 3.12 (Nir)**.** The above proof technically does not work because we have not ruled out the possibility that g might vanish arbitrarily close to z_0, thus making the limits not actually make sense. We will not fix this problem, but we will remark that a holomorphic function will only have finitely many zeroes on a compact set, which we could use to create a neighborhood for z_0 on which g doesn't vanish.

And here is our chain rule.

Proposition 3.13 (Chain rule)**.** Fix Ω and U open subsets of $\mathbb{C}$ with functions $f\colon \Omega \to U$ and $g\colon U \to \mathbb{C}$. Further, suppose that f is differentiable at $z_0 \in \Omega$ and that g is differentiable at $f(z_0) \in U$. Then $(g \circ f)$ is differentiable at z_0 with derivative

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. This proof is long, so we will try to be brief. The main idea is to consider the auxiliary function $r\colon U \setminus \{f(z_0)\} \to \mathbb{C}$ defined by

$$r(w) := \frac{g(w) - g(f(z_0))}{w - f(z_0)} - g'(f(z_0)).$$

We extend r to $f(z_0)$ by setting $r(f(z_0)) := 0$. Now, the differentiability of g at $f(z_0)$ implies that

$$\lim_{z\to z_0} \frac{g(z) - g(f(z_0))}{z - z_0} = g'(f(z_0)),$$

so in particular rearranging implies that r is continuous on at $f(z_0) \in U$.

The reason we used the letter r is that we should think of r as a remainder term. Indeed, we see

$$g(w) - g(f(z_0)) = g'(f(z_0))(w - f(z_0)) + r(w)(w - f(z_0)).$$

Plugging in $w = f(z)$, we get

$$g(f(z)) - g(f(z_0)) = g'(f(z_0))(f(z) - f(z_0)) + r(f(z))(f(z) - f(z_0)),$$

so

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = g'(f(z_0)) \cdot \frac{f(z) - f(z_0)}{z - z_0} + r(f(z)) \cdot \frac{f(z) - f(z_0)}{z - z_0}.$$

Sending $z \to z_0$ makes the rightmost term vanish by continuity because $r(f(z_0)) = 0$ and the limit is $f'(z_0)$, so we are left with

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0),$$

which is what we wanted. ■

Remark 3.14 (Nir). Let's complete the proof of quotient rule. Note that the derivative of $\frac{1}{g(z)}$ will be, by the chain rule, $-\frac{1}{g(z)^2} \cdot g'(z)$. Thus, the derivative of $\frac{f(z)}{g(z)} = f(z) \cdot \frac{1}{g(z)}$ will be

$$f'(z) \cdot \frac{1}{g(z)} - f(z) \cdot \frac{g'(z)}{g(z)^2} = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

And we finish with a result which is less common in real analysis, essentially saying that differentiable functions are "approximately" linear.

Proposition 3.15 (Carathéodory). Fix $\Omega \subseteq \mathbb{C}$ an open subset with a function $f\colon \Omega \to \mathbb{C}$ and point $z_0 \in \Omega$. Then f is differentiable at z_0 if and only if there exists a function $h\colon \Omega \to \mathbb{C}$ which is continuous at z_0 such that

$$f(z) - f(z_0) = h(z)(z - z_0).$$

In particular, $h(z_0) = f'(z_0)$.

Proof. We show the directions independently.

- Suppose f is differentiable at z_0. We construct the function h manually. We define

$$h(z) := \begin{cases} (f(z) - f(z_0))/(z - z_0) & z \in \Omega \setminus \{z_0\}, \\ f'(z_0) & z = z_0. \end{cases}$$

  In particular, we note that h is continuous at z_0 because $h(z) \to f'(z_0)$ as $z \to z_0$ by differentiability of f.

- Suppose h is such a function. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} h(z) = h(z_0)$$

  by continuity. Formally, the first equality is holding for the limit in $\Omega \setminus \{z_0\}$, and the second equality is continuity for $h|_{\Omega \setminus \{z_0\}}$.

To finish, we note that the second part shows that $h(z_0) = f'(z_0)$. ■

3.2 February 14

Happy Valentine's Day, I suppose. Homework #4 is due on Sunday. Homework #5 will be released on Friday.

3.2.1 Motivating Cauchy–Riemann Equations

Today we're talking about the Cauchy–Riemann equations.

Idea 3.16. The Cauchy–Riemann equations are necessary conditions for a function to be holomorphic.

In fact, they will be sufficient as well, but we will only see this next class.

Throughout today's class, we will fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset. We recall that a function $f\colon \Omega \to \mathbb{C}$ is "differentiable" at some $z_0 \in \Omega$ if and only if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If it exists, we denoted it by $f'(z_0)$, though we will not assume it exists yet. If we fix $\Delta z := z - z_0$, then we can write the above as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{\Delta z}.$$

Now, to motivate our discussion, we recall that under the isomorphism $\mathbb{C} \cong \mathbb{R}^2$ with basis $\{1, i\}$, we can define $u(x,y) := \operatorname{Re} f(x+yi)$ and $v(x,y) := \operatorname{Im} f(x+yi)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ so that

$$f(x+yi) = u(x,y) + iv(x,y).$$

The point of this is to encode some geometry directly into our set-up.

Example 3.17. Given $f(z) = z^2$, we can plug in

$$f(x+yi) = (x+yi)^2 = \underbrace{x^2 - y^2}_{u} + i \cdot \underbrace{2xy}_{v}.$$

Now that we're moving things to $\mathbb{R}^2$, we will fix $z_0 := x_0 + y_0 i$ for $x_0, y_0 \in \mathbb{R}$ with $z = x + yi$ so that $\Delta z = (x - x_0) + (y - y_0)i = \Delta x + i\Delta y$. And for a little more convenience, we fix $\Delta w := f(z_0 + z) - f(z_0)$ so that

$$f'(z_0) \stackrel{?}{=} \lim_{z\to z_0} \frac{\Delta w}{\Delta z},$$

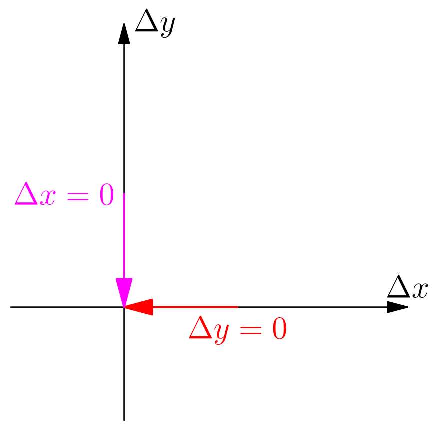
if the limit exists. Expanding out f into real and imaginary parts, we find

$$\Delta w := \big(u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\big) + i\big(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\big).$$

Now assume that f is inf act differentiable at z_0 so that $f'(z_0)$ will actually exist. Our key idea to continue is to split up the limit into real and imaginary parts because it will exist if and only if the limits of the real and imaginary parts exist. So we note

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right) \end{aligned} \tag{$*$}$$

We will now compute this limit in two ways to get the Cauchy–Riemann equations, as follows.



These are probably the easiest two limits that we could think of, so it's nice that they will be so useful. Anyways, here is our working out.

- We set $\Delta y = 0$ so that $\Delta z = \Delta x$. This gives

$$\frac{\Delta w}{\Delta x} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \cdot \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

On one hand, we can use (∗) to show the real part comes out to

$$\operatorname{Re} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta x}\right) = \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}.$$

This limit must exist because f is differentiable at z_0, and when this limit exists, the rightmost limit is called the partial derivative $u_x(x_0, y_0)$.

On the other hand, the imaginary part comes out to

$$\operatorname{Im} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta x}\right) = \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x},$$

which comes out to $v_x(x_0, y_0)$ because we know that the limit exists.

So in total, we see $f'(z_0) = u_x(x_0, y_0) + i \cdot v_x(x_0, y_0)$.

- We set $\Delta x = 0$ so that $\Delta z = i\Delta y$. Be warned that an unexpected sign is about to appear from this i. This time we get

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \cdot \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}.$$

To "rationalize" the deminators, we write

$$\frac{\Delta w}{\Delta z} = \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y} - i \cdot \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y},$$

where we are using $1/i = -i$. Note that the us and vs have swapped from the last computation!

We now compute our limits. On one hand,

$$\operatorname{Re} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y},$$

which is $v_y(x_0, y_0)$ because the limit exists. On the other hand,

$$\operatorname{Im} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0} -\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y},$$

which is $-u_y(x_0, y_0)$ because the limit exists.

So in total, we see $f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Remark 3.18. Either equation itself is pretty useful to actually compute formulae for the derivatives.

Synthesizing, we see

$$f'(z_0) = u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Comparing real and imaginary parts, we get the following.

Theorem 3.19 (Cauchy–Riemann)**.** Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f\colon \Omega \to \mathbb{C}$ a function differentiable at some $z_0 = x_0 + y_0 i \in \mathbb{C}$. If we write $f(x+yi) = u(x,y) + i(x,y)$, then

$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0), \\ v_x(x_0,y_0) = -u_y(x_0,y_0). \end{cases}$$

In fact, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Proof. This follows from the above discussion. ■

3.2.2 Examples

Let's see some examples to be convinced of the utility of Theorem 3.19. Let's start by checking that something is differentiable.

Example 3.20. Take $f(z) = z^2$ so that

$$(x + yi) = (x + yi)^2 = \left(x^2 - y^2\right) + i(2xy)$$

so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ has $f(x + yi) = u(x, y) + iv(x, y)$. We know that f is entire (it's impossible), so picking up any $z = x + yi \in \mathbb{C}$, we compute

$$u_x(x, y) = 2x = v_y(x, y) \qquad \text{and} \qquad v_x(x, y) = 2y = -(-2y) = -u_y(x_0, y_0),$$

verifying Theorem 3.19. In fact, we can see that $f'(z) = u_x(x, y) + v_x(x, y) = 2x + 2yi = 2z$.

And now let's see something which isn't differentiable.

Example 3.21. Take $f(z) = |z|^2$ so that

$$f(x + yi) = |x + yi|^2 = (x + yi)(x - yi) = x^2 + y^2,$$

which only has a real part! Namely, we have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ to make $f(x+yi) = u(x, y) + iv(x, y)$. Now suppose for the sake of contradiction that f were differentiable at some $z = x + yi \in \mathbb{C}$. Then we are forced into

$$2x = u_x(x, y) = v_x(x, y) = 0 \qquad \text{and} \qquad 0 = v_x(x, y) = -u_y(x, y) = -2y,$$

which means $x = y = 0$. So f is differentiable at nowhere outside $\mathbb{C} \setminus \{0\}$.

Observe that the above example does not show that f is differentiable at $0 \in \mathbb{C}$, though this is true. To be explicit, Theorem 3.19 does not tell us that satisfying the Cauchy–Riemann equations implies differentiability.

Remark 3.22. Extending Example 3.21, we can show that the only entire real-valued function is constant.

Let's also close with an application of Theorem 3.19.

Corollary 3.23. Fix $\Omega \subseteq \mathbb{C}$ a connected nonempty open subset and $f : \Omega \to \mathbb{C}$ a function differentiable on all of Ω so that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.

Proof. By Theorem 3.19, we see that, for any $z = x + yi$, we see

$$u_x(x, y) = v_y(x, y) = \operatorname{Re} f'(z) = 0 \qquad \text{and} \qquad v_x(x, y) = -u_y(x, y) = \operatorname{Im} f'(z) = 0.$$

In particular, for some function $g : C \to \mathbb{R}$ for some $C \subseteq \mathbb{R}^2$ connected and open, having $g_x = 0$ forces g to be constant as a function of x on any connected horizontal line, and $g_y = 0$ forces g to be constant as a function of y.

Now, because any path between two points in an open subset can be approximated by vertical and horizontal line segments contained in neighborhoods of points, we see that the endpoints of any path in C must have the same value.[1] But C is open and connected and hence path-connected, so C any two points can be connected by path, so g must be constant on all of C.

Returning to f, we see that u and v will be constant on the embedding of Ω into $\mathbb{R}^2$ (recall that $\mathbb{C} \cong \mathbb{R}^2$ topologically, so $\Omega \subseteq \mathbb{R}^2$ remains open and connected), so f is constant on Ω. This is what we wanted. ■

[1] Please don't ask me to rigorize this.

Remark 3.24. We do need the connected hypothesis: we could take $\Omega = \mathbb{C} \setminus \mathbb{R}$ and with $f(z) = 1_{\operatorname{Re} z > 0}$.

3.3 February 16

We talk more about the Cauchy–Riemann equations today. For our announcements, Homework #4 is due on Sunday. There is a midterm next Friday; we will get a review sheet and some practice problems in the next few days. There will be no homework, and there will be extra office hours.

3.3.1 Introducing Sufficient Conditions

The slogan for today as follows.

Idea 3.25. The Cauchy–Riemann equations provide a sufficient condition for differentiability.

Recall our theorem.

Theorem 3.19 (Cauchy–Riemann)**.** Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f \colon \Omega \to \mathbb{C}$ a function differentiable at some $z_0 = x_0 + y_0 i \in \mathbb{C}$. If we write $f(x+yi) = u(x,y) + i(x,y)$, then

$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0), \\ v_x(x_0,y_0) = -u_y(x_0,y_0). \end{cases}$$

In fact, $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$.

These are sufficient conditions for differentiability. Today we are discussing necessary conditions for differentiability.

Theorem 3.26. Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f \colon \Omega \to \mathbb{C}$ a function. Writing $f(x+yi) = u(x,y) + iv(x,y)$ and fixing some $z_0 := x_0 + y_0 i$, then suppose we have the following.

- We have u_x, u_y, v_x, v_y all exist and are continuous (!).
- We have
$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0), \\ v_x(x_0,y_0) = -u_y(x_0,y_0). \end{cases}$$

Then f is differentiable at z_0.

Remark 3.27. It is possible to construct functions which are differentiable at z_0 but do not have continuous first partial derivatives.

Let's do some examples of Theorem 3.26 to see its utility.

Example 3.28. Fix $f(x+yi) = x^2 + y + i\left(y^2 - x\right)$. Here, $u(x,y) = x^2 + y$ and $v(x,y) = y^2 - x$, so we see

$$u_x(x,y) = 2x, \quad u_y(x,y) = 1, \quad v_x(x,y) = -1, \quad \text{and} \quad v_y(x,y) = 2y.$$

So all first partial derivatives are continuous. To satisfy the Cauchy–Riemann equations, we see that we need $u_x = v_y$ and $u_y = -v_x$, which is equivalent to $2x = 2y$ and $1 = --1$. It follows from Theorem 3.26 that f is differentiable on the line $y = x$, and f is not differentiable anywhere else by Theorem 3.19.

Remark 3.29. Another type of question is to be given $u(x,y)$ and be asked for what $v(x,y)$ might be.

3.3.2 Proving Sufficient Conditions

Let's go ahead and prove Theorem 3.26.

Proof of Theorem 3.26. As with last time, we fix $\Delta z := z - z_0$ and $\Delta x = x - x_0$ and $\Delta y = y - y_0$ so that our difference quotient is

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \underbrace{\frac{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)}{\Delta z}}_{\Delta u/\Delta z :=} + i \cdot \underbrace{\frac{v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)}{\Delta z}}_{\Delta v/\Delta z :=}.$$

So our goal is to show that

$$\lim_{\Delta z\to 0}\left(\frac{\Delta u}{\Delta z} + i\cdot\frac{\Delta v}{\Delta y}\right)$$

exists and is equal to $u_x(x_0,y_0)+iv_x(x_0,y_0)$. So we need to force our first partial derivatives into the limit.

We start with $\Delta u/\Delta z$. To make our partial derivatives appear, we write

$$\begin{aligned}\frac{\Delta u}{\Delta z} &= \frac{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)}{\Delta z}\\ &= \frac{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)}{\Delta z} + \frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta z}.\end{aligned}$$

To get our partial derivatives, we apply the Mean value theorem (!): define

$$F(x) := u(x, y_0+\Delta y) \qquad \text{and} \qquad F(y) := u(x_0, y).$$

We do our applications one at a time.

- Note that $F(x)$ is differentiable everywhere from x_0 to $x_0+\Delta x$, so the Mean value theorem provides some x_0^* between x_0 and $x_0+\Delta x$ such that
$$F(x_0+\Delta x) - F(x_0) = F'(x_0^*)\Delta x.$$
- Similarly, $F(y)$ is differentiable everywhere from y_0 to $y_0+\Delta y$, so the Mean value theorem provides some y_0^* between y_0 and $y_0+\Delta y$ such that
$$F(y_0+\Delta x) - F(y_0) = F'(y_0^*)\Delta y.$$

Synthesizing and plugging in, we get

$$\frac{\Delta u}{\Delta z} = \frac{u_x(x_0^*,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0^*)\Delta y}{\Delta z}.$$

We now use continuity of our first partial derivative. Our hope is that sending $\Delta z\to 0$ will send $u_x(x_0^*,y_0)\to u_x(x_0,y_0)$ and $u-y(x_0,y_0^*)\to u_y(x_0,y_0)$. To show this, we show the difference will be small. We write

$$\frac{\Delta u}{\Delta z} = \frac{u_x(x_0,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0)\Delta y}{\Delta z} + E_{ux} + E_{uy},$$

where

$$E_{ux} = \big(u_x(x_0^*,y_0) - u_x(x_0,y_0)\big)\frac{\Delta x}{\Delta z} \quad \text{and} \quad E_{uy} = \big(u_y(x_0,y_0^*) - u_y(x_0,y_0)\big)\frac{\Delta y}{\Delta z}.$$

We now remark that we can repeat the entire above argument for $\frac{\Delta v}{\Delta z}$. Namely, running the above machinery lets us write

$$\frac{\Delta v}{\Delta z} = \frac{v_x(x_0,y_0)\Delta x}{\Delta z} + \frac{v_y(x_0,y_0)\Delta y}{\Delta z} + E_{vx} + E_{vy},$$

where
$$E_{vx} = \left(u_x(x_0^{**}, y_0) - u_x(x_0, y_0)\right)\frac{\Delta x}{\Delta z} \quad \text{and} \quad E_{vy} = \left(u_y(x_0, y_0^{**}) - u_y(x_0, y_0)\right)\frac{\Delta y}{\Delta z}.$$
We now show that the various $E_\bullet$ terms vanish as $\Delta z \to 0$. Note that, as $\Delta z \to 0$, the following happen.

- Because x_0^* and x_0^{**} are bounded between x_0 and $x_0 + \Delta x$, they will approach x_0.
- Because y_0^* and y_0^{**} are bounded between y_0 and $y_0 + \Delta y$, they will approach y_0.
- We will have $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$ by direct expansion of the norm because $\operatorname{Re}\Delta z = \Delta x$ and $\operatorname{Im}\Delta z = \Delta y$.

It follows that each of the $E_\bullet$ do indeed vanish as $\Delta z \to 0$. For example,
$$\left|\left(u_x(x_0^*, y_0) - u_x(x_0, y_0)\right)\frac{\Delta x}{\Delta z}\right| \le \left|u_x(x_0^*, y_0) - u_x(x_0, y_0)\right|$$
will go to 0 as $\Delta z \to 0$ by the continuity of u_x at (x_0, y_0).

Now we return to our difference quotient. We see
$$\begin{aligned}\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \lim_{\Delta z\to 0}\left(\frac{\Delta u}{\Delta z} + i\cdot\frac{\Delta v}{\Delta z}\right)\\ &= \lim_{\Delta z\to 0}\left(\frac{u_x(x_0,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0)\Delta y}{\Delta z} + i\cdot\frac{v_x(x_0,y_0)\Delta x}{\Delta z} + i\cdot\frac{v_y(x_0,y_0)\Delta y}{\Delta z}\right)\\ &\qquad + \lim_{\Delta z\to 0}E_{ux} + \lim_{\Delta z\to 0}E_{uy} + \lim_{\Delta z\to 0}E_{vx} + \lim_{\Delta z\to 0}E_{vy}\\ &= \lim_{\Delta z\to 0}\left(\frac{u_x(x_0,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0)\Delta y}{\Delta z} + i\cdot\frac{v_x(x_0,y_0)\Delta x}{\Delta z} + i\cdot\frac{v_y(x_0,y_0)\Delta y}{\Delta z}\right),\end{aligned}$$
using the fact that our error terms all vanish. At this point we use the Cauchy–Riemann equations. We see
$$\begin{aligned}\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \lim_{\Delta z\to 0}\left(\frac{u_x(x_0,y_0)\Delta x}{\Delta z} - \frac{v_x(x_0,y_0)\Delta y}{\Delta z} + i\cdot\frac{v_x(x_0,y_0)\Delta x}{\Delta z} + i\cdot\frac{u_x(x_0,y_0)\Delta y}{\Delta z}\right)\\ &= \lim_{\Delta z\to 0}\left(u_x(x_0,y_0)\cdot\frac{\Delta x + i\Delta y}{\Delta z}\right) + i\cdot\lim_{\Delta z\to 0}\left(v_x(x_0,y_0)\cdot\frac{\Delta x+i\Delta y}{\Delta z}\right),\end{aligned}$$
which finishes after evaluating our first partial derivatives. ■

3.4 February 18

Good morning everyone. Here are some announcements.

- Homework #4 is due on Sunday.
- Next Friday is our midterm. A review sheet has been posted. Some practice problems and a practice midterm will be released today or tomorrow.
- Next week will have office hours every day.
- Next Wednesday will be a review class.

3.4.1 Power Series

Today we are building towards a discussion of analytic functions. We won't actually define what "analytic" means, but it will be important, so we will spend today setting up the definitions and results.

Definition 3.30 (Complex power series)**.** A *complex power series* is a formal expression of the form

$$S(z) := \sum_{k=0}^{\infty} a_k x^k$$

where $\{a_k\}_{k\in\mathbb{N}} \subseteq \mathbb{C}$ and z is a (formal) variable taking complex values.

Our main goal for today is to be able to answer the following question.

Question 3.31. For which z will $S(z)$ converge?

The answer to this is essentially the same as for real analysis: it's the radius of convergence.

Definition 3.32 (Radius of convergence)**.** The *radius of convergence* of a complex power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ is defined to be equal to the radius of convergence of the real power series

$$T(x) = \sum_{k=0}^{n} |a_k| x^k.$$

Concretely, we define

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

We should probably check convergence in the radius of convergence.

Proposition 3.33. Fix a complex power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ with radius of convergence R. Then the following hold.

(a) The sequence of partial sums $\sum_{k=0}^{n} \left|a_k z^k\right|$ converge for any z with $|z| < R$. In other words, $S(z)$ converges absolutely.

(b) The series $S(z)$ will diverge for z with $|z| > R$.

Proof. We take these one at a time. The point is to imitate the proofs from real analysis.

(a) We note that, if $R = 0$, there is nothing to prove here. Otherwise, fix z with $|z| < R$ so that there exists some $\rho \in \mathbb{R}$ with $|z| < \rho < R$. For example, $\rho := \frac{|z|+R}{2}$ will do.

Now, because $\rho < R$, we see that $\frac{1}{\rho} > \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (this is legal because $R \neq 0$), so there exists some N for which any fixed $n > N$ has

$$\sup_{k\geq n} \sqrt[k]{|a_k|} < \frac{1}{\rho}.$$

In particular, each $k > N$ will have $\sqrt[k]{|a_k|} < 1/\rho$, so $|a_k|\rho^k < 1$. So, setting

$$M := \max\left(\{1\} \cup \{|a_k| : k \leq N\}\right),$$

we see that $|a_k|\rho^k \leq M$ for each $k \in \mathbb{N}$.

But because $|z| < \rho$, we note that $|z|/\rho < 1$, so we bound

$$\left|a_k z^k\right| = \left|a_k \rho^k\right| \cdot \left|\frac{z^n}{\rho^n}\right| \leq M \left|\frac{z}{\rho}\right|^n.$$

However, $|z/\rho| < 1$, so the series $\sum_{k=0}^{\infty} |z/\rho|^k$ will converge as a geometric series, so we are done by the comparison test.

(b) We proceed by contraposition. Suppose that $S(z)$ converges, so by Lemma 2.71, $a_k z^k \to 0$ as $k \to \infty$. We show that $|z| \leq R$. If $z = 0$, there is nothing to say; otherwise, it will suffice to show that

$$\frac{1}{|z|} \overset{?}{\geq} \limsup_{k\to\infty} \sqrt[k]{|a_k|}.$$

For this, fix $\varepsilon = 1$, so we are granted some N for which $k > N$ has

$$\left|a_k z^k\right| < 1.$$

In particular, this rearranges into $1/|z| > \sqrt[k]{|a_k|}$. So for each $n > N$, we see $1/|z| > \sqrt[k]{|a_k|}$ for $k > n$, so $1/|z| \geq \sup\{\sqrt[k]{|a_k|} : k > n\}$, so

$$\frac{1}{|z|} \geq \lim_{n\to\infty} \sup\left\{\sqrt[k]{|a_k|} : k > n\right\} = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

which is what we wanted. ■

Remark 3.34 (Nir)**.** The proof of (b) might feel weird because we are not using the full power of $S(z)$ converging, just that its terms go to 0. However, a power series will "essentially" converge whenever its terms go to 0 (aside from boundary cases), so it is not too surprising that this is all that we need.

We have the following warning.

Warning 3.35. Proposition 3.33 is agnostic to the case of $|z| = R$.

In general, the behavior need not be uniform, as with $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$.

3.4.2 Series of Functions

We will be interested in series of functions, which generalize power series.

Definition 3.36 (Series of functions)**.** Fix $X \subseteq \mathbb{C}$ a nonempty set and $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then we define the *series of functions*

$$S(z) = \sum_{k=0}^{\infty} f_k(z)$$

for each $z \in \mathbb{C}$.

Observe that the partial sums of some $S(z) = \sum_{k=0}^{m} f_k(z)$ will look like some finite sum

$$S_m(z) = \sum_{k=0}^{m} f_k(z),$$

which defines a sequence of functions $\{S_m\}_{m\in\mathbb{N}}$ where $S_m : X \to \mathbb{C}$. We are interested in the convergence of S as a function.

Definition 2.69 (Converge, diverge)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

Uniform convergence will be nice because (say) it will preserve continuity, but before talking about utility, we discuss a way to check uniform convergence.

Theorem 3.37 (Weierstrass M-test)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset and $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ defining a series of functions $S(z) = \sum_{k=0}^{\infty} f_k(z)$. Further, suppose that, for each $k \in \mathbb{N}$, there exists some M_k such that

$$|f_k(z)| \le M_k$$

for each $z \in X$. If $\sum_{k=0}^{\infty} M_k$ converges, then $S(z)$ converges uniformly.

In other words, we can determine uniform convergence of a series of functions by bounding the functions individually.

Proof of Theorem 3.37. This is not as hard as it looks. Let S_m denote the mth partial sum. By Proposition 2.133, it suffices to show that, for each $\varepsilon > 0$, there exists some N such that $n \ge m > N$ implies

$$\sup_{x\in X}\{|S_n(z) - S_m(z)|\} < \varepsilon.$$

Well, we know that the series $\sum_{k=0}^{\infty} M_k$ converges, so its partial sums are Cauchy, so there exists some N such that $n \ge m > N$ implies

$$\sum_{k=m+1}^{n} M_k < \varepsilon,$$

where the left-hand side is the difference between the nth and mth partial sums. So now we bound

$$|S_n(z) - S_m(z)| = \left|\sum_{k=m+1}^{n} f_k(z)\right| \le \sum_{k=m+1}^{n} |f_k(z)| \le \sum_{k=m+1}^{n} M_k,$$

for any $z \in X$. Thus,

$$\sup_{z\in X}\{|S_n(z) - S_m(z)|\} \le \sum_{k=m+1}^{n} M_k < \varepsilon.$$

This finishes the proof. ∎

And now let's apply the Weierstrass M-test to power series.

Corollary 3.38. Fix $S(z) = \sum_{k=0}^{\infty} a_k z^k$ a power series with positive radius of convergence $R > 0$. We have the following.

(a) For any r such that $0 < r < R$, the power series $S(z)$ converges uniformly in $\overline{B(0,r)}$.

(b) The power series $S(z)$ is continuous on $B(0,r)$.

Proof. Most of our work will be done in (a), which comes from the Weierstrass M-test.

(a) Fix some r with $0 < r < R$. Note that $S(r)$ converges absolutely by Proposition 3.33. To use the Weierstrass M-test, we set $f_k(z) := a_k z^k$ with $M_k := |a_k| r^k$ so that $|z| \le r$ implies

$$|f_k(z)| = \left|a_k z^k\right| = |a_k| \cdot |z|^k \le |a_k| r^k.$$

But we know that $S(r)$ converges absolutely, so

$$\sum_{k=0}^{\infty} \left|a_k r^k\right| = \sum_{k=0}^{\infty} M_k$$

converges, so now Theorem 3.37 promises that $S(z)$ will converge uniformly for each $z \in \overline{B(0,r)}$.

(b) Note that, for every k, the function $f_k(z) = a_k z^k$ is a polynomial and hence entire and hence continuous on $B(0, R)$.

The trick is to apply (a) by starting with a fixed $z_0 \in B(0, R)$ with r such that $|z_0| < r < R$. In particular, by restriction, it suffices to show that $S|_{B(0,r)}$ is continuous at z_0. (For example, $r = \frac{|z_0|+R}{2}$ will work.) So now we note that the continuous partial sums of $S(z)$ converge uniformly to $S(z)$ on $B(0, r)$ by (a), so Proposition 2.136 forces $S(z)$ itself to be continuous on $B(0, r)$. This finishes. ■

We remark that the restriction to $S|_{B(0,r)}$ only works because $B(0, r)$ is an open set. Here is the exact lemma we just used.

Lemma 3.39. Fix $f : X \to \mathbb{C}$ a function and $U \subseteq \mathbb{C}$ an open subset X with $x \in U \cap X$. Then f is continuous at x if and only if the restriction $f|_{U\cap X} : U \cap X \to \mathbb{C}$ is continuous at x.

An alternate way to give the hypothesis on U is that $U \cap X$ is an open subset of X.

Proof. We show the directions independently.

- Suppose that f is continuous at x. We show that $f|_{U\cap X}$ is continuous at x. Well, for any $\varepsilon > 0$, we are promised some $\delta > 0$ so that any $z \in X$ has

$$|z - x| < \delta \implies |f(z) - f(x)| < \varepsilon.$$

In particular, any $z \in X \cap U$ has

$$|z - x| < \delta \implies \big|f|_{U\cap X}(z) - f|_{U\cap X}(x)\big| = |f(z) - f(x)| < \varepsilon.$$

- Suppose that $f|_{U\cap X}$ is continuous at x. Fix any $\varepsilon > 0$. Because $x \in U$ and U is open, there exists $r > 0$ such that $B(x, r) \subseteq U$. Because $f|_{U\cap X}$ is continuous at x, there exists some $\delta_0 > 0$ such that

$$|z - x| < \delta_0 \implies |f(z) - f(x)| = |f_{U\cap X}(z) - f_{U\cap X}(z)| < \varepsilon$$

for $z \in U \cap X$. However, taking $\delta := \min\{r, \delta\}$, we see that any $z \in X$ with $|x - z| < \delta$ will have $z \in B(x, \delta) \subseteq U$, so $z \in U \cap X$ automatically. So $|z - x| < \delta$ will still imply

$$|f(z) - f(x)| < \varepsilon,$$

and we are done. ■

Remark 3.40 (Nir). More generally, if we have a sequence of continuous functions $f_k : X \to \mathbb{C}$ such that the series $S(z) := \sum_{k=0}^{\infty} f_k(z)$ converges uniformly on X, then S is a continuous function on X. Indeed, fix some $z_0 \in X$ and $\varepsilon > 0$. We have the following.

- There is N so that $n > N$ has $|S(z) - f_n(z)| < \varepsilon/3$ for $z \in X$. Fix some $n > N$.
- There is $\delta > 0$ so that $|z - z_0| < \delta$ has $|f_n(z) - f_n(z_0)| < \varepsilon/3$.

Thus, $|z - z_0| < \delta$ will have

$$|S(z) - S(z_0)| < |S(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - S(z_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

3.5 February 23

Good morning everyone. We are doing review today.

3.5.1 Review Highlights

Here are some of the answers to questions asked in class.

- The midterm will be like the practice midterm.
- You do not need to be stressed about the midterm.
- Things proven in class we will not be asked to prove on the midterm.
- We will probably will not be asked to do anything involving summation by parts.
- Professor Morrow will not curve downward.
- Please write more on the exam for the sake of partial credit.
- For words with multiple definitions (like continuity and compactness), the first definition is preferred, though other definitions will likely be accepted.
- We may cite facts from real analysis, which is a requirement for this class; e.g., $[0,1]$ is compact.
- Lemmas elided from class we will not be responsible for. Essentially, please know the things on the review.
- Please know the definitions of things is important. They will be graded fairly harshly because these are critical to know to going forwards.

Let's do a practice problem.

Exercise 3.41. Find the possible functions $v(x,y)\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(z) = f(x+iy) = x^2 - y^2 + iv(x,y)$$

is entire and $f(0) = 0$.

Proof 1. The point is to use the Cauchy–Riemann equations. We set $u(x,y) := x^2 - y^2$ so that $f(x+yi) = u(x,y) + iv(x,y)$. If we want this to be differentiable, we want

$$u_x(x,y) = 2x = v_y(x,y)$$

by Theorem 3.19. This means that $v(x,y) = 2xy + h(x)$ for some function $h(x)\colon \mathbb{R} \to \mathbb{R}$. Again, we note

$$u_y(x,y) = -2y = -v_x(x,y) = -2y - h'(x),$$

so we want $h'(x) = 0$. So h is a constant function, so we set $h(x) = c$ for some $c \in \mathbb{R}$.

It remains to determine c. Well, so far the story is that

$$f(x+iy) = x^2 - y^2 + i(2xy + c).$$

Plugging in $x = y = 0$ forces $c = 0$, so we see that we get $\boxed{f(x+iy) = x^2 - y^2 - i \cdot 2xy}$. ■

Remark 3.42. The current form of the answer is fine: we do not have to simplify in terms of z or something. More generally, we will not have to spend large amounts of time simplifying on the exam.

Let's present another proof.

Proof 2. The point is to use the x information to fully piece together $f'(z)$. As before, set, $u(x,y) = x^2 - y^2$. Namely, the Cauchy–Riemann equations promise

$$f'(z) = f'(x+yi) = u_x(x,y) + iv_x(x,y) = u_x(x,y) - iu_y(x,y).$$

Taking partial derivatives of u implies that

$$f(z) = 2x - i(-2y) = 2x + i \cdot 2y = 2(x+yi) = 2z.$$

So from here, we can take the "antiderivative" (i.e., guess) that $f(z) = z^2 + c$. Lastly, plugging in $f(0) = 0$, we get $c = 0$, so $\boxed{f(z) = z^2}$. ■

Remark 3.43. We can rigorize that this is the only possible solution because any other solution $g(z)$ must have $g(z) - z^2$ with constant derivative 0, from which we can argue that $g(z) - z^2$ is constant using the Cauchy–Riemann equations and the fact that $\mathbb{C}$ is path-connected. To be explicit, we are using Corollary 3.23.

3.6 February 25

There was no lecture today because we had a midterm.

3.7 February 28

Good morning, everyone. Here are some announcements.

- Midterm grades will be posted today or tomorrow, on `bCourses`.
- Class on Wednesday will be a recording. Professor Morrow will be giving a talk, at 9AM as decided by the powers that be.
- There is no homework due Friday because we haven't covered anything since the midterm.

3.7.1 Holomorphic Power Series

Today we actually talk about analytic functions. Professor Morrow promises that it is actually complex analysis today, and once we talk about analytic functions and path integration, we will prove the Cauchy integral formula, which is one of the major results of the course.

We recall the following definition.

Definition 3.30 (Complex power series)**.** A *complex power series* is a formal expression of the form

$$S(z) := \sum_{k=0}^{\infty} a_k x^k$$

where $\{a_k\}_{k \in \mathbb{N}} \subseteq \mathbb{C}$ and z is a (formal) variable taking complex values.

So far we've talked about the radius of convergence of a power series as well as some properties of series of functions in general (e.g., the Weierstrass M-test).

Today we are showing the following result.

Proposition 3.44. Fix $S(z) = \sum_{k=0}^{\infty} a_k z^k$ a (complex) power series with radius of convergence $R > 0$. Then $S(z)$ is holomorphic on $B(0,R)$ with derivative

$$S'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

Further, $S'(z)$ also has radius of convergence R.

Note that this derivative is essentially the "term-wise" derivative of $S(z)$, so it is more or less the best thing that we could want.

Proof. We will symbolically define

$$S'(z) := \sum_{k=1}^{\infty} k a_k z^{k-1}$$

and show that it is equal to the requested derivative. We start by noting the radius of convergence of S' is

$$\frac{1}{\lim_{k\to\infty} \sqrt[k]{|(k+1)a_k|}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} \cdot \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} = 1 \cdot R = R,$$

so at the very least our radius of convergence matches, as claimed.

Fix $0 < r < R$ a real number (i.e., we don't want to deal with $R = +\infty$), so that it suffices to show S is holomorphic with the given derivative on $B(0,r)$. (Namely, for a given $w \in B(0,R)$, choose any r with $|w| < r < R$.)

Indeed, given $w \in B(0,r)$, it suffices to show that S is differentiable at w with the requested derivative, for which we claim

$$\left(\lim_{z\to w} \frac{S(z) - S(w)}{z - w}\right) - S'(w) \stackrel{?}{=} 0,$$

where $S'(z)$ is the claimed derivative. To set up our computation, we fix a positive integer m and work with the mth partial sum, computing

$$\begin{aligned}
\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) &= \sum_{k=0}^{m} \frac{a_k z^k - a_k w^k}{z - w} - \sum_{k=1}^{m} k a_k w^{k-1} \\
&= (a_0 - a_0) + \sum_{k=1}^{m} a_k \left(\frac{z^k - w^k}{z - w} - k w^{k-1}\right) \\
&= \sum_{k=1}^{m} a_k \left(\sum_{a+b=k-1} z^b w^a - \sum_{a+b=k-1} w^{k-1}\right) \\
&= \sum_{k=1}^{m} a_k \left(\sum_{a+b=k-1} \left(z^b w^a - w^{k-1}\right)\right) \\
&= \sum_{k=1}^{m} a_k \left(\sum_{a+b=k-1} w^a \left(z^b - w^b\right)\right).
\end{aligned}$$

With this in mind, we set

$$h_k(z) = \sum_{a+b=k-1} w^a \left(z^b - w^b\right),$$

which we note is a polynomial in $z \in B(0,r)$ because we fixed w to be constant. In particular, we have

$$\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) = \sum_{k=1}^{m} a_k h_k(z).$$

We now show that this series converges uniformly as $m \to \infty$; we will use Theorem 3.37. For this, we bound

$$|h_k(z)| = \left| \sum_{a+b=k-1} w^a \left(z^b - w^b\right) \right| \le \sum_{a+b=k-1} |w|^a \left(|z|^b + |w|^b\right) < \sum_{a+b=k-1} r^a \left(r^b + r^b\right) = 2(k-1)r^{k-1},$$

so we bound $|a_k h_k(z)| < |a_k| \cdot 2(k-1)r^{k-1}$. Namely, by Theorem 3.37, it suffices to show that the series

$$\sum_{k=1}^{\infty} 2(k-1)|a_k|r^{k-1}$$

converges. Well, $\sum_{k=1}^{\infty} 2(k-1)|a_k|x^{k-1}$ is a power series with radius of convergence

$$\frac{1}{\lim_{k\to\infty}\left(\sqrt[k]{2k} \cdot \sqrt[k]{|a_{k+1}|}\right)} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{2k}} \cdot \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_{k+1}|}} = R,$$

so indeed the power series $\sum_{k=1}^{\infty} 2(k-1)|a_k|x^{k-1}$ converges at $x = r < R$.

So in total, we see that the series of functions

$$\sum_{k=1}^{\infty} a_k h_k(z)$$

uniformly converges as $m \to \infty$. Because each component function $a_k h_k(z)$ is continuous, we see that the entire series will converge to a continuous function by Remark 3.40. In other words, we can evaluate

$$\lim_{z\to w} \lim_{m\to\infty} \left(\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) \right) = \lim_{z\to w} \sum_{k=1}^{\infty} a_k h_k(z) = \sum_{k=1}^{\infty} a_k h_k(w).$$

But now we notice that $h_k(w) = 0$ for each h_k, so this sum does indeed vanish.

We are now essentially done. We compute

$$\begin{aligned} \lim_{z\to w} \frac{S(z) - S(w)}{z - w} &= \lim_{z\to w} \left(\frac{S(z) - S(w)}{z - w} - S'(w) \right) + \lim_{z\to w} S'(w) \\ &= \lim_{z\to w} \left(\frac{\lim_{m\to\infty} S_m(z) - \lim_{m\to\infty} S_m(w)}{z - w} - \lim_{m\to\infty} S_m'(w) \right) + S'(w) \\ &= \lim_{z\to w} \lim_{m\to\infty} \left(\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) \right) + S'(w) \\ &= S'(w), \end{aligned}$$

so we are done. ■

So indeed, power series are holomorphic. Here is nice application of this fact.

Corollary 3.45. Fix

$$S(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad T(z) = \sum_{k=0}^{\infty} b_k z^k$$

two complex power series with radius of convergence $R > 0$. If $S(z) = T(z)$ for all $z \in B(0, R)$, then $a_k = b_k$ for each k.

Proof. We proceed inductively, in spirit. For example $a_0 = S(0) = T(0) = b_0$, so these are equal as our base case. Further, we could take one derivative to see that

$$S'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \text{and} \qquad T'(z) = \sum_{k=1}^{\infty} k b_k z^{k-1},$$

so $a_1 = S'(0) = T'(0) = b_1$. More generally, setting $S^{(m)}$ to be the mth derivative, we can see that

$$S^{(m)}(z) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)a_k z^{k-m} \qquad \text{and} \qquad T^{(m)} = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)a_k z^{k-m},$$

and both of these have the same radius of convergence. So now $a_m = \frac{1}{m!}S^{(m)}(0) = \frac{1}{m!}T^{(m)}(0) = b_m$. ■

3.7.2 Analytic Functions

To define analytic, we need one more definition.

> **Definition 3.46** (Power series expansion)**.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f\colon X \to \mathbb{C}$ a function. We say that f has a *power series expansion centered at* $z_0 \in X$ if and only if there is a positive real number r such that $B(z_0, r) \subseteq X$ and further there is a power series defined by $\{a_k\}_{k\in\mathbb{N}}$ which has
>
> $$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$
>
> for each $z \in B(z_0, r)$.

And here is our definition.

> **Definition 3.47** (Analytic)**.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f\colon X \to \mathbb{C}$ a function. Then f is *analytic* at $z_0 \in \mathbb{C}$ if and only if f has a power series expansion at z_0. Explicitly, there is a power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ and positive real number $r > 0$ (less than the radius of convergence) such that
>
> $$f(z) = S(z-z_0) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$
>
> for any $z \in B(z_0, r)$. Then f is *analytic* if and only if it is analytic at each $z_0 \in \mathbb{C}$.

Here is the idea.

> **Idea 3.48.** Analytic functions are locally power series.

Being analytic is a very nice condition. For example, we have the following.

> **Proposition 3.49.** Analytic functions are holomorphic on their domain.

Proof. Fix $f\colon X \to \mathbb{C}$ an analytic function. For each $x \in X$, we note that f is locally equal to a power series at x (i.e., $f|_{B(x,r)}$ is a power series), which is holomorphic by Proposition 3.44. Because f is locally differentiable at each point, f will be actually differentiable at each point. ■

> **Remark 3.50.** It will turn out that the converse is also true, but this is a pretty deep result. We will prove it from the Cauchy integral formula. The main obstacle is how we should construct the power series, which the Cauchy integral formula will tell us how to do.

Anyways, let's prove something of substance.

> **Lemma 3.51.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f\colon X \to \mathbb{C}$ an analytic function. Then f' is also analytic.

Proof. Fix $z_0 \in X$. Because f is analytic, there is a positive real number $r > 0$ and power series $S(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ (with radius of convergence at least r) such that

$$f(z) = S(z-z_0) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

for each $z \in B(z_0, r)$. By Proposition 3.44, we see that

$$f'(z) = S'(z-z_0) = \sum_{k=1}^{\infty} k a_k(z-z_0)^{k-1}$$

for each $z \in B(0, r)$. So we see that f' has a power series expansion at our arbitrarily chosen $z_0 \in X$, so f' is analytic at each $z_0 \in X$, so f' is analytic. ■

Remark 3.52. We can iterate the above lemma to show that an analytic function is infinitely differentiable.

Remark 3.53. In fact, because analytic will turn out to be equivalent to holomorphic, we will see that being once differentiable implies being analytic implies being infinitely differentiable. This is pretty nice.

Next class we will start talking about the exponential function, a very important analytic function.

3.8 March 2

This lecture was recorded.

3.8.1 Definition of the Exponential

For the next couple lectures we will be discussing the very special functions $\exp$ and $\log$. For now, we will focus on $\exp$, defined as follows.

Definition 3.54 ($\exp$)**.** We define the *complex exponential* $\exp\colon \mathbb{C} \to \mathbb{C}$ by the power series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

In particular, we are going to be building our exponentiation from scratch. Nevertheless, we promise that it will work fine.

As such, we have the following checks.

Lemma 3.55. We have that $\exp$ is analytic and entire with derivative $\exp'(z) = \exp(z)$.

Proof. Very quickly, we note that the radius of convergence of $\exp$ is lower-bounded by

$$\left(\lim_{n\to\infty} \sqrt[n]{|1/n!|}\right)^{-1} \ge \left(\lim_{n\to\infty} \sqrt[n]{n^{-n/2}}\right)^{-1} = \left(\lim_{n\to\infty} n^{-1/2}\right)^{-1} = \infty,$$

so our radius of convergence is actually ∞. As such Proposition 3.44 tells us that $\exp$ is holomorphic on $B(0,\infty) = \mathbb{C}$ (i.e., entire) with derivative

$$\exp'(z) = \sum_{k=1}^{\infty} \frac{k}{k!} z^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} z^{k-1} = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

where we have shifted indices in the last step. So indeed, $\exp'(z) = \exp(z)$.

Lastly, to show that $\exp$ is analytic, we need to show that $\exp$ can be locally expanded as a power series. For this, we appeal to the following lemma.

Lemma 3.56. Fix $S(z) := \sum_{k=0}^{\infty} a_k z^k$ a power series with radius of convergence $R > 0$. Then $S(z)$ is analytic on $B(0, R)$.

Proof. There is actually something to show here: given $z_0 \in \mathbb{C}$, we need to expand $S(z)$ locally at a power series at z_0. In particular, we need to be able to write

$$S(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k,$$

where the series on the right converges for any $z \in B(z_0, r)$ for some $r > 0$. For this, we expand

$$S(z + z_0) = \sum_{n=0}^{\infty} a_k (z + z_0)^n,$$

under the assumption $z, z_0, z+z_0, |z|+|z_0| \in B(0, R)$. (We will discuss how to ensure these conditions later.)

The short version of what we are about to do is that we will expand out this power series in terms of z and then collect terms of the same degree. Making this rigorous requires some care to the uniform convergence, but everything is okay because we converge absolutely.

Heuristically, we have

$$\sum_{n=0}^{\infty} a_n (z + z_0)^n = \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} \binom{n}{k} a_n z^k z_0^\ell \right) \stackrel{*}{=} \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{\infty} \binom{n}{k} a_n z_0^\ell \right) z^k,$$

where $\stackrel{*}{=}$ is the equality which requires attention. To rigorize $\stackrel{*}{=}$, we use Lemma 2.81.[2] Indeed, to make the application clearer, we set

$$a_{n,k} := \begin{cases} \binom{n}{k} a_n z^k z_0^{n-k} & k \le n, \\ 0 & k > n \end{cases}$$

so that we are interested in exchanging the order of the summation

$$\sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} \binom{n}{k} a_n z^k z_0^\ell \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}.$$

Well, for fixed n, we see that $\sum_{k=0}^{\infty} |a_{n,k}|$ is a finite sum and hence converges. And further, we see that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{n,k}| = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_n |z|^k |z_0|^{n-k} \right) = \sum_{n=0}^{\infty} a_n (|z| + |z_0|)^n = S(|z| + |z_0|),$$

which converges because $|z| + |z_0| \in B(0, R)$. As such, Lemma 2.81 tells us that

$$S(z + z_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{k,\ell} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{k,\ell} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} a_n z^k z_0^{n-k}.$$

The inner sums we may simplify as $\sum_{n=k}^{\infty} \binom{n}{k} a_n z^k z_0^{n-k} = z^k \sum_{\ell=0}^{\infty} \binom{n}{k} a_n z_0^\ell$, so we do indeed find that

$$S(z + z_0) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{\infty} \binom{n}{k} a_n z_0^\ell \right) z^k,$$

[2] Yes, I, too, am impressed that this lemma is seeing use.

for any $z \in \mathbb{C}$. In particular, plugging in $z - z_0$ tells us that

$$S(z) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{\infty} \binom{n}{k} a_n z_0^{\ell} \right) (z - z_0)^k,$$

which gives us our power series expansion at z_0.

It remains to show the power series expansion will hold in some neighborhood $B(z_0, r)$. Translating back, we need to know that the power series expansion for $S(z + z_0)$ will hold in some neighborhood $S(0, r)$. To review, our hypotheses were that

$$z, z_0, z + z_0, |z| + |z_0| \in B(0, R).$$

Recalling that $z_0 \in B(0, R)$ automatically, we set $r := R - |z_0| > 0$. Then $r < R$, so $z \in B(0, R)$. Similarly,

$$|z + z_0| \leq |z| + |z_0| < r + |z_0| = R,$$

so we get $z + z_0, |z| + |z_0| \in B(0, R)$ as well. So we have constructed our neighborhood and have verified that $S(z)$ is analytic at z_0. ■

Thus, because we defined $\exp$ as a power series with infinite radius of convergence, we see that $\exp$ is analytic everywhere on $\mathbb{C}$. ■

3.8.2 Basic Properties of the Exponential

Now that we know $\exp'(z) = \exp(z)$, we can begin actually building some theory. We pick up the following nice properties of $\exp$.

Proposition 3.57. Fix $z, w \in \mathbb{C}$.

(a) We have that $\exp(z + w) = \exp(z) \exp(w)$.

(b) We have that $\exp(z) \neq 0$.

(c) We have that $\exp(-z) = 1/\exp(z)$.

Proof. Parts (b) and (c) will follow from (a), so we will focus our attention on (a). Fixing some $\alpha \in \mathbb{C}$, the trick is to consider

$$f(z) = \exp(z) \exp(\alpha - z).$$

Observe that $z \mapsto z$ and so $\alpha - z$ are entire, so the chain rule promises each factor of f is entire, so f is entire by the product rule. Tracking all this through, we can compute the derivative as

$$\begin{aligned} f'(z) &= \exp'(z) \exp(\alpha - z) + \exp(z) \exp'(\alpha - z) \cdot (-1) \\ &= \exp(z) \exp(\alpha - z) - \exp(z) \exp(\alpha - z) \\ &= 0. \end{aligned}$$

Thus, f' is constantly 0 everywhere (and $\mathbb{C}$ is connected by Corollary 2.112), so f is constant on $\mathbb{C}$ by Corollary 3.23. However, we can plug in $z = \alpha$ into f to see that

$$f(\alpha) = \exp(\alpha) \cdot \exp(0) = \exp(\alpha),$$

where $\exp(0) = 1$ by construction of $\exp$. In particular, we see that

$$\exp(z) \exp(\alpha - z) = \exp(\alpha)$$

for any $z, \alpha \in \mathbb{C}$. Setting $\alpha := w + z$ recovers $\exp(z + w) = \exp(z) \exp(w)$, which is part (a).

We now show (b) and (c). Setting $z = -w \in \mathbb{C}$ in (a), we see that

$$1 = \exp(0) = \exp(z + -z) = \exp(z)\exp(-z).$$

Thus, because $\mathbb{C}$ is an integral domain, we see that $\exp(z) \neq 0$ automatically, which is (b). So, using the field structure of $\mathbb{C}$ to divide by $\exp(z)$, we conclude that

$$\exp(-z) = 1/\exp(z),$$

which proves (c). ■

Remark 3.58 (Nir). In other words, $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a homomorphism: $\exp$ does map to $\mathbb{C}^\times$ by (c) of the proposition, and $\exp$ satisfies the needed homomorphism property by (a).

In fact, $\exp$ will behave with our complex analytic structure.

Lemma 3.59. Fix any $z \in \mathbb{C}$. Then

$$\overline{\exp(z)} = \exp(\overline{z}).$$

Proof. The main point is that $z \mapsto \overline{z}$ is continuous on $\mathbb{C}$, say by Example 2.123. Thus, we compute

$$\overline{\exp(z)} = \overline{\lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^k}{k!}} \stackrel{*}{=} \lim_{n\to\infty} \overline{\sum_{k=0}^{n} \frac{z^k}{k!}} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\overline{z}^k}{k!} = \exp(\overline{z}),$$

where we have used the continuity of $z \mapsto \overline{z}$ in $\stackrel{*}{=}$. In particular, the point is that the sequence of partial sums $S_n := \sum_{k=0}^{n} \frac{z^k}{k!}$ approach $\exp(z)$, so by continuity, $\overline{S_n}$ (which goes to $\overline{\exp(z)}$ definitionally) must approach $\exp(\overline{z})$. ■

Our next goal is to study certain outputs of $\exp$. Like a good algebraist, we will particularly be interested in the "kernel" of $\exp$ (as a homomorphism). For now, we will avoid saying the word "kernel" and instead simply solve for the output 1.

Lemma 3.60. Fix any $t \in \mathbb{R}$. Then $|\exp(it)| = 1$.

Proof. Note that

$$\overline{\exp(it)} = \exp(\overline{it}) = \exp(-it) = 1/\exp(it),$$

where we have used Lemma 3.59 followed by Proposition 3.57. Thus,

$$|\exp(it)|^2 = \exp(it)\cdot\overline{\exp(it)} = 1,$$

so $|\exp(it)| = 1$ follows because the norm is always a positive real number. ■

In fact, we can do better than the above.

Corollary 3.61. Fix any $z \in \mathbb{C}$. Then $|\exp(z)| = 1$ if and only if $\operatorname{Re}(z) = 0$.

Proof. We show our implications separately.

- Suppose that $\operatorname{Re}(z) = 0$. Then we can write $z = it$ for some $t \in \mathbb{R}$, from which Lemma 3.60 tells us that $|\exp(z)| = |\exp(it)| = 1$ for free.

- Suppose that $|\exp(z)| = 1$. Writing $z = x + yi$ with $x, y \in \mathbb{R}$, we compute

  $$\exp(z) = \exp(x)\exp(iy) = \exp(x),$$

  where we have used Proposition 3.57 and Lemma 3.60. Now, taking norms, we see that $|\exp(x)| = |\exp(z)| = 1$.

  However, $\exp|_\mathbb{R}$ is a strictly increasing function: it is differentiable with continuous nonzero derivative (using Proposition 3.57), so the Intermediate value theorem implies that the derivative must stay the same sign for all $x_0 \in \mathbb{R}$. So noting $\exp(0) = 1$ is enough to conclude $\exp'(x_0) > 0$ for any $x_0 \in \mathbb{R}$, so $\exp$ is strictly increasing from a Mean value theorem argument.[3]

  Thus, if $x < 0$, then $|\exp(x)| = \exp(x) < 1$, and if $0 < x$, then $1 < \exp(x) = |\exp(x)|$. So we see that $x = 0$ with $|\exp(x)| = 1$ is our only way to hit 1, so $\operatorname{Re} z = x = 0$ follows. ■

So far we understand $|\exp(z)|$ pretty well. It is time to turn to $\exp$.

Definition 3.62 (Kernel of $\exp$)**.** We define the *kernel of* $\exp$ as

$$\ker \exp := \{z \in \mathbb{C} : \exp(z) = 1\}.$$

Remark 3.63. This is intended to align with abstract algebra: viewing $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ as a homomorphism, we see that we are asking for the values of $z \in \mathbb{C}$ which go to the identity of $\mathbb{C}^\times$, which is 1.

Example 3.64. We have that $\exp(0) = 1$, so $0 \in \ker \exp$.

To better access the kernel, we will want to talk about the real and imaginary parts of $\exp(it)$.

Definition 3.65 (Sine, cosine)**.** Given $z \in \mathbb{C}$, we define the (complex) $\sin$ and $\cos$ functions as

$$\cos z := \frac{\exp(iz) + \exp(-iz)}{2} \qquad \text{and} \qquad \sin z := \frac{\exp(iz) - \exp(-iz)}{2i}.$$

We can see pretty directly that

$$\cos z + i \sin z = \frac{\exp(iz) + \exp(-iz)}{2} - \frac{\exp(iz) - \exp(-iz)}{2} = \exp(iz).$$

In the case where z is real, we get to say a little more.

Remark 3.66. Using Proposition 2.8 with Lemma 3.59, we see that, for when $t \in \mathbb{R}$,

$$\cos t = \frac{\exp(it) + \exp(-it)}{2} = \frac{\exp(it) + \overline{\exp(it)}}{2} = \operatorname{Re} \exp(it),$$

and

$$\sin t = \frac{\exp(it) - \exp(-it)}{2i} = \frac{\exp(it) - \overline{\exp(it)}}{2i} = \operatorname{Im} \exp(it).$$

In particular $\exp(it) = \cos t + i \sin t$ is our decomposition into real and imaginary parts.

3.8.3 Some Trigonometry

Before we go any further, we do some trigonometry. We want to establish that $\exp(it)$ is periodic, but this requires a little effort; we follow sx63102.

[3] If $a < b$, then use the Mean value theorem to find $x \in (a, b)$ with $f(b) - f(a) = (b - a)f'(x) > 0$, so $f(a) < f(b)$.

Lemma 3.67. For each $z \in \mathbb{C}$, we have $\cos^2 z + \sin^2 z = 1$.

Proof. We directly compute

$$\cos^2 z + \sin^2 z = \frac{\exp(iz)^2 + 2\exp(iz)\exp(-iz) + \exp(-iz)^2}{4} + \frac{\exp(iz)^2 - 2\exp(iz)\exp(-iz) + \exp(-iz)^2}{-4}.$$

After the dust settles, we are left with

$$\cos^2 z + \sin^2 z = \exp(iz)\exp(-iz),$$

which is 1 by Proposition 3.57. ■

More or less by just staring at $\cos$ and $\sin$, we can see that they are entire.

Lemma 3.68. For each $z \in \mathbb{C}$, we have $\frac{d}{dz}\cos z = -\sin z$ and $\frac{d}{dt}\sin z = \cos z$.

Proof. We directly compute

$$\frac{d}{dz}\frac{\exp(iz) + \exp(-iz)}{2} = \frac{i\exp(iz) - i\exp(iz)}{2} = -\frac{\exp(iz) - \exp(-iz)}{2i} = -\sin z,$$

and

$$\frac{d}{dz}\frac{\exp(iz) - \exp(-iz)}{2i} = \frac{i\exp(iz) + i\exp(iz)}{2} = \frac{\exp(iz) + \exp(-iz)}{2} = \cos z,$$

which is what we wanted. ■

Lemma 3.69. For $z \in \mathbb{C}$, we have

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \qquad \text{and} \qquad \sin z = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k+1)!} z^{2k+1}.$$

Proof. We directly compute, for any $z \in \mathbb{C}$, we have

$$\cos z = \frac{1}{2}(\exp(iz) + \exp(-iz)) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^k}{k!} z^k + \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} z^k\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{i^k + (-i)^k}{k!} z^k.$$

Here, we were allowed to merge the two sums because they are just limits which converge. Now, we note that

$$i^k + (-i)^k = \begin{cases} 2 & k \equiv 0 \pmod 4, \\ 0 & k \equiv 1 \pmod 2, \\ -2 & k \equiv 2 \pmod 4, \end{cases}$$

so all the odd terms vanish, leaving us with

$$\cos z = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k},$$

which is what we wanted.

On the other hand, we note that $\cos z$ is an entire function, and its power series will converge everywhere because the power series for $\exp$ also converges everywhere. In particular, Proposition 3.44 tells us that

$$\sin z = -\frac{d}{dz}\cos z = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2k}{(2k)!} z^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} z^{2k-1},$$

which gives the power series for $\sin$ after shifting over our indices. Notably, Proposition 3.44 assures us that this also has infinite radius of convergence. ■

To continue, we have to do a little real analysis.

Lemma 3.70. There exists the smallest positive real number θ such that $\cos\theta = 0$.

Proof. On one hand, note $\cos 0 = 1$. On the other hand, using the Alternating series bound, we note

$$\cos 2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot 2^{2k} \le 1 - \frac{4}{2} + \frac{16}{24} = -\frac{1}{3} < 0.$$

Thus, there certainly exists some $t \in [0,2]$ such that $\cos t = 0$, so we define

$$\theta := \inf\{t > 0 : \cos t = 0\}.$$

Because $\cos$ is continuous, we note that the set $\{t : \cos t = 0\}$ will be closed and hence contain all of its limit points, so we do have $\cos\theta = 0$.

Further, $\cos 0 = 1$ implies there is some δ such that $|t| < \delta$ has $|\cos t - 1| < 1$, meaning there is an open neighborhood around 0 for which $\cos t \ne 0$. In particular, we must have $\theta \ge \delta > 0$, so θ is a positive real number. So lastly, we note that any $t > 0$ for which $\cos t = 0$ must have $t \ge \theta$ by construction, so θ is indeed the smallest positive real number with $\cos\theta = 0$. ■

And now we get to define π.

Definition 3.71 (π)**.** We define $\pi \in \mathbb{R}$ so that $\pi/2$ is the smallest positive real number such that $\cos\pi/2 = 0$.

And now let's show our periodicity.

Lemma 3.72. We have that $\exp(z + 2\pi i) = \exp(z)$ for any $z \in \mathbb{C}$. In fact, 2π is the smallest positive real number θ such that $\exp(i\theta) = 1 = \exp 0$.

Proof. We start with the second sentence. We are given that $\cos\pi/2 = 0$ already, and $\pi/2$ is the smallest such positive real number. From Lemma 3.67, we see that this requires $\sin\pi/2 \in \{\pm 1\}$. However,

$$\frac{d}{dt}\sin t = \cos t$$

must be positive in the interval $(0, \pi/2)$ because $\cos 0 = 1 > 0$ and $\cos$ is nonzero on $(0, \pi/2)$. In particular, a Mean value theorem argument tells us that $\sin$ is strictly increasing on $(0, \pi/2)$, so we have

$$\sin\pi/2 > \sin 0 = 0,$$

so $\sin\pi/2 = 1$. Plugging into Remark 3.66, we get that $\exp(i\pi/2) = i$, so

$$\exp(2\pi i) = \exp(4 \cdot i\pi/2) = \exp(i\pi/2)^4 = i^4 = 1.$$

It remains to show that 2π is the smallest such positive real number. Well, suppose that $\theta > 0$ has $\exp(\theta i) = 1$ and is the smallest such positive real number; we get for free that $\theta \le 2\pi$ by the above. On the other hand, we compute

$$\exp(\theta/4 \cdot i)^4 = \exp(\theta i) = 1,$$

but we can factor $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$, so $\exp(\theta/4 \cdot i) \in \{\pm 1, \pm i\}$. Certainly if $\exp(\theta/4 \cdot i) \in \{\pm 1\}$, then $\exp(\theta/2 \cdot i) = \exp(\theta/4 \cdot i)^2 = 1$, but $\theta/2 < \theta/4$, so this cannot be. So instead, we have that

$$\exp(\theta/4 \cdot i) = \pm i,$$

so in particular, Remark 3.66 tells us that $\cos(\theta/4) = \operatorname{Re}\exp(\theta/4 \cdot i) = 0$. Thus, $\theta/4 \ge \pi/2$ by the definition of π, so $\theta \ge 2\pi$. It follows $\theta = 2\pi$.

We now show the first sentence. By Proposition 3.57, we merely have to compute

$$\exp(z + 2\pi i) = \exp(z)\exp(2\pi i) = \exp z,$$

so we are done. ■

While we're here, we note that also get access to the kernel from our work.

Proposition 3.73. We have that $\ker \exp = \{2\pi i n : n \in \mathbb{Z}\}$.

Proof. In one direction, certainly

$$\exp(2\pi i n) = \exp(2\pi i)^n = 1$$

by Lemma 3.72. In the other direction, suppose $\exp z = 1$. Then Corollary 3.61 forces $\operatorname{Re} z = 0$, so we can write $z = it$. By the division algorithm, we can write

$$t = 2\pi q + r,$$

where $q \in \mathbb{Z}$ and $r \in [0, 2\pi)$, from which we see

$$1 = \exp z = \exp(it) = \exp(2\pi i q + ir) = \exp(2\pi i q)\exp(ir) = \exp(ir).$$

However, $r < 2\pi$ is smaller than the smallest positive real number for which $\exp(ir) = 1$, so r cannot be a positive real number at all. But we do know $r \geq 0$, so $r = 0$ is forced. Thus, $t = 2\pi i q$, as needed. ■

Remark 3.74 (Nir). As a last remark, it would be a crime to note say that $\exp(i\pi) = -1$. Indeed,

$$\exp(i\pi)^2 = \exp(2\pi i) = 1,$$

but we can factor $z^2 - 1 = (z+1)(z-1)$, s $\exp(i\pi) \in \{\pm 1\}$. But $\pi < 2\pi$, so we cannot have $\exp(i\pi) = 1$, so $\exp(i\pi) = -1$ is forced.

3.8.4 Polar Coordinates

We would like to talk about polar coordinates, so for this we would like to access the arctangent function. This requires a little care.

Lemma 3.75. We have that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ for any $z \in \mathbb{C}$.

Proof. This comes down to computing

$$\cos(-z) = \frac{\exp(i(-z)) + \exp(-i(-z))}{2} = \frac{\exp(iz) + \exp(-iz)}{2} = \cos z.$$

Similarly,

$$\sin(-z) == \frac{\exp(i(-z)) - \exp(-i(-z))}{2i} = -\frac{\exp(iz) - \exp(-iz)}{2i} = -\sin z,$$

which is what we wanted. ■

So now we note that $\cos$ is, by definition of $\pi/2$, nonzero on $[0, \pi/2)$. The above lemma lets us extend this nonzero region to $(-\pi/2, \pi/2)$, permitting the following definition.

Definition 3.76. Given a real number $t \in (-\pi/2, \pi/2)$, we define $\tan t := \frac{\sin t}{\cos t}$. Note that this definition is legal because $\cos t \neq 0$ for $(-\pi/2, \pi/2)$.

Lemma 3.77. The function $\tan$ is real differentiable and strictly increasing.

Proof. That $\tan$ is real differentiable follows from the quotient rule, which applies because the denominator $\cos$ is nonzero on all of $(-\pi/2, \pi/2)$.[4] In fact, we can compute the derivative as

$$\frac{d}{dt}\tan t = \frac{d}{dt}\frac{\sin t}{\cos t} = \frac{(\cos t)(\cos t) - (\sin t)(-\sin t)}{(\cos t)^2},$$

where we have used Lemma 3.68. So from Lemma 3.67, we see that $\frac{d}{dt}\tan t = \frac{1}{(\cos t)^2}$, which is positive for real numbers t. Thus, $\tan t$ is in fact strictly increasing. ■

We would like to show that $\tan$ surjects onto $\mathbb{R}$. To start, we note $\tan 0 = \sin 0/\cos 0 = 0/1 = 0$.

Lemma 3.78. For $t \in (-\pi/2, \pi/2)$, we have that $\tan(-z) = -\tan z$.

Proof. By brute force, Lemma 3.75 tells us that

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = \tan t,$$

which is what we wanted. ■

Lemma 3.79. The function $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection.

Proof. We already know that $\tan$ is injective because it is strictly increasing by Lemma 3.77, so we have left to show the surjection. Additionally, Lemma 3.78 implies that we merely have to show that $\tan$ surjects onto $\mathbb{R}_{\geq 0}$, and because $\tan 0 = 0$, we merely have to show that $\tan$ surjects onto $\mathbb{R}^+$.

Now, $\tan$ is continuous (by Lemma 3.77), so the Intermediate value theorem means that we merely need to show $\tan$ takes on arbitrarily large values in $\mathbb{R}^+$. For this, we claim that

$$\lim_{t\to\pi/2} \tan t = \infty,$$

which will be enough. So fix any $M > 0$. Well, because $\cos$ is continuous, we see that

$$\lim_{t\to\pi/2} \cos t = \cos \pi/2 = 0.$$

Thus, for $\varepsilon = 1/(2M)$, there exists some $\delta_1 > 0$ so that $\pi/2 - \delta_1 < t < \pi/2$ will have $\cos t < \varepsilon$. Because $\cos$ must be positive for $t < \pi/2$, we actually have $0 < \cos t < \varepsilon$. Additionally, because $\sin$ is continuous, we see that

$$\lim_{t\to\pi/2} \sin t = \sin \pi/2 = 1.$$

Thus, there exists some $\delta_2 > 0$ so that $\pi/2 - \delta_2 < t < \pi/2$ will have $\sin t > 1/2$. In particular, setting $\delta := \min\{\delta_1, \delta_2\}$, we see $\pi/2 - \delta < t < \pi/2$ implies that

$$\tan t = \frac{\sin t}{\cos t} > \frac{1/2}{\varepsilon} = \frac{1}{2\varepsilon} = M.$$

This finishes. ■

The above check permits the following definition.

[4] Technically, we should extend $\tan$ to a small open strip around $(-\pi/2, \pi/2)$ in order to make the complex quotient rule work and then restrict $\tan$ afterwards. We will settle for merely saying that we should do this instead of actually doing it.

Definition 3.80 (Arctangent)**.** We define $\arctan\colon \mathbb{R} \to (-\pi/2, \pi/2)$ to be the inverse function of $\tan$.

Note that the above definition makes sense because $\tan$ is a bijection $(-\pi/2, \pi/2) \to \mathbb{R}$. In fact, the proof of Lemma 3.79 lets us say

$$\lim_{t\to\infty} \arctan t = \frac{\pi}{2}.$$

In fact, we see $\tan(-t) \to -\pi/2$ as $t \to \infty$, so

$$\lim_{t\to-\infty} \arctan t = -\frac{\pi}{2}.$$

We are now ready to give polar form.

Remark 3.81. Very quickly, we note that $\arctan$ is a continuous function. This is true because it is strictly increasing (it is the inverse function of the strictly increasing function $\tan$) and it satisfies the intermediate value property ($\arctan$ is in fact bijective because it is an inverse function).

Proposition 3.82 (Polar form)**.** For any $z \in \mathbb{C}^\times$, there exist unique real numbers $r > 0$ and $\theta \in [-\pi, \pi)$ such that $z = r\exp(i\theta)$.

Proof. We start by showing uniqueness because it is easier: if $r_1 \exp(i\theta_1) = r_2 \exp(i\theta_2)$, then taking magnitudes tells us that

$$|r_1| = |r_1 \exp(i\theta_1)| = |r_2 \exp(i\theta_2)| = |r_2|,$$

where we have used Corollary 3.61. Because r_1 and r_2 are positive real numbers, we conclude $r_1 = r_2$. So now

$$\exp(i(\theta_1 - \theta_2)) = \exp(i\theta_1)/\exp(i\theta_2) = 1$$

using Proposition 3.57. By Proposition 3.73, this forces $\theta_1 - \theta_2 \in 2\pi i\mathbb{Z}$. However, $-\pi \le \theta_1, \theta_2 < \pi$ implies that

$$-2\pi < \theta_1 - \theta_2 < 2\pi,$$

so $\theta_1 - \theta_2 = 0$ is forced, so $\theta_1 = \theta_2$.

We now show that the r and θ actually exist for any $z \in \mathbb{C}^\times$. As above, we take $r = |z|$, so we need to set θ. Well, we see that Remark 3.66 gives

$$r\exp(i\theta) = r\cos\theta + ir\sin\theta.$$

So we want a value $\theta \in [-\pi, \pi)$ such that $\operatorname{Re} z = r\cos\theta$ and $\operatorname{Im} z = r\sin\theta$. Noting that $z \ne 0$ implies $r \ne 0$, we want to choose θ such that

$$(\cos\theta, \sin\theta) \stackrel{?}{=} (\operatorname{Re} z/r, \operatorname{Im} z/r).$$

In particular, we set $a := \operatorname{Re} z/r$ and $b := \operatorname{Im} z/r$ so that $a^2 + b^2 = \frac{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}{r^2} 1$. So, given $(a, b) \in \mathbb{R}^2$ such that $a^2 + b^2 = 1$, we need to find θ such that

$$(\cos\theta, \sin\theta) \stackrel{?}{=} (a, b).$$

We set θ by hand. We do casework.

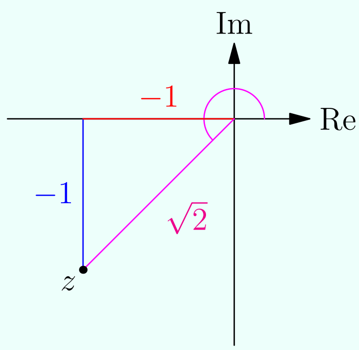
- If $a = 0$, then $\cos\theta = 0$ and $b = \pm 1$. Well, for $b = \pm 1$, we set $\theta = \pm\frac{\pi}{2}$ so that $\cos \pm\frac{\pi}{2} = \cos\frac{\pi}{2}$ and $\sin \pm\frac{\pi}{2} = \pm\sin\frac{\pi}{2} = \pm 1$ by Lemma 3.75.
- If $a > 0$, then we choose $\theta = \arctan(b/a) \in (-\pi/2, \pi/2)$. In particular, we see that $\tan\theta = \frac{b}{a}$, so we have the system of equations

$$\frac{\sin\theta}{\cos\theta} = \frac{b}{a} \qquad \text{and} \qquad (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

In particular, $\sin\theta = \frac{b}{a}\cos\theta$, so $\left(1 + \frac{b^2}{a^2}\right)(\cos\theta)^2 = 1$, so $\cos\theta = \pm a$ because $a^2 + b^2$. But $\cos\theta$ is positive on $(-\pi/2, \pi/2)$, so we see that $\cos\theta = a$, from which we can read $\sin\theta = \frac{b}{a} \cdot a = b$.

- If $a < 0$, we note $-a > 0$, so we use the above argument to choose $\gamma = \arctan(b/-a) \in (-\pi/2, \pi/2)$ so that
$$\cos\gamma = -a \qquad \text{and} \qquad \sin\gamma = b.$$
In particular, we see that $-\gamma$ has
$$\exp(i(-\gamma)) = \overline{\exp(i\gamma)} = \overline{-a+bi} = -a-bi.$$
In particular, multiplying this through by $\exp(i\pi) = -1$, we see that $\exp(i(\pi-\gamma)) = a+bi$, giving $\cos(\pi-\gamma) = a$ and $\sin(\pi-\gamma) = b$.
It remains to force $\pi-\gamma$ into $[-\pi,\pi)$. However, $\exp(it)$ is periodic with period 2π, so we can callously shift $2\pi-\gamma$ into $[0,2\pi)$ via the division algorithm and then subtract γ to get a representative of $\pi-\gamma$ in $[-\pi,\pi)$. This finishes. ■

Example 3.83. We take $z = -1-i$. Here, $|z| = \sqrt{1+1} = \sqrt{2}$; further $\operatorname{Re} z < 0$, so we compute
$$\pi - \arctan(-1/-(-1)) = \pi - -\frac{\pi}{4} = \frac{5\pi}{4},$$
so we take $\theta = -3\pi/4$ after shifting. So the above argument assures us that $z = \sqrt{2}\exp(-i\cdot 3\pi/4)$. Here is the image.



3.9 March 4

Good morning everyone. Today's lecture was not recorded.

- Homework #5 will be uploaded today, due next Friday.
- The class average on the midterm was a 74; it might have been a little long. There will probably be something approximately equal to a 6-point curve.

Before continuing, we make some remarks, as a review from real analysis.

Remark 3.84 (Nir). Today, we will want to pick up some properties of the real logarithm. We define $\log\colon \mathbb{R}^+ \to \mathbb{R}$ as the inverse of $\exp\colon \mathbb{R} \to \mathbb{R}^+$, for which we need to know $\exp\colon \mathbb{R} \to \mathbb{R}^+$ is a bijection.

- Note $\exp r > 0$ for $r \ge 0$ by the power series, and $\exp(r) = 1/\exp(-r) > 0$ for $r < 0$. Thus, $\exp'(r) = \exp(r) > 0$ everywhere, so $\exp$ is strictly increasing and therefore injective.
- For surjective, by continuity and $\exp(-r) = 1/\exp(r)$, we need $\exp r \to \infty$ as $r \to \infty$, for which we note $\exp(1) > \exp(0) = 1$ gives $\exp(1) > 1+\varepsilon$ for some $\varepsilon > 0$, so $\exp(n) > (1+\varepsilon)^n > 1+n\varepsilon$.

Remark 3.85 (Nir)**.** We will also want to know that $\log\colon \mathbb{R}^+ \to \mathbb{R}$ is continuous. Well, $x < y$ if and only if $\exp(x) < \exp(y)$ implies that $x < y$ requires $\log x < \log y$, so $\log$ is strictly increasing. Thus, it suffices to show that $\log$ satisfies the intermediate value property, but $\log$ is surjective (it's the inverse function of a bijection and hence a bijection), so we are done.

3.9.1 Arguments

Today we talk more about the exponential function. Last time we proved the following.

Proposition 3.82 (Polar form)**.** For any $z \in \mathbb{C}^\times$, there exist unique real numbers $r > 0$ and $\theta \in [-\pi, \pi)$ such that $z = r\exp(i\theta)$.

As a brief review, we recall that we took $r = |z|$, and we computed θ in terms of some $\arctan$s. Essentially, this means that we can effectively compute polar form without tears.

Remark 3.86. The interval $[-\pi, \pi)$ is somewhat arbitrary; we can choose any set of representatives for $\mathbb{R}/2\pi\mathbb{Z}$. To see this, we note that the unique $\theta \in [-\pi, \pi)$ will have a unique representative in any set of representatives for $\mathbb{R}/2\pi\mathbb{Z}$ and vice versa. For example, any half-open interval of length 2π (such as $[0, 2\pi)$) will do the trick. To see this,

We can in fact use polar form to talk about the exponential map.

Corollary 3.87. For any $z \in \mathbb{C}^\times$, there exists some $w \in \mathbb{C}$ such that $\exp(w) = z$.

Proof. To start, we know that we can write $z = r\exp(i\theta)$ by Proposition 3.82. So, using real analysis, we set

$$w := \log r + i\theta,$$

where $\log\colon \mathbb{R}^+ \to \mathbb{R}$ is the real logarithm. Thus,

$$z = r\exp(i\theta) = \exp(\log r)\exp(i\theta) = \exp(\log r + i\theta) = \exp(w),$$

which is what we wanted. ■

Continuing to talk about polar form, we have the following definition.

Definition 3.88 (Argument)**.** Given a complex number $z \in \mathbb{C}^\times$, we define the *principal argument* $\arg z \in [-\pi, \pi)$ by writing $z := |z|\exp(i\theta)$ (using Proposition 3.82) and taking $\arg z := \theta$. More generally, for any $\eta \in \mathbb{R}$, we define

$$\arg_\eta\colon \mathbb{C}^\times \to [\eta, \eta + 2\pi)$$

by $\arg_\eta(z) := \arg z + \pi + \eta$.

Remark 3.89 (Nir)**.** By definition, we see that $|z|\exp(i\arg z) = z$.

Example 3.90. We have that $\arg_{-\pi} = \arg$.

3.9.2 Branches of the Complex Logarithm

The logarithm is somewhat subtle, so we have to be careful. We take the following definition.

Definition 3.91 (Branch of the logarithm). Fix $\Omega \subseteq \mathbb{C} \setminus \{0\}$ an open, connected subset. A *branch of the logarithm* is a continuous function $f \colon \Omega \to \mathbb{C}$ such that

$$\exp(f(z)) = z.$$

Intuitively, f will "look like" an inverse for $\exp$.

Nevertheless, there is a fairly standard choice of branch.

Definition 3.92 (Log). Taking $\Omega := \mathbb{C} \setminus \mathbb{R}_{\le 0}$, we define the *principal branch of the logarithm* as $\operatorname{Log} \colon \Omega \to \mathbb{C}$ by

$$z \mapsto \log|z| + i \arg z.$$

Remark 3.93. It is not too hard to check that $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ is connected. Indeed, it is path-connected: for any $a + bi \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$, we define $\gamma \colon [0,1] \to \mathbb{C} \setminus \mathbb{R}_{\le 0}$ by

$$\gamma(t) := (1-t)(a+bi) + t.$$

Notably, $\operatorname{Im}\gamma(t) = (1-t)b$, so $\gamma(t) \in \mathbb{R}_{\le 0}$ would imply that $\operatorname{Im}\gamma(t) = 0$ so that $t = 1$ or $b = 0$. We cannot have $t = 0$ because $\gamma(1) = 1$; we cannot have $b = 0$ because $b = 0$ requires $a > 0$, so $\operatorname{Re}\gamma(t) = (1-t)a + t > 0$ always.

Remark 3.94. We can check directly that $\exp \operatorname{Log} z = z$ for $z \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$. In particular, Remark 3.89 lets us write

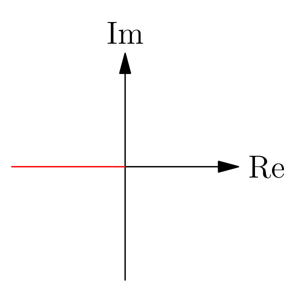
$$\exp \operatorname{Log} z = \exp(\log|z| + i \arg z) = \exp(\log|z|)\exp(i \arg z) = |z| \exp(i \arg z) = z.$$

We will check that Log is actually a continuous later, in Corollary 3.98.

Remark 3.95. Again, $\log \colon \mathbb{R}^+ \to \mathbb{R}$ here is the real logarithm, which is legal because $z \ne 0$ so that $|z| > 0$.

In particular, we are essentially using the construction from back in Corollary 3.87.

As some brief geometric commentary, we are calling these "branches" our open sets Ω are typically $\mathbb{C}$ minus a single line, and the subtlety of why we have to do this is to make the logarithm continuous. For example, in the principal branch, we deleted $\mathbb{R}_{\le 0}$, which has the following image.



We should probably check that Log is actually well-formed; namely, it turns out that we had some choice in our construction of Log.

Lemma 3.96. Fix $z, w \in \mathbb{C}$ such that $\exp(z) \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$ and $\operatorname{Log}\exp(z) = w$. Then there is a $k \in \mathbb{Z}$ such that $z = w + 2\pi i k$.

Proof. Write $z = x+iy$ so that $\exp z = \exp(x)\exp(iy)$. Now, we know that $\exp(\alpha) = 0$ if and only if $\alpha \in 2\pi i\mathbb{Z}$, so for example, we can write

$$\exp(yi + 2\pi in) = \exp(iy)$$

for any $n \in \mathbb{Z}$. So, by the division algorithm, we choose a $k \in \mathbb{Z}$ so that

$$\widetilde{y} := y + 2\pi k$$

has $\widetilde{y} \in [-\pi, \pi)$. But now, because $\exp(z) \notin \mathbb{R}_{\ge 0}$, we see that we cannot have $\widetilde{y} = -\pi$ because this would make $\exp(iy) = -1$ and therefore $\exp z = -\exp(x) \in \mathbb{R}_{\le 0}$.

The point of choosing this $\widetilde{y}$ is that we still have $\exp(z) = \exp(x)\exp(iy) = \exp(x) = \exp(i\widetilde{y})$, but now $\widetilde{y} \in (-\pi, \pi)$, so we are assured

$$\arg\exp(z) = \widetilde{y}.$$

At this point, we just write out

$$w = \operatorname{Log}\exp(z) = \operatorname{Log}\exp(x+iy) = \log(|\exp(x)\exp(it)|) + i\arg\exp(z) = x + i\widetilde{y}.$$

So now we can write $w = x + iy - 2\pi ik$, which is what we wanted. ■

Let's return to our discussion of branches. There are a few reasons why we want "branches" for Log. Roughly speaking, here is the reasoning.

- The function exp is not injective: it has kernel $\ker\exp = 2\pi i\mathbb{Z}$. In particular, if we wanted to define Log on $1 \in \mathbb{C}$, then we need to make a choice among the representatives in $2\pi i\mathbb{Z}$.
- In order to avoid having to make a choice, we chose Log to have imaginary part in $[-\pi, \pi)$ always (in fact, $-\pi$ is illegal because Log doesn't take inputs in $\mathbb{R}_{\le 0}$).
- But making this choice makes Log not continuous at values in $\mathbb{R}_{\le 0}$ because (notably!) $\arg z$ is not continuous on $\mathbb{R}_{\le 0}$. In particular, $z \to -1$ from above gives $\arg z \to \pi$ while $z \to -1$ from below gives $\arg z \to -\pi$.
- So the point of introducing the branch is to simply throw out the $\mathbb{R}_{\le 0}$ and recover our continuity.

3.9.3 The Principal Branch

We now finish the checks that Log is actually a branch of the logarithm. For this, it remains to check that Log is continuous; in fact, we will extend and show that Log is holomorphic. As discussed when we were talking about branches, the issue with extending the continuity of Log to all of $\mathbb{C}$ is arg, so we pay arg some special attention.

Lemma 3.97. The restricted argument function $\arg\colon \mathbb{C} \setminus \mathbb{R}_{\le 0} \to [-\pi, \pi)$ is continuous.

Proof. Fix some $z \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$, and we show that $\arg z$ is continuous at z. We do casework because we have to back-track through the definition of arg and therefore back through Proposition 3.82

- Suppose $\operatorname{Re} z > 0$. Then it suffices to show that arg is continuous on $B(z, \operatorname{Re} z) \subseteq \{w : \operatorname{Re} w > 0\}$. Well, on this region we defined $\arg w$ by

$$\arg w = \arctan\left(\frac{\operatorname{Im} w}{\operatorname{Re} w}\right).$$

On $\{w : \operatorname{Re} w > 0\}$, we see that $\operatorname{Re} w \ne 0$, so the continuity of Re and Im promise that $\operatorname{Im} w / \operatorname{Re} w$ is continuous. So because arctan is continuous, we conclude $\arctan(\operatorname{Im} w / \operatorname{Re} w)$ is continuous at z.

- Suppose $\operatorname{Re} z < 0$ and $\operatorname{Im} z > 0$. Then it suffices to show that $\arg$ is continuous on

$$B(z, \min\{-\operatorname{Re} z, \operatorname{Im} z\}) \subseteq \{w : \operatorname{Re} w < 0, \operatorname{Im} z > 0\}.$$

  Here, we defined $\arg z$ by shifting $\pi - \arctan(\operatorname{Im} w / -\operatorname{Re} w)$ into $[-\pi, \pi)$. But now, $\operatorname{Im} w / \operatorname{Re} w > 0$, so $\arctan(\operatorname{Im} w / -\operatorname{Re} w) \in (0, \pi/2)$, so

$$\arg w = \pi - \arctan(\operatorname{Im} w / -\operatorname{Re} w) \in [-\pi, \pi).$$

  The function $\arctan(\operatorname{Im} w / \operatorname{Re} w)$ is continuous for the same reasons as before, so the total function is continuous at z.

- Suppose $\operatorname{Re} z < 0$ and $\operatorname{Im} z < 0$. Then it suffices to show that $\arg$ is continuous on

$$B(z, \min\{-\operatorname{Re} z, -\operatorname{Im} z\}) \subseteq \{w : \operatorname{Re} w < 0, \operatorname{Im} z < 0\}.$$

  On this region, we defined $\arg z$ by shifting $\pi - \arctan(\operatorname{Im} w / -\operatorname{Re} w)$ into $[-\pi, \pi)$. However, $\operatorname{Im} w / -\operatorname{Re} w < 0$, so $\arctan(\operatorname{Im} w / -\operatorname{Re} w) \in (-\pi/2, 0)$, so

$$\arg w = -\pi - \arctan(\operatorname{Im} w / -\operatorname{Re} w) \in [-\pi, \pi).$$

  The function $\arctan(\operatorname{Im} w / \operatorname{Re} w)$ is continuous for the same reasons as before, so the total function is continuous at z.

- Suppose $\operatorname{Re} z = 0$ and $\operatorname{Im} z > 0$. Then we defined $\arg z = \frac{\pi}{2}$. To check continuity here, we note that it suffices to look in the ball $B(0, \operatorname{Im} z) \subseteq \{w : \operatorname{Im} w > 0\}$. Then

$$\lim_{\substack{w \to z \\ \operatorname{Re} w > 0, \operatorname{Im} w > 0}} \arg w = \lim_{\substack{w \to z \\ \operatorname{Re} w > 0, \operatorname{Im} w > 0}} \arctan\left(\frac{\operatorname{Im} w}{\operatorname{Re} w}\right) = \lim_{x \to \infty} \arctan x = \frac{\pi}{2}$$

  while

$$\lim_{\substack{w \to z \\ \operatorname{Re} w < 0, \operatorname{Im} w > 0}} \arg w = \lim_{\substack{w \to z \\ \operatorname{Re} w < 0, \operatorname{Im} w > 0}} \pi - \arctan\left(\frac{\operatorname{Im} w}{-\operatorname{Re} w}\right) = \pi - \lim_{x \to \infty} \arctan x = \pi - \frac{\pi}{2} = \frac{\pi}{2},$$

  which both match $\arg z$. So, fixing some $\varepsilon > 0$, we can use the two limits above to find suitable δ_1, δ_2 in each region, and then we take $\delta := \min\{\delta_1, \delta_2\}$.

- Suppose $\operatorname{Re} z = 0$ and $\operatorname{Im} z < 0$. We repeat the previous argument. Then we defined $\arg z = -\frac{\pi}{2}$. To check continuity here, we note that it suffices to look in the ball $B(0, \operatorname{Im} z) \subseteq \{w : \operatorname{Im} w < 0\}$. Then

$$\lim_{\substack{w \to z \\ \operatorname{Re} w > 0, \operatorname{Im} w < 0}} \arg w = \lim_{\substack{w \to z \\ \operatorname{Re} w > 0, \operatorname{Im} w > 0}} \arctan\left(\frac{\operatorname{Im} w}{\operatorname{Re} w}\right) = \lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

  while

$$\lim_{\substack{w \to z \\ \operatorname{Re} w < 0, \operatorname{Im} w < 0}} \arg w = \lim_{\substack{w \to z \\ \operatorname{Re} w < 0, \operatorname{Im} w < 0}} -\pi - \arctan\left(\frac{\operatorname{Im} w}{-\operatorname{Re} w}\right) = -\pi - \lim_{x \to -\infty} \arctan x = -\pi + \frac{\pi}{2} = -\frac{\pi}{2},$$

  which both match with $\arg z$. So, fixing some $\varepsilon > 0$, we can use the two limits above to find suitable δ_1, δ_2 in each region, and then we take $\delta := \min\{\delta_1, \delta_2\}$.

The above casework finishes the proof. ■

Corollary 3.98. The function $\operatorname{Log}\colon \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$ is continuous.

Proof. Well, we write

$$\operatorname{Log} z = \log|z| + i\arg z,$$

and we now know that each component is continuous, so the total function is continuous. To be explicit, the function $\log|z|$ is the composite of two continuous functions and is therefore continuous; the function $\arg z$ is continuous by the previous lemma. So we may finish by Proposition 2.95. ■

In fact, we get that Log is holomorphic, essentially inherited from $\exp$.

Lemma 3.99. Fix $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ connected and open subsets. Further, suppose we have a continuous function $f\colon \Omega_1 \to \Omega_2$ and a holomorphic function $g\colon \Omega_2 \to \Omega_1$ such that $g(f(z)) = z$ and $g'(z) \neq 0$ for each $z \in \Omega_1$. Then f is holomorphic on Ω_1 with derivative

$$f'(z) = \frac{1}{g'(f(z))}.$$

Proof. We quickly observe that f is injective: if $z, w \in \Omega_1$ have $f(z) = f(w)$, then $z = g(f(z)) = g(f(w)) = w$. Now, the trick is that, for distinct $z, w \in \Omega_1$, we may write

$$\frac{g(f(z)) - g(f(w))}{z - w} = \frac{g(f(z)) - g(f(w))}{f(z) - f(w)} \cdot \frac{f(z) - f(w)}{z - w}.$$

In particular, note $z \neq w$ implies $f(z) \neq f(w)$ because f is injective. We see that the left-hand side is merely 1 because $g \circ f = \operatorname{id}_{\Omega_1}$. In particular, we may write

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{z \to w} \frac{1}{\frac{g(f(z)) - g(f(z))}{f(z) - f(w)}}.$$

Notably, the denominator here is legal because $z \neq w$ implies $f(z) \neq f(w)$ and $g(f(z)) \neq g(f(w))$.

To finish, imagine some sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq \Omega_1 \setminus \{w\}$ such that $z_n \to w$. By continuity of f, we see that $f(z_n) \to f(w)$. However, we know that

$$\lim_{z' \to f(w)} \frac{g(z') - g(f(w))}{z' - f(w)} = g'(f(w)),$$

so $f(z_n) \to f(w)$ tells us that $\frac{g(f(z_n)) - g(f(w))}{f(z_n) - f(w)} \to g'(f(z))$. Because our sequence $\{z_n\}_{n \in \mathbb{N}}$ was arbitrary, we may conclude

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{z \to w} \frac{1}{\frac{g(f(z)) - g(f(z))}{f(z) - f(w)}} = \frac{1}{\lim_{z \to w} \frac{g(f(z)) - g(f(z))}{f(z) - f(w)}} = \frac{1}{g'(f(z))}.$$

This finishes. ■

Proposition 3.100. The function Log is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with derivative

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}.$$

Proof. We simply apply Lemma 3.99 with $\Omega_1 = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ and $\Omega_2 = \mathbb{C}$ and $f = \operatorname{Log}$ and $g = \exp$. We quickly check the hypotheses.

- Note Ω_2 is connected and open, as discussed before.

- Note Ω_1 is connected by Remark 3.93 and open because
$$\Omega_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cup \{z \in \mathbb{C} : \operatorname{Im} z \neq\} = \operatorname{Re}^{-1}(\mathbb{R}_{>0}) \cup \operatorname{Im}^{-1}(\mathbb{R} \setminus \{0\})$$
is the union of two open sets by the continuity of Re and Im.
- The function f is continuous by Corollary 3.98.
- The function g is holomorphic on Ω_2 by Lemma 3.55.
- We have $g(f(z)) = z$, essentially by construction; see Remark 3.94.
- The function $g' = \exp$ is nonzero everywhere on Ω_2 because $\exp(z)\exp(-z) = 1$ for $z \neq 0$.

Now, applying Lemma 3.99, we see that
$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{\exp'(\operatorname{Log} z)} = \frac{1}{\exp(\operatorname{Log} z)} = \frac{1}{z},$$
where we have used the facts that $\exp' = \exp$ by Lemma 3.55 and that $\exp(\operatorname{Log} z) = z$ as shown above. ■

THEME 4

INTEGRATION

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

4.1 March 7

Good morning everyone.

4.1.1 Smooth Paths

Today we are going to build some theory of paths. We recall the definition.

Definition 2.99 (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ where $a < b$ are real numbers.

Now that we have access to some differentiation, we can talk about the smoothness of our paths.

Definition 4.1 (Differentiable for paths)**.** Fix $[a,b] \subseteq \mathbb{R}$. A path $\gamma\colon [a,b] \to \mathbb{C}$ is *differentiable* if and only if the limit

$$\lim_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0}$$

exists. If the limit exists, we set it equal to $\gamma'(t)$ and call it the *derivative*. Further, γ is *differentiable* if and only if γ is differentiable at all points $t \in [a,b]$.

Remark 4.2. When computing $\gamma'(a)$ and $\gamma'(b)$, the above limit is one-sided.

There are still going to be some pathological paths that are differentiable, so we add more smoothness conditions.

Definition 4.3 (C^1)**.** Fix $[a,b] \subseteq \mathbb{R}$. A path $\gamma\colon [a,b] \to \mathbb{C}$ is C^1 or *smooth* if and only if γ is differentiable and γ' is continuous.

This is perhaps a little too strong, but it is the correct notion. Here is a slightly weaker version.

Definition 4.4 (Piecewise C^1). Fix $[a,b] \subseteq \mathbb{R}$. A path $\gamma\colon [a,b] \to \mathbb{C}$ is *piecewise* C^1 if and only if there exists a sequence $\{a_k\}_{k=0}^n$ with $a_0 = a$ and $a_n = b$ such that

$$\gamma|_{[a_k, a_{k+1}]}$$

is C^1 for each $0 \le k < n$.

The point is that we are going to want to glue C^1 paths together in the future, and the resulting path need not be C^1.

This is a math class, so we should probably prove something today, so have some lemmas.

Lemma 4.5. Fix $s\colon [a,b] \to [c,d]$ a function differentiable at $t_0 \in [a,b]$. Further, if $\gamma\colon [c,d] \to \mathbb{C}$ is differentiable at $\gamma(t_0)$, then $\gamma \circ s$ is differentiable at t_0 with

$$(\gamma \circ s)'(t_0) = \gamma'(s(t_0))s'(t_0).$$

Proof. As usual with compositions, we consider the difference quotient $v\colon [c,d] \to \mathbb{C}$ defined as

$$v(x) := \begin{cases} \frac{\gamma(x)-\gamma(s(t_0))}{x-s(t_0)} - \gamma'(s(t_0)) & t \neq s(t_0), \\ 0 & x = s(t_0). \end{cases}$$

Now, by definition of the differentiability of γ at $s(t_0)$, we know that $v(x) \to 0$ as $x \to s(t_0)$. Rearranging, we see that

$$\gamma(x) - \gamma(s(t_0)) = (x - s(t_0)) \cdot \big(\gamma'(s(t_0)) + v(x)\big)$$

for all $x \in [c,d]$, so plugging in $s(t) \in [c,d]$, we see that

$$\gamma(s(t)) - \gamma(s(t_0)) = (s(t) - s(t_0)) \cdot \big(\gamma'(s(t_0)) + v(s(t))\big).$$

So now we rearrange backwards to see

$$\frac{\gamma(s(t)) - \gamma(s(t_0))}{t - t_0} = \frac{s(t) - s(t_0)}{t - t_0} \cdot \big(\gamma'(s(t_0)) + v(s(t))\big).$$

Upon taking the limit as $t \to t_0$, the differentiability of s at t_0 assures us that

$$\lim_{t \to t_0} \frac{\gamma(s(t)) - \gamma(s(t_0))}{t - t_0} = \left(\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}\right)\left(\lim_{t \to t_0} \gamma'(s(t_0)) + v(s(t))\right) = s'(t_0)\gamma'(s(t_0)).$$

Notably, we are using the fact that $v \circ s$ is continuous at t_0 because s is continuous and v is continuous at $s(t_0)$. ■

Lemma 4.6. Fix $\gamma\colon [a,b] \to \mathbb{C}$ a path differentiable at $c \in (a,b)$. Then $\overline{\gamma}$ is differentiable on $\mathbb{C}$ with derivative $\overline{\gamma'(c)}$.

Proof. Note that the function $z \mapsto \overline{z}$ is continuous, so we compute

$$\lim_{t \to c} \frac{\overline{\gamma}(t) - \overline{\gamma}(c)}{t - c} = \lim_{t \to c} \overline{\left(\frac{\gamma(t) - \gamma(c)}{t - c}\right)} = \overline{\lim_{t \to c} \frac{\gamma(t) - \gamma(c)}{t - c}} = \overline{\gamma'(c)},$$

which is what we wanted. ■

Remark 4.7. Lemma 4.6 might seem surprising because conjugation itself is usually not complex differentiable. However, this is okay because we are only really taking limits in $\mathbb{R}$, so the extra dimension of $\mathbb{C}$ does not impede us.

As a side remark, we note the following: we can approximate any path reasonably well.

Theorem 4.8. For any path $\gamma\colon [a,b] \to \mathbb{C}$, there exists a sequence of piecewise C^1 paths $\{\gamma_k\}_{k\in\mathbb{N}}$ such that $\gamma_k \to \gamma$ uniformly.

Proof. The main point is to use the Stone–Weierstrass theorem. We will not prove this in class, for it would sidetrack us somewhat significantly. ■

The reason why we bring up the above result is that we can, roughly speaking, understand paths (and integration on paths) by reducing them to piecewise C^1 paths and then studying the C^1 paths individually.

4.1.2 Reparameterization

We are going to want to adjust our "speed" along a path, for which we have the following definition.

Definition 4.9 (Reparameterization)**.** Fix $s\colon [a,b] \to [c,d]$ a continuously differentiable function with $s(a) = c$ and $s(b) = d$. Then, given a path $\gamma\colon [c,d] \to \mathbb{C}$ be a C^1 path. Then the path

$$\widetilde{\gamma} := \gamma \circ s\colon [a,b] \to \mathbb{C}$$

is again a C^1 path. We call $\widetilde{\gamma}$ a *reparameterization* of γ.

Remark 4.10. We can also check that, in the context of the above definition, $\operatorname{im}\gamma = \operatorname{im}\widetilde{\gamma}$. Indeed, it suffices to show that φ is surjective, for which we note $\gamma(a) = c$ and $\gamma(b) = d$ gives us surjectivity onto $[c,d]$ by the Intermediate value theorem.

Reparameterization allows us a notion of equivalence.

Definition 4.11 (Equivalent)**.** Two paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are *equivalent* if and only if there is a continuously differentiable, bijective function $s\colon [a,b] \to [c,d]$ such that $s' > 0$ and $\gamma_1 = \gamma_2 \circ s$. We denote this by $\gamma_1 \sim_e \gamma_2$

One can check that $\sim_e$ defined above is an equivalence relation.

Lemma 4.12. The relation $\sim_e$ defined on paths is an equivalence relation.

Proof. We have the following checks.

- Reflexive: given a path $\gamma\colon [a,b] \to \mathbb{C}$, we show $\gamma \sim_e \gamma$. Indeed, the function $s\colon [a,b] \to [a,b]$ defined by $s(x) := x$ is bijective and continuously differentiable (with constant derivative 1) and $\gamma(t) = \gamma(s(t))$. So s witnesses $\gamma \sim_e \gamma$.
- Symmetric: fix paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ with $\gamma_1 \sim_e \gamma_2$; i.e., we are given a bijective and continuously differentiable $s\colon [a,b] \to [c,d]$ such that $\gamma_1 = \gamma_2 \circ s$.

  Because s is bijective, it has an inverse function $r\colon [c,d] \to [a,b]$, which we can check is also continuously differentiable by real analysis; the idea is to copy the proof of Lemma 3.99 to show that $r'(t) = \frac{1}{s'(r(t))}$ which is continuously differentiable (using the condition $s' > 0$).

  Thus, $\gamma_1 = \gamma_2 \circ s$ implies $\gamma_2 = \gamma_1 \circ r$, so $\gamma_2 \sim_e \gamma_1$.

- Transitive: fix paths $\gamma_1\colon [a,b]\to\mathbb{C}$ and $\gamma_2\colon [c,d]\to\mathbb{C}$ and $\gamma_3\colon [e,f]\to\mathbb{C}$ such that $\gamma_1\sim_e\gamma_2$ and $\gamma_2\sim_e\gamma_3$; i.e., we have bijective, continuously differentiable functions $r\colon [a,b]\to[c,d]$ and $s\colon [c,d]\to[e,f]$ such that $r',s'>0$ and
$$\gamma_1=\gamma_2\circ s\qquad\text{and}\qquad\gamma_2=\gamma_3\circ r.$$
But then we see $r\circ s$ is bijective and continuously differentiable (by the chain rule) with $(r\circ s)'>0$, so $\gamma_1=\gamma_3\circ(r\circ s)$ witnesses $\gamma_1\sim_e\gamma_3$.

The above sketchy checks finish the proof. ■

Definition 4.13 (Oriented curve)**.** An equivalence class $[\gamma]_e$ of paths is an *oriented curve*.

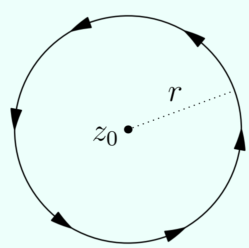
Here are two basic curves.

Example 4.14. Given $z_0\in\mathbb{C}$, the set of constant paths $\gamma\colon [a,b]\to\mathbb{C}$ by $\gamma\equiv z_0$ is an oriented curve.

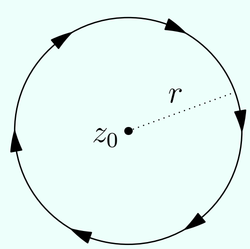
Example 4.15. Given $\alpha,\beta\in\mathbb{C}$, we define the line segment $\gamma\colon [0,1]\to\mathbb{C}$ by
$$\gamma(t):=(1-t)\alpha+t\beta,$$
which we can check is differentiable with constant derivative $(-\alpha+\beta)$ and is therefore continuously differentiable.

There might not be a nice, canonical way to define a curve. Here are two circles.

Example 4.16. Fix $z_0\in\mathbb{C}$ and $r\in\mathbb{R}_{>0}$. Then we define the circle of radius r centered at z_0 by the path $\gamma\colon [0,2\pi]\to\mathbb{C}$ by the path
$$\gamma(t):=z_0+r\exp(it).$$
Here is the image.



Example 4.17. Fix $z_0\in\mathbb{C}$ and $r\in\mathbb{R}_{>0}$. Then we define the circle of radius r centered at z_0 by the path $\gamma\colon [0,2\pi]\to\mathbb{C}$ by the path
$$\gamma_0(t):=z_0+r\exp(-it).$$
Here is the image.



We can generalize the above example.

Definition 4.18 (Opposite path)**.** Given a path $\gamma\colon [a,b]\to\mathbb{C}$, we define the *opposite path* $\gamma^-\colon [a,b]\to\mathbb{C}$ by $\gamma^-(t):=\gamma(b+a-t)$.

4.1.3 Conformal Maps

We close with the following theorem.

Theorem 4.19. Fix $\Omega\subseteq\mathbb{C}$ an open and connected subset. Further, fix two paths $\gamma_1,\gamma_2\colon [a,b]\to\mathbb{C}$ two C^1 paths and some holomorphic function $f\colon\Omega\to\mathbb{C}$. Now, suppose that $t_1,t_2\in[a,b]$ have $z_0:=\gamma_1(t_1)=\gamma_2(t_2)\neq 0$ with $\gamma_1(t_1),\gamma_2(t_2)\neq 0$ and $f'(z_0)\neq 0$. Then

$$\frac{\gamma_1'(t_1)}{\gamma_2'(t_2)}=\frac{(f\circ\gamma_1)'(t_1)}{(f\circ\gamma_2)'(t_2)}.$$

Proof. The main tool that we need is a version of Lemma 4.5 to deal with composition.

Lemma 4.20. Fix $\gamma\colon [a,b]\to\mathbb{C}$ a path differentiable at $t_0\in[a,b]$. Further, set a nonempty open subset $\Omega\subseteq\mathbb{C}$ with $\operatorname{im}\gamma\subseteq\Omega$ with a function $f\colon [c,d]\to\mathbb{C}$ differentiable at $\gamma(t_0)$. Then $f\circ\gamma$ is differentiable at t_0 with

$$(f\circ\gamma)'(t_0)=f'(\gamma(t_0))\gamma'(t_0).$$

Proof. We repeat the proof of Lemma 4.5. We consider the difference quotient $v\colon [c,d]\to\mathbb{C}$ defined as

$$v(z):=\begin{cases}\frac{f(z)-f(\gamma(t_0))}{z-\gamma(t_0)}-f'(s(t_0)) & z\neq\gamma(t_0),\\ 0 & z=\gamma(t_0).\end{cases}$$

Now, by definition of the differentiability of f at $s(t_0)$, we know that $v(z)\to 0$ as $z\to\gamma(t_0)$. Rearranging, we see that

$$f(z)-f(\gamma(t_0))=(z-\gamma(t_0))\cdot\big(f'(\gamma(t_0))+v(z)\big)$$

for all $z\in\Omega$, so plugging in $\gamma(t)\in\Omega$, we see that

$$f(\gamma(t))-f(\gamma(t_0))=(\gamma(t)-\gamma(t_0))\cdot\big(f'(\gamma(t_0))+v(\gamma(t))\big).$$

So now we rearrange backwards to see

$$\frac{f(\gamma(t))-f(\gamma(t_0))}{t-t_0}=\frac{\gamma(t)-\gamma(t_0)}{t-t_0}\cdot\big(f'(\gamma(t_0))+v(\gamma(t))\big).$$

Upon taking the limit as $t\to t_0$, the differentiability of γ at t_0 assures us that

$$\lim_{t\to t_0}\frac{f(\gamma(t))-f(\gamma(t_0))}{t-t_0}=\left(\lim_{t\to t_0}\frac{\gamma(t)-\gamma(t_0)}{t-t_0}\right)\left(\lim_{t\to t_0}f'(\gamma(t_0))+v(\gamma(t))\right)=\gamma'(t_0)f'(\gamma(t_0)).$$

Notably, we are using the fact that $v\circ\gamma$ is continuous at t_0 because γ is continuous and v is continuous at $\gamma(t_0)$. ■

Using the above lemma, we can compute

$$\frac{(f\circ\gamma_1)'(t_1)}{(f\circ\gamma_2)'(t_2)}=\frac{f'(\gamma(t_1))\gamma'(t_1)}{f'(\gamma(t_2))\gamma'(t_2)}=\frac{\gamma'(t_1)}{\gamma'(t_2)}.$$

Note that we have successfully used the hypotheses that $f'(z_0)\neq 0$ and $\gamma_2'(t_2)\neq 0$; the last hypothesis that $\gamma_1'(t_1)\neq 0$ is added for aesthetic reasons. ■

We should probably explain why we named this subsection "conformal maps." We pick up the following corollary.

Corollary 4.21. In the context of the Theorem 4.19, we have that

$$\arg \gamma_1'(t_1) - \arg \gamma_2'(t_2) \equiv \arg(f \circ \gamma_1)'(t_1) - \arg(f \circ \gamma_2)'(t_2) \pmod{2\pi}$$

Proof. Everything involved is nonzero by hypothesis; this time $\gamma_1'(t_1) \neq 0$ is not aesthetic. By taking $\arg$ of both sides of the conclusion of Theorem 4.19, we see that it will suffice to show

$$\arg(z/w) - \arg z + \arg w \stackrel{?}{\equiv} 0 \pmod{2\pi}$$

for $z, w \in \mathbb{C}^\times$ after some rearranging. Well, we write $z = r\exp(i\alpha)$ and $w = s\exp(i\beta)$ so that

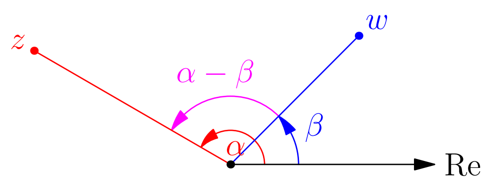
$$z/w = (r/s)\exp(i(\alpha - \beta))$$

using Proposition 3.57. In particular, $\alpha \equiv \arg z$ and $\beta \equiv \arg w$, so

$$\arg(z/w) \equiv \alpha - \beta \equiv \arg z - \arg w \pmod{2\pi},$$

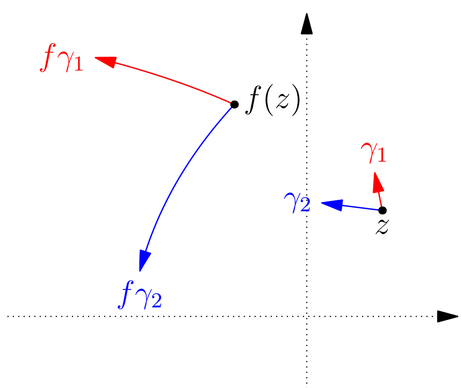
which is what we wanted. ■

Now, we recall that $\arg z$ is intended to be the (counterclockwise) angle from the real axis to a complex number z. As such, $\arg z - \arg w$ should be the (counterclockwise) angle from $w := s\exp(i\beta)$ to $z := r\exp(i\alpha)$, as in the following diagram.



Thus, Corollary 4.21 is saying that, at a point $z_0 \in \mathbb{C}$, the angle between two tangent vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ remains the same if we pass the tangent vectors through f. This angle-preserving property is called being "conformal."

It would be a crime to give this description and then not actually show this for some holomorphic function, so we pass two rays from $z := 1 + i$ through $f(z) := z^2$.



Indeed, we can check visually that it looks like the angle got preserved (even though lengths are quite scaled).

4.2 March 9

Here we go.

4.2.1 Integrals from the Reals

Today we start talking about path integration.

Definition 4.22 (Integrable). Fix $\psi\colon [a,b]\to\mathbb{C}$ a function (such as a path) with $\psi(t)=u(t)+iv(t)$, where $u,v\colon\mathbb{R}\to\mathbb{R}$. Then ψ is *integrable* over $[a,b]$ if and only if u and v are both integrable over $[a,b]$ (as real functions!). In this case, we define

$$\int_a^b \psi(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

We have the following sanity checks.

Lemma 4.23. Fix $\psi_1,\psi_2\colon [a,b]\to\mathbb{C}$ integrable functions with $\alpha_1,\alpha_2\in\mathbb{C}$. Then

$$\int_a^b (\alpha_1\psi_1(t)+\alpha_2\psi_2(t))\,dt = \alpha_1\int_a^b \psi_1(t)\,dt + \alpha_2\int_a^b \psi_2(t)\,dt.$$

Proof. This is by brute force. Let $\alpha_1=x_1+y_1i$ and $\alpha_2=x_2+y_2i$ and $\psi_1(t)=u_1(t)+iv_1(t)$ and $\psi_2(t)=u_2(t)+iv_2(t)$. Then we see that

$$\begin{aligned}\alpha_1\psi_1(t)+\alpha_2\psi_2(t) &= (x_1+y_1i)(u_1(t)+iv_1(t))+(x_2+y_2i)(u_2(t)+iv_2(t))\\ &= (x_1u_1(t)+x_2u_2(t)-y_1v_1(t)-y_2v_2(t))+i(x_1v_1(t)+x_2v_2(t)+y_1u_1(t)+y_2u_2(t))\end{aligned}$$

has integrable components because u_1,v_1,u_2,v_2 are all integrable by hypothesis, and the components are just $\mathbb{R}$-linear combinations of these. Doing a lot of expansion, the fact that linear combinations of real-valued integrals is legal, we see

$$\begin{aligned}\int_a^b (\alpha_1\psi_1(t)+\alpha_2\psi_2(t))\,dt &= \int_a^b (x_1u_1(t)+x_2u_2(t)-y_1v_1(t)-y_2v_2(t))\,dt\\ &\quad + i\int_a^b (x_1v_1(t)+x_2v_2(t)+y_1u_1(t)+y_2u_2(t))\,dt\\ &= x_1\int_a^b u_1(t)\,dt + x_2\int_a^b u_2(t)\,dt - y_1\int_a^b v_1(t)\,dt - y_2\int_a^b v_2(t)\,dt\\ &\quad + ix_1\int_a^b v_1(t)\,dt + ix_2\int_a^b v_2(t)\,dt + iy_1\int_a^b u_1(t)\,dt + iy_2\int_a^b u_2(t)\,dt\\ &= (x_1+y_1i)\left(\int_a^b u_1(t)\,dt + i\int_a^b v_1(t)\,dt\right)\\ &\quad + (x_2+y_2i)\left(\int_a^b u_2(t)\,dt + i\int_a^b v_2(t)\,dt\right)\\ &= \alpha_1\int_a^b \psi_1(t)\,dt + \alpha_2\int_a^b \psi_2(t)\,dt,\end{aligned}$$

which is what we wanted. ■

Lemma 4.24. Fix $\psi\colon [a,b]\to\mathbb{C}$ an integral function. Then

$$\left|\int_a^b \psi(t)\,dt\right| \le \int_a^b |\psi(t)|\,dt.$$

Proof. There is approximately one idea to this proof: the point is to create a real-valued integral equal to the norm. Note that $\int_a^b \psi(t)\,dt = 0$ means we are done for free. Thus, we can put $\int_a^b \psi(t)\,dt = 0$ into polar form as

$$r\exp(i\theta) = \int_a^b \psi(t)\,dt$$

for $r > 0$. We would like to factor out a $\exp(i\theta)$ from this integral, so we compute (using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ from Lemma 3.75) that

$$\begin{aligned}\psi(t)\exp(-i\theta) &= (u(t)+iv(t))(\cos(-\theta)+i\sin(-\theta))\\ &= (u(t)+iv(t))(\cos\theta - i\sin\theta)\\ &= \underbrace{(u(t)\cos\theta + v(t)\sin\theta)}_{\alpha(t)} + i\cdot\underbrace{(v(t)\cos\theta - u(t)\sin\theta)}_{\beta(t)}\end{aligned}$$

In particular, $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$, and so by Proposition 3.57, we write

$$\psi(t) = \alpha(t)\exp(i\theta) + i\beta(t)\exp(i\theta) = \exp(i\theta)(\alpha(t) + i\beta(t)).$$

Thus, we can write

$$r\exp(i\theta) = \int_a^b \psi(t)\,dt = \int_a^b \exp(i\theta)(\alpha(t)+i\beta(t))\,dt = \exp(i\theta)\int_a^b (\alpha(t)+i\beta(t))\,dt.$$

Upon cancelling out the $\exp(i\theta)$, we see that

$$r = \int_a^b (\alpha(t)+i\beta(t))\,dt = \int_a^b \alpha(t)\,dt + i\int_a^b \beta(t)\,dt.$$

Because β is still a real function, that integral evaluates to a real number, but because we have no imaginary part, we conclude

$$r = \int_a^b \alpha(t)\,dt.$$

So now we appeal to real analysis. We see

$$\left|\int_a^b \psi(t)\,dt\right| = r = \int_a^b \alpha(t)\,dt \le \int_a^b |\psi(t)|\,dt,$$

where $\alpha(t) \le |\psi(t)|$ is because

$$\alpha(t) = \operatorname{Re}\psi(t)\exp(-i\theta) \le |\psi(t)|\cdot|\exp(-i\theta)| = |\psi(t)|,$$

where $|\exp(-i\theta)| = 1$ by Corollary 3.61. ∎

4.2.2 Path Integration

We have the following definition.

Definition 4.25 (Integration)**.** Fix $\Omega \subseteq \mathbb{C}$ an open and connected subset with a C^1 path $\gamma \colon [a,b] \to \Omega$. Now, given a continuous function $f \colon \Omega \to \mathbb{C}$, we define the *integral*

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt,$$

if the integral exists.

Lemma 4.26. Under the hypotheses of Definition 4.25, the integral $\int_\gamma f(z)\,dz$ actually exists.

Proof. Note that f and γ are both continuous, so $f \circ \gamma$ is continuous. Similarly, γ' is continuous because γ is C^1. In total, we can expand

$$f(\gamma(t))\gamma'(t)$$

to be a product of continuous functions and therefore must be continuous. It follows that $\operatorname{Re}(f(\gamma(t))\gamma'(t))$ and $\operatorname{Im}(f(\gamma(t))\gamma'(t))$ is also a continuous function, so these components are integrable, so the total integral

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt$$

exists. ■

Our goal is to show that the integral itself only depends on the equivalence class of γ.

We can extend this definition a little to piecewise C^1 paths.

Definition 4.27 (Integration)**.** Fix $\Omega \subseteq \mathbb{C}$ an open and connected subset with a piecewise C^1 path $\gamma\colon [a,b] \to \Omega$, where we have the strictly increasing sequence $\{a_k\}_{k=1}^n$ such that $a = a_0$ and $b = a_n$ and $\gamma|_{a_k,a_{k+1}}$ are C^1. Then, given a continuous function $f\colon \Omega \to \mathbb{C}$, we define the *integral*

$$\int_\gamma f(z)\,dz := \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(\gamma(t))\gamma'(t)\,dt.$$

Note that this integral exists because each component integral exists because $\gamma|_{[a_k,a_{k+1}]}$ is in fact C^1.

Example 4.28. Fix $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) := \frac{1}{z}$ and $\gamma\colon [0,2\pi] \to \mathbb{C}$ by $\gamma(t) = \exp(it)$ so that $\gamma'(t) = i\exp(it)$. It follows

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \left(\frac{1}{\exp(it)} \cdot i\exp(it)\right)\,dt = \int_0^{2\pi} i\,dt = 2\pi i t.$$

Now let's show that the integral does not change on reparameterization.

Lemma 4.29. Fix $\gamma_1\colon [a,b] \to \Omega$ and $\gamma_2\colon [c,d] \to \Omega$ two equivalent piecewise C^1 paths. Then, for any continuous function $f\colon \Omega \to \mathbb{C}$,

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Proof. By equivalence, we are promised a function $s\colon [c,d] \to [a,b]$ which is continuously differentiable, bijection, and has positive derivative everywhere such that $\gamma_2 = \gamma_1 \circ s$.

We will in the case where γ_1 is C^1, and the general case will follow. Then we compute

$$\int_{\gamma_2} f(z)\,dz = \int_c^d f(\gamma_1(s(t)))(\gamma_1 \circ s)'(t)\,dt.$$

Applying Lemma 4.5, we see

$$\int_{\gamma_2} f(z)\,dz = \int_c^d f(\gamma_1(s(t)))\gamma_1'(s(t))s'(t)\,dt$$

By applying a u-substitution along s (notably, this is now an integral from a real variable!),[1] we see

$$\int_{\gamma_2} f(z)\,dz = \int_a^b f(\gamma_1(s))\gamma_1'(s)\,ds = \int_{\gamma_1} f(z)\,dz,$$

which is what we wanted. ■

4.2.3 Path Arithmetic

Let's blast through some lemmas.

Warning 4.30. In the following statements, we will merely require our paths to be piecewise C^1, but the proofs will deal with the C^1 case. This can be amended by partitioning all the intervals to make everything C^1, but we will not write this out formally.

Lemma 4.31. Fix an open subset $\Omega \subseteq \mathbb{C}$. If $\gamma\colon [a,b] \to \Omega$ is a piecewise C^1 path and $f, g\colon \Omega \to \mathbb{C}$ are continuous functions and $\alpha, \beta \in \mathbb{C}$, then we have

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$

Proof. We write

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \int_a^b (\alpha f(\gamma(t)) + \beta g(\gamma(t)))\gamma'(t)\,dt$$

by definition, which expands by Lemma 4.23 into

$$\alpha \int_a^b f(\gamma(t))\gamma'(t)\,dt + \beta \int_a^b g(\gamma(t))\gamma'(t)\,dt = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma f(z)\,dz,$$

which is what we wanted. ■

Lemma 4.32. Fix an open subset $\Omega \subseteq \mathbb{C}$. Further, fix $\gamma\colon [a,b] \to \Omega$ is a piecewise C^1 path with $\gamma^-(t) := \gamma(b+a-t)$ the opposite path. Then, for $f\colon \Omega \to \mathbb{C}$ a continuous function,

$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$

Proof. The point is to do a u-substitution $t \mapsto b+a-t$. Indeed, we compute

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_b^a f(\gamma(b+a-t))\gamma'(b+a-t)\,dt,$$

where in the last step we have applied our u-substitution, legal from real analysis because our integral is from a real variable.[2] However, we see $\gamma(b+a-t) = \gamma^-(b+a-t)$, so the right-hand integral is the desired one; notably, $(\gamma^-)'(t) = -\gamma(b+a-t)$, but this inherited minus sign reverses the order of the time to be from $t=a$ to $t=b$, as it should be. ■

[1] Technically, we should expand out this integral into real and imaginary parts and then apply the u-substitution. Please don't make me do this.

[2] Again, to be formal, we should expand this into real and imaginary parts and then apply the u-substitution, but we won't bother.

Lemma 4.33. Fix an open subset $\Omega \subseteq \mathbb{C}$. Further, fix $\gamma\colon [a,b] \to \Omega$ and $\eta\colon [c,d] \to \Omega$ to be a piecewise C^1 paths such that $\gamma(b) = \eta(c)$. Then, for a continuous function $f\colon \Omega \to \mathbb{C}$, we have

$$\int_{\gamma*\eta} f(z)\,dz = \int_\gamma f(z)\,dz + \int_\eta f(z)\,dz.$$

Proof. Note that

$$\int_{\gamma*\eta} f(z)\,dz = \int_a^{b+d-c} f((\gamma*\eta)(t))(\gamma*\eta)'(t)\,dt = \int_a^b f(\gamma(t))\gamma'(t)\,dt + \int_b^{b+d-c} f(\eta(t-b+c))\eta'(t-b+c)\,dt.$$

This is what we want as soon as we apply the change of variables $t-b+c \mapsto t$. ■

For our last lemma, we have the following definition.

Definition 4.34 (Length)**.** Fix a C^1 path $\gamma\colon [a,b] \to \mathbb{C}$. Then we define the *length of* γ as

$$\ell(\gamma) := \int_a^b |\gamma'(t)|\,dt.$$

More generally, if γ is piecewise C^1, then we are promised a strictly increasing sequence $\{a_k\}_{k=0}^n$ where $a_0 = a$ and $a_n = b$ such that $\gamma|_{[a_k,a_{k+1}]}$ is C^1. So we define the *length* as

$$\ell(\gamma) := \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} |\gamma'(t)\,dt.$$

Let's use this definition a little.

Proposition 4.35. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$. Then, for $\gamma\colon [a,b] \to \mathbb{C}$ a piecewise C^1 path and a continuous function $f, g\colon \Omega \to \mathbb{C}$, we have the following. Then we have

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{t\in[a,b]}\{|f(\gamma(t))|\} \cdot \ell(\gamma).$$

Proof. By composition, $|f \circ \gamma|$ is a continuous function. In particular, because $[a,b]$ is a compact set, the supremum will actually exist, thus bounding f on $\gamma([a,b])$. Now, estimating, we see

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(z))\gamma'(z)\,dz\right| \le \int_a^b |f(\gamma(z))| \cdot |\gamma'(z)|\,dz.$$

By real analysis, we bound this last integral (from real analysis) as

$$\sup_{t\in[a,b]}\{|f(\gamma(t))|\} \cdot \int_a^b |\gamma'(t)|\,dt,$$

which is what we wanted. ■

We close with a definition, to advertise the fundamental theorem of calculus.

Definition 4.36 (Primitive)**.** Fix a nonempty, open subset $\Omega \subseteq \mathbb{C}$. Then, given two continuous functions $F, f\colon \Omega \to \mathbb{C}$, we say that F is a *primitive* on f if and only if F is holomorphic on Ω and $F' = f$.

4.3 March 11

Good morning everyone.

4.3.1 The Fundamental Theorem of Calculus

Today we continue talking about path integration. We want to talk about a Fundamental theorem of calculus, so we pick up the following definition.

Definition 4.36 (Primitive)**.** Fix a nonempty, open subset $\Omega \subseteq \mathbb{C}$. Then, given two continuous functions $F, f\colon \Omega \to \mathbb{C}$, we say that F is a *primitive* on f if and only if F is holomorphic on Ω and $F' = f$.

As promised, we have the following statement.

Theorem 4.37 (Fundamental theorem of calculus)**.** Fix an open, connected, nonempty subset $\Omega \subseteq \mathbb{C}$ with continuous functions $F, f\colon \Omega \to \mathbb{C}$ such that F is a primitive of f. If $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise C^1, then we can compute

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. We proceed by force. Write

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)),$$

where the last step is separating out $F \circ \gamma$ into real and imaginary parts and using the Fundamental theorem of calculus from $\mathbb{R}$. ■

Remark 4.38. Importantly, Theorem 4.37 asserts that the exact path γ does not matter to this integral—only its endpoints!

Corollary 4.39. Fix an open, connected, nonempty subset $\Omega \subseteq \mathbb{C}$ with continuous functions $F, f\colon \Omega \to \mathbb{C}$ such that F is a primitive of f. If $\gamma\colon [a,b] \to \mathbb{C}$ is a closed path, then

$$\int_\gamma f(z)\,dz = 0.$$

Proof. We compute

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0,$$

which is what we wanted. ■

Example 4.40. The function $f(z) = \frac{1}{z}$ does not have a primitive on $\mathbb{C} \setminus \{0\}$, which we can see formally because

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i,$$

for $\gamma\colon [0, 2\pi]$ defined by $\gamma(t) := e^{it}$. Less formally, we would like $f(z) = \frac{1}{z}$ to have primitive given by $\operatorname{Log} z$, but $\operatorname{Log} z$ is only defined on $\mathbb{C} \setminus \mathbb{R}_{<0}$.

So we would like to determine when a function has a primitive root.

To start our discussion, we have the following technical result.

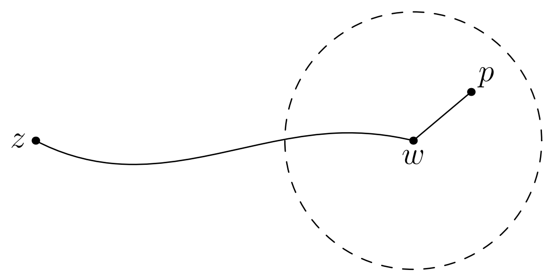
Lemma 4.41. Fix a nonempty, open, and connected subset $\Omega \subseteq \mathbb{C}$. Then any two points in $\mathbb{C}$ are connected by a piecewise C^1 path contained in Ω.

Proof. The idea is to build path-connected components as in Proposition 2.110, but this time, we force our paths to be piecewise C^1.

Fix $z \in \Omega$, and let $U \subseteq \Omega$ denote the set of points $w \in \Omega$ such that there exists a piecewise C^1 path from z to w contained in Ω. We want to show $U = \Omega$. The key is the following lemma.

Lemma 4.42. Fix everything as above. If $X \subseteq \Omega$ is convex with $X \cap U \neq \varnothing$, then $X \subseteq U$.

Proof. Fix any $w \in X \cap U$ so that we want to show $X \subseteq U$. In other words, for any $p \in X$, we want to show $p \in U$. The point, like with Proposition 2.110, is the following image.



Indeed, because $w \in U$, there exists a piecewise C^1 path $\gamma\colon [a,b] \to \Omega$ from z to w. To finish, we set $\eta\colon [0,1] \to B(w,r)$ to be

$$\eta(t) := w + t(p - w)$$

so that $\eta(0) = w$ and $\eta(1) = p$ and $\eta'(t) = p - w$ is a constant function and therefore continuous. Because X is convex, we see that η lives in X and therefore in Ω.

Thus, η is a C^1 path from w to p, so $\gamma * \eta$ is a piecewise C^1 path from z to w to p.[3] Because γ and η both output to Ω, we see that $\gamma * \eta$ does as well, so $p \in U$ follows. ■

We now have the following checks on U.

- We see that U is nonempty because $z \in U$. Namely, the path $\gamma_z\colon [0,1] \to \Omega$ by

$$\gamma_z(t) := z$$

  has derivative $\gamma_z'(t) = 0$, which is constant and hence continuous. Thus, γ_z is a C^1 path from z to z.

- We show that U is open in Ω. Indeed, suppose that $w \in U$. We need to find an open neighborhood around w which lives in U; well, Ω is open, so there exists some $r > 0$ such that $B(w,r) \subseteq \Omega$.

  But now, $B(w,r)$ is convex (by Example 2.16) and intersects U nontrivially at $w \in U$, so $B(w,r) \subseteq U$ by Lemma 4.42, so we are done.

- We show that U is closed in Ω. For this, we show that $\Omega \setminus U$ is open in Ω. Well, given $w \in \Omega \setminus U$, we need to find an open neighborhood around w contained in $\Omega \setminus U$; because Ω is open, we certainly may find some $r > 0$ such that $B(w,r) \subseteq \Omega$.

  So we claim that $B(w,r) \subseteq \Omega \setminus U$ or equivalently that $B(w,r) \cap U = \varnothing$. Well, supposing for the sake of contradiction that we can find $p \in B(w,r) \cap U$, we see that $B(w,r)$ is convex by Example 2.16 and intersects U nontrivially at U, so $B(w,r) \subseteq U$ and $w \in U$ will follow by Lemma 4.42. But this contradicts the construction of w, so we are done.

[3] Technically, we should provide a partition for $\gamma * \eta\colon [a, b+1] \to \Omega$. Well, partition $[a,b]$ by the partition for γ, and then take $[b, b+1]$ to be the last portion.

Thus, U is a nonempty closed and open subset of Ω, so because Ω is connected, we must have $U = \Omega$: we see that $\Omega = U \sqcup (\Omega \setminus U)$ is a disjoint union into open sets, so because U is nonempty, we must have $\Omega \setminus U$ be empty, so $U = \Omega$. But this is exactly what we wanted, so we are done. ■

Remark 4.43. We can strengthen this to having a C^1 path, with a little more technical care.

As such, we have the following.

Theorem 4.44. Fix a nonempty, open, and connected subset $\Omega \subseteq \mathbb{C}$. Further, fix a continuous function $f \colon \Omega \to \mathbb{C}$ such that

$$\int_\gamma f(z)\,dz = 0$$

for all closed paths γ. Then f admits a primitive F.

Proof. We construct our primitive F by hand. Fix $z_0 \in \Omega$. Then, for any $z \in \Omega$, we choose some piecewise C^1 path $\gamma \colon [a,b] \to \mathbb{C}$ with $\gamma(a) = z_0$ and $\gamma(b) = z$ so that we can define

$$F(z) := \int_\gamma f(z)\,dz.$$

Of course, it is not immediately obvious that F does not depend on the exact choice of path γ, but it does not: suppose $\gamma_1 \colon [a,b] \to \mathbb{C}$ and $\gamma_2 \colon [c,d] \to \mathbb{C}$ have $\gamma_1(a) = \gamma_2(c) = z_0$ and $\gamma_1(b) = \gamma_2(d) = z$. Now, the key observation is that

$$\gamma := \gamma_2 * \gamma_1^-,$$

which we can see is well-defined because $\gamma_1^-(b) = z_0$ and $\gamma_2(c) = z_0$ as well. Further, γ is closed because $\gamma_1^-(a) = z$ while $\gamma_2(d) = z$, so we see that

$$\int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz + \int_{\gamma_1^-} f(z)\,dz = \int_{\gamma_2 * \gamma_1^-} f(z)\,dz = 0,$$

where we have used (in order) Lemma 4.32 and Lemma 4.33 and the hypothesis.

It remains to show that F is holomorphic on $\mathbb{C}$ with $F' = f$. Well, fix $w \in \Omega$ and $\varepsilon > 0$ such that $B(w,\varepsilon) \subseteq \Omega$, which is legal because Ω is open. Now, we are promised a piecewise C^1 path $\gamma \colon [a,b] \to \Omega$ such that

$$\gamma(a) = z_0 \qquad \text{and} \qquad \gamma(b) = w.$$

Now, for any $z_1 \in B(w,\varepsilon)$, we set $s_1 \colon [0,1] \to B(w,\varepsilon)$ be the line segment connecting w to z_1; explicitly, we have

$$s_1(t) = w + t(z_1 - w).$$

Then, we define $\gamma_1 * = \gamma * s_1$, a path from z_0 to w to z_1. In particular, we find

$$F(z_1) - F(w) = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{s_1} f(z)\,dz,$$

where we have used Lemma 4.33 again. In particular, for $z_1 \neq w$, we find

$$\begin{aligned}\left|\frac{F(z_1) - F(w)}{z_1 - w} - f(w)\right| &= \left|\frac{1}{z_1 - w}\int_0^1 f(w + t(z_1 - w))(z_1 - w)\,dt - f(w)\right| \\ &= \left|\int_0^1 f(w + t(z_1 - w))\,dt - f(w)\right|.\end{aligned}$$

By Proposition 4.35, we see

$$\left|\frac{F(z_1) - F(w)}{z_1 - w} - f(w)\right| \le \ell(s_1) \cdot \sup_{t \in [0,1]} \{|f(w + t(z_1 - w)) - f(w)|\}.$$

We now need to show that this values goes to 0 as $\varepsilon > 0$ goes to 0. Well, for some $\varepsilon' > 0$, there is a $\delta > 0$ such that $|z - w| < \delta$ for which

$$|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon'.$$

In particular, we see that $|z_1 - w| < \delta'$ implies that

$$|w + t(z_1 - w) - w| = |t(z_1 - w)| = |z_1 - w| < \delta' \implies |f(w + t(z_1 - w)) - f(w)| < \varepsilon',$$

so

$$\sup_{t\in[0,1]} \{|f(w + t(z_1 - w)) - f(w)|\} \leq \varepsilon'.$$

Putting this together, we see that

$$F'(w) = \lim_{z_1 \to w} \frac{F(z_1) - F(w)}{z_1 - w} = f(w),$$

so we are done. ∎

Remark 4.45. This criterion might appear useless, but we promise that it isn't. It will turn out that we don't really have to check all paths.

4.3.2 Winding Numbers

We start with a continuous version of the polar form of a complex number. This will be the major technical step in our construction of the winding number.

Lemma 4.46. Fix $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ a path. Then there is a continuous function $\theta_\gamma\colon [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)| \exp(2\pi i \theta_\gamma(t)).$$

Furthermore, if we have two such functions θ_γ and ψ_γ, then $\theta_\gamma - \psi_\gamma$ differ by a constant integer.

Proof. The point is to choose θ_γ with various branches of Log. We proceed with the following steps.

1. For psychological reasons, we replace $\gamma(t)$ with $\frac{\gamma(t)}{|\gamma(t)|}$ so that $|\gamma(t)| = 1$, and we are looking for a function $\theta\colon [0,1] \to \mathbb{R}$ so that
$$\gamma(t) = \exp(2\pi i \theta(t)).$$

2. We now temper the speed of γ by partitioning its interval $[0,1]$. Because $[0,1]$ is compact γ is continuous, γ is in fact, uniformly continuous by Proposition 2.124. So, for example, we can find some $\delta > 0$ such that $s, t \in [0,1]$ has
$$|s - t| < \delta \implies |\gamma(s) - \gamma(t)| < 1.$$
As such, we set some $n \in \mathbb{N}$ exceeding $\frac{1}{\delta}$ and partition $[0,1]$ by $\{a_k\}_{k=0}^n$ defined by $a_k := k/n$ (note $a_0 = 0$ and $a_n = 1$) so that $|a_{k+1} - a_k| = \frac{1}{n} < \delta$. The point is that our partition $\{a_k\}_{k=0}^n$ forces γ to move at a reasonable pace.

3. We now define θ piecewise by $\overline{\theta}_k\colon [a_k, a_{k+1}] \to \mathbb{C}$ by
$$\overline{\theta}_k(t) := \frac{\arg\gamma(a_k) + \arg(\gamma(t)/\gamma(a_k))}{2\pi}.$$
Notably, $|\gamma(t) - \gamma(a_k)| < 1/2$ implies that $\gamma(t)/\gamma(a_k)$ cannot be in $\mathbb{R}_{<0}$ because γ lives on the unit circle in $\mathbb{C}$, so $\gamma(t)/\gamma(a_k) \in \mathbb{R}_{<0}$ would imply that $\gamma(t)/\gamma(a_k) = -1$ and so $|\gamma(t) - \gamma(a_k)| = |1 - -1| = 2$.

Thus, $\overline{\theta}_k$ defined above is in fact a continuous function by Lemma 3.97 because it is the composite of continuous functions.

Importantly, we can check that $i \arg \gamma(a_k) = \operatorname{Log} \gamma(a_k)$ because $|\gamma(a_k)|$, so we see

$$\begin{aligned}\exp(2\pi i \overline{\theta}_k(t)) &= \exp(i \arg \gamma(a_k) + i \arg(\gamma(t)/\gamma(a_k))) \\ &= \exp(\operatorname{Log} \gamma(a_k)) \exp(\operatorname{Log} \gamma(t)/\gamma(a_k)) \\ &= \gamma(a_k) \cdot \gamma(t)/\gamma(a_k) = \gamma(t),\end{aligned}$$

so our $\overline{\theta}_k$ is chosen correctly.

4. Next we glue our $\overline{\theta}_k$ functions. Fixing some a_m for $0 < m < n$, we see

$$\exp(2\pi i \overline{\theta}_{m-1}(a_m)) = \exp(2\pi i \overline{\theta}_m(a_m)),$$

so $2\pi i(\overline{\theta}_m(a_m) - \overline{\theta}_{m-1}(a_m)) \in \ker \exp$ by Proposition 3.57, so $\overline{\theta}_m(a_m) = s_m + \overline{\theta}_{m-1}(a_m)$ for some $s_m \in \mathbb{Z}$ by Proposition 3.73. As such, we define

$$\theta(t) := \overline{\theta}_m(t) + \sum_{k=1}^{m} s_k \qquad \text{where} \qquad t \in [a_m, a_{m+1}],$$

where $t \in [a_m, a_{m+1}]$. Note that this function is well-defined on the endpoints a_m for $0 < m < n$ because $\overline{\theta}_m(a_m) + s_m = \overline{\theta}_{m-1}(a_m)$. On one hand,

$$\exp(2\pi i \theta(t)) = \exp(2\pi i \overline{\theta}_m(t)) \exp\left(2\pi i \cdot \sum_{k=1}^{m} s_k\right) = \gamma(t) \cdot 1 = \gamma(t)$$

as we showed above (using Proposition 3.57 and Proposition 3.73), so we see that this θ satisfies the needed equation.

Lastly, to see that θ is continuous, we note that θ is continuous within each interval (a_m, a_{m+1}) because this turns into a shifted version of $\overline{\theta}_m$, which we know is continuous by construction. Then at each endpoint the well-definedness check shows that we can glue these intervals together.

Thus, we have exhibited our continuous function θ. It remains to show that this θ is unique up to shifting by an integer. Well, suppose θ and ψ both satisfy

$$\gamma(t) = |\gamma(t)| \exp(2\pi i \theta(t)) = |\gamma(t)| \exp(2\pi i \psi(t)).$$

Using Proposition 3.57, we see that

$$\exp\big(2\pi i(\theta(t) - \gamma(t))\big) = 1,$$

so $\theta(t) - \gamma(t) \in \mathbb{Z}$ by Proposition 3.73. However, $t \mapsto \theta(t) - \gamma(t)$ is a continuous function from the connected set $[0, 1]$ to the set $\mathbb{Z}$, but because the image must be connected by Proposition 2.98, so the image must be a single point.[4] Thus,

$$\theta(t) = \gamma(t) + n$$

for any $t \in [0, 1]$ for some fixed integer n. ■

This gives us the winding number.

[4] Any connected subset $S \subseteq \mathbb{Z}$ containing a point $a \in \mathbb{Z}$ cannot be disconnected by the open sets $(a - 1/2, a + 1/2) \cap \mathbb{Z}$ and $(-\infty, a - 1/2) \cup (a + 1/2, \infty) \cap \mathbb{Z}$, so the latter set must be empty, so $S \subseteq (a - 1/2, a + 1/2)$, so $S = \{a\}$.

Definition 4.47 (Winding number)**.** Fix $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ a closed path and $\theta_\gamma\colon [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)| \exp(2\pi i \theta_\gamma(t)).$$

Then we define the *winding number of γ around 0* by

$$\operatorname{Ind}(\gamma, 0) := \frac{\theta_\gamma(1) - \theta_\gamma(0)}{2\pi i}.$$

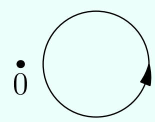
More generally, for a given path $\gamma\colon [0,1] \to \mathbb{C} \setminus \{z_0\}$, the winding number of γ around z_0 is

$$\operatorname{Ind}(\gamma, z_1) := \operatorname{Ind}(\gamma - z_1, 0).$$

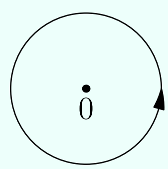
Remark 4.48. Because γ is closed, we see that $\exp(\theta_\gamma(0)) = \exp(\theta_\gamma(1))$, so $\theta_\gamma(0) \equiv \theta_\gamma(1) \pmod{2\pi i}$, so the winding number is in fact an integer. In fact, the winding number is independent of the chosen θ_γ because any two such functions differ by a constant integer, by Lemma 4.46.

Pictorially, the winding number of $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ is intended to be the number of times γ "winds" around 0. We have the following examples, which we will not justify formally.

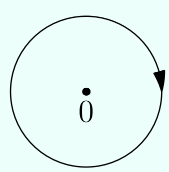
Example 4.49. The following path has winding number 0.



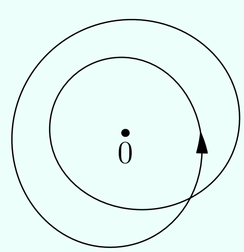
Example 4.50. The following path has winding number 1.



Example 4.51. The following path has winding number -1.



Example 4.52. The following path has winding number 2.



4.4 March 14

Good morning everyone. It's π day. Here are some house-keeping notes.

- Homework #6 is due on Friday, at midnight.
- Class on Friday will be recorded.
- Next week is spring break!

4.4.1 Winding Numbers by Integrals

Today we finish our discussion of path integration; soon we will transition over to the Cauchy integral formula. We recall the following lemma.

Lemma 4.46. Fix $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ a path. Then there is a continuous function $\theta_\gamma\colon [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)| \exp(2\pi i \theta_\gamma(t)).$$

Furthermore, if we have two such functions θ_γ and ψ_γ, then $\theta_\gamma - \psi_\gamma$ differ by a constant integer.

We quickly recall that the function θ_γ in the statement is, roughly speaking, the composition of $\gamma(t)$ (normalized) with a suitably chosen branch of the logarithm.

This gave us the following definition.

Definition 4.47 (Winding number)**.** Fix $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ a closed path and $\theta_\gamma\colon [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)| \exp(2\pi i \theta_\gamma(t)).$$

Then we define the *winding number of* γ *around* 0 by

$$\operatorname{Ind}(\gamma, 0) := \frac{\theta_\gamma(1) - \theta_\gamma(0)}{2\pi i}.$$

More generally, for a given path $\gamma\colon [0,1] \to \mathbb{C} \setminus \{z_0\}$, the winding number of γ around z_0 is

$$\operatorname{Ind}(\gamma, z_1) := \operatorname{Ind}(\gamma - z_1, 0).$$

Remark 4.53. It is advisable to not really care about the definition given in Lemma 4.46 because we are about to give a more computational view of it. To be more explicit, Lemma 4.46 is bad for computation.

Here's a better way to compute the winding number.

Lemma 4.54. Fix $\gamma\colon [0,1] \to \mathbb{C}$ a closed, piecewise C^1 path. Further, fix $z_0 \in \mathbb{C} \setminus \operatorname{im} \gamma$. Then

$$\operatorname{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0}\, dz.$$

Here are some example.

Example 4.55. Fix $\gamma\colon [0,1] \to \mathbb{C}$ by $\gamma(t) := \exp(2\pi i t)$ to be the unit circle. We can compute that

$$\oint_\gamma \frac{1}{z}\, dz = \int_0^1 \frac{2\pi i \exp(2\pi i t)}{2\exp(2\pi i t)}\, dt = 2\pi i,$$

so the winding number of γ around $z_0 = 0$ is 1.

Example 4.56. Fix $\gamma\colon[0,1]\to\mathbb{C}$ by $\gamma(t):=\exp(-2\pi it)$ to be the clockwise unit circle. We can then compute that

$$\oint_\gamma \frac{1}{z}\,dz=\int_0^1\frac{-2\pi i\exp(-2\pi it)}{\exp(-2\pi it)}\,dt=-2\pi i,$$

so the winding number of γ around $z_0=0$ is -1.

And here is our proof.

Proof of Lemma 4.54. This proof is a little slick. The point is to write $a(t)$ in terms of a branch of the logarithm. As from Lemma 4.46, we have that

$$\gamma(t)=z_0+r(t)\exp(2\pi ia(t))$$

where $r(t):=|\gamma(t)-z_0|$ and $a(t)\colon[0,1]\to\mathbb{R}$ is a continuous function.

Fix some $t_0\in[0,1]$. The idea is to show that everything in sight is differentiable. Because $\gamma(t_0)\neq z_0$ and γ is continuous, we can find some $\delta>0$ and a suitable branch of the logarithm Log so that $\operatorname{Log}(\gamma(t)-z_0)$ is defined on all $B(\gamma(t_0),\delta)$. Here, we can compute

$$r(t)=\exp\left(\frac{1}{2}\log\left((\gamma(t)-z_0)(\overline{\gamma}(t)-\overline{z_0})\right)\right).$$

Notably, this is the real-valued logarithm, so all of our standard logarithm rules apply (i.e., we are allowed to move the $\frac{1}{2}$ outside without concern). Thus, we see that r is a composite of continuous functions and therefore continuous here. Now, by the continuity of $r(t)$, we can build a branch of the logarithm so that $\operatorname{Log}\left(\frac{\gamma(t)-z_0}{r(t)}\right)$ is defined near $\gamma(t_0)$. Because

$$\exp\left(\operatorname{Log}\left(\frac{\gamma(t)-z_0}{r(t)}\right)\right)=\exp(2\pi ia(t)),$$

we conclude from Proposition 3.73 that

$$a(t)-\frac{1}{2\pi i}\cdot\frac{\gamma(t)-z_0}{r(t)}$$

is always an integer for each $t\in[0,1]$. But because $[0,1]$ is connected and $\mathbb{Z}$ is discrete, this must be constant, so there is a fixed integer $n\in\mathbb{Z}$ such that

$$a(t)=\frac{1}{2\pi i}\cdot\frac{\gamma(t)-z_0}{r(t)}+n.$$

Now we integrate. We see that

$$\oint_\gamma\frac{1}{z-z_0}\,dz=\int_0^1\frac{\gamma'(t)}{\gamma(t)-z_0}\,dz=\int_0^1\frac{\left(r'(t)+2\pi ir(t)a'(t)\right)\exp(2\pi it)}{r(t)\exp(2\pi it)}\,dt.$$

At this point, we notice that the exponential functions cancel, so we have that

$$\oint_\gamma\frac{1}{z-z_0}\,dz=\int_0^1\frac{r'(t)+2\pi ir(t)a'(t)}{r(t)}\,dt=\int_0^1\frac{r'(t)}{r(t)}\,dt+2\pi i\int_0^1 a'(t)\,dt.$$

Now these integrals are completely real-valued. So we compute

$$\int_0^1\frac{r'(t)}{r(t)}\,dt=\log r(1)-\log r(0)=0$$

because $\gamma(1)=\gamma(0)$ (it's a closed path). Thus, we are left with

$$\oint_\gamma\frac{1}{z-z_0}\,dz=2\pi i\int_0^1 a'(t)\,dt=2\pi i(a(1)-a(0)),$$

so the conclusion follows. ■

Here are some corollaries.

Corollary 4.57. Fix a closed piecewise C^1 path $\gamma\colon [0,1] \to \mathbb{C}$ and $z_0 \in \mathbb{C} \setminus \operatorname{im}\gamma$. Then $\operatorname{Ind}(\gamma^-, z_0) = -\operatorname{Ind}(\gamma, z_0)$.

Proof. Applying Lemma 4.32 to Lemma 4.54, we get

$$\operatorname{Ind}(\gamma^-, z_0) = \oint_{\gamma^-} \frac{1}{z}\,dz = -\oint_\gamma \frac{1}{z}\,dz = -\operatorname{Ind}(\gamma, z_0),$$

which is what we wanted. ■

Corollary 4.58. Fix closed piecewise C^1 paths $\gamma, \eta\colon [0,1] \to \mathbb{C}$ such that $\gamma(1) = \eta(0)$, and pick up some $z_0 \in \mathbb{C} \setminus (\operatorname{im}\gamma \cup \operatorname{im}\eta)$. Then $\operatorname{Ind}(\gamma * \eta, z_0) = \operatorname{Int}(\gamma, z_0) + \operatorname{Ind}(\eta, z_0)$.

Proof. Applying Lemma 4.33 to Lemma 4.54, we get

$$\operatorname{Ind}(\gamma * \eta, z_0) = \oint_{\gamma*\eta} \frac{1}{z}\,dz = \oint_\gamma \frac{1}{z}\,dz + \oint_\eta \frac{1}{z}\,dz = \operatorname{Ind}(\gamma, z_0) + \operatorname{Ind}(\eta, z_0),$$

which is what we wanted. ■

4.4.2 More General Indices

We will want a slightly more general version of the winding number for where we're going.

Definition 4.59 (Index)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ and a closed piecewise C^1 path $\gamma\colon [a,b] \to \Omega$. Given a function $f\colon \Omega \to \mathbb{C}$ which is continuous on $\operatorname{im}\gamma$, we define

$$\operatorname{Ind}_f(\gamma, w) := \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-w}\,dz$$

Remark 4.60. This integral is equal to

$$\int_a^b \frac{f(\gamma(t))}{\gamma(t)-w} \cdot \gamma'(t)\,dt,$$

which is now more obviously well-defined. In particular, the inner function is piecewise continuous, so its real and imaginary parts are integrable.

Proposition 4.61. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ and a closed, piecewise C^1 path $\gamma\colon [a,b] \to \Omega$. Given a function $f\colon \Omega \to \mathbb{C}$ a function continuous on $\operatorname{im}\gamma$, the function $\operatorname{Ind}_f(\gamma, -)$ is analytic (!) at w with power series around z_0 given by

$$\operatorname{Ind}_f(\gamma, z_0) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right) (z-z_0)^n$$

This is our first major step towards showing that all holomorphic functions are analytic: here we have been granted a way to conjure some magical power series.

For Proposition 4.61, we will need the following lemma.

Lemma 4.62. Fix Ω and γ as in Proposition 4.61. Given a sequence of continuous functions $\{f_k\}_{k=1}^\infty$ which uniformly converge to f on $\operatorname{im}\gamma$, then f is integrable and

$$\lim_{n\to\infty}\oint_\gamma f_n(z)\,dz=\oint_\gamma f(z)\,dz.$$

Proof. Roughly speaking, the point is to look at

$$\oint_\gamma (f-f_n)(z)\,dz$$

and use Proposition 4.35 and uniform convergence to show that this vanishes as $n\to\infty$.

Let's be a little more precise. We need to show that

$$\lim_{n\to\infty}\oint_\gamma \big(f(z)-f_n(z)\big)\,dz=0.$$

By Proposition 4.35, we can say

$$\left|\oint_\gamma \big(f(z)-f_n(z)\big)\,dz\right|\le \sup_{t\in[a,b]}\{|f(\gamma(t))-f_n(\gamma(t))|\}\cdot\ell(\gamma).$$

If $\ell(\gamma)=0$, there is nothing to say. Otherwise, we set any $\varepsilon>0$ and note that uniform convergence of $f_n\to f$ promises us some N for which $n>N$ has

$$|f(z)-f_n(z)|<\frac{\varepsilon}{2\ell(\gamma)}$$

for all $z\in\operatorname{im}\gamma$. In particular, we find that

$$\left|\oint_\gamma \big(f(z)-f_n(z)\big)\,dz\right|\le \sup_{t\in[a,b]}\{|f(\gamma(t))-f_n(\gamma(t))|\}\cdot\ell(\gamma)\le\frac{\varepsilon}{2\ell(\gamma)}\cdot\ell(\gamma)<\varepsilon,$$

so we have established the needed limit. ■

And here is our proof.

Proof of Proposition 4.61. Fix $z_0\in\mathbb{C}\setminus\operatorname{im}\gamma$. For psychological reasons, we translate our path and Ω so that $z_0=0$. Now, $[a,b]$ is compact, so $\gamma([a,b])$ is compact and therefore closed, so $\mathbb{C}\setminus\operatorname{im}\gamma$ is open, so we can find an $r>0$ such that $B(0,2r)\subseteq\mathbb{C}\setminus\operatorname{im}\gamma$.

Now, for any $w\in B(0,r)$ and $z\in\operatorname{im}\gamma$, we have $|w|<r$ and $|z|2r$, so we have $|w/z|<1/2$. Continuing with our estimation, we set

$$M:=\sup_{t\in[a,b]}\{|f(\gamma(t))|\}$$

which exists because $[a,b]$ is compact (namely, real-valued continuous functions always maximums on compact sets). Thus, we bound

$$\left|\frac{f(z)w^n}{z^{n+1}}\right|=\left|\frac{f(z)}{z}\right|\cdot\left|\frac{w}{z}\right|^n\le\frac{M}{2r}\cdot\left(\frac{1}{2}\right)^n. \qquad (*)$$

Noting that $|w/z|<1$, it follows that

$$\frac{f(z)}{z-w}=\frac{f(z)}{z}\cdot\frac{1}{1-w/z}=\sum_{n=0}^\infty\frac{f(z)}{z}\left(\frac{w}{z}\right)^n=\sum_{n=0}^\infty\frac{f(z)w^n}{z^{n+1}},$$

by how we sum geometric series. In fact, by the Weierstrass M-test, this sum converges uniformly: by $(*)$, we can write

$$\sum_{n=0}^\infty\left|\frac{f(z)w^n}{z^{n+1}}\right|\le\frac{M}{2r}\sum_{n=0}^\infty\left(\frac{1}{2}\right)^n<\infty.$$

Thus, Lemma 4.62 tells us that

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\oint_\gamma \left(\sum_{n=0}^\infty \frac{w^n f(z)}{z^{n+1}}\right) dz = \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}\,dz\right) w^n,$$

which gives the desired power series expansion. ■

4.5 March 16

Good morning everyone. Here are some house-keeping notes.

- Homework #6 is still due Friday.
- Class on Friday will be recorded.

4.5.1 Cauchy Integral Formula Primer

Today we're start with the Cauchy integral formula. Here's the statement.

Theorem 4.63 (Cauchy integral formula)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and some $z_0 \in \Omega$ with $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$. Further, fix the path $\gamma\colon [0,1] \to \Omega$ given by

$$\gamma(t) := z_0 + r\exp(2\pi i t).$$

Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic, then any $w \in B(z_0, r)$ has

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \operatorname{Ind}_f(\gamma, w).$$

Namely, evaluating a holomorphic function f at a point w can be determined only from the value of f on the path γ!

Here is a nice consequence.

Corollary 4.64. Holomorphic functions are analytic.

Proof. Use Theorem 4.63 to show that any function f differentiable at a point in an open set is equal to $\operatorname{Ind}(w,\gamma)$ locally, from which Proposition 4.61 provides the local power series expansion. ■

4.5.2 The Cauchy–Goursat Theorem

To prove Theorem 4.63, we will proceed in steps. Here is one major step.

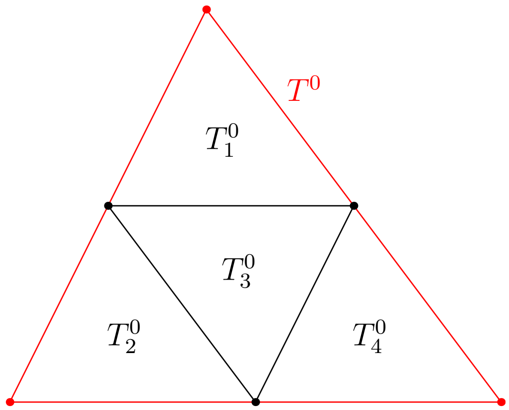
Theorem 4.65 (Cauchy–Goursat)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and T a triangle in Ω (i.e., a closed path defined as the concatenation of three segments). If $f\colon \Omega \to \mathbb{C}$ is holomorphic, then

$$\oint_T f(z)\,dz = 0.$$

Proof. Suppose for the sake of contradiction that the integral is nonzero. Set

$$I := \left|\oint_T f(z)\,dz\right| \ne 0.$$

Here is the image. The idea is to subdivide our triangle $T := T^0$ by midpoints.



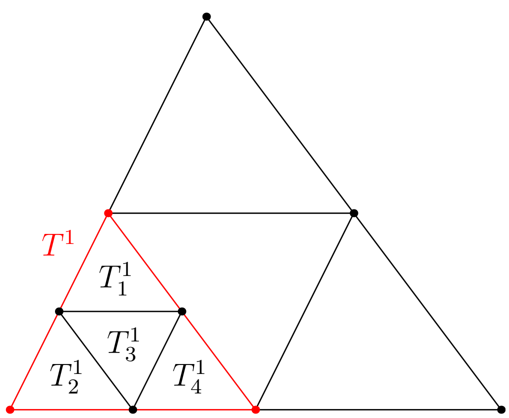
By orienting everything properly, we get cancellation along the overlapped regions, so

$$\oint_T f(z)\,dz = \sum_{i=1}^{4} \oint_{T_i^0} f(z)\,dz.$$

Because the norm here is nonzero, there is an index i such that

$$\frac{I}{4} \le \left| \oint_{T_i^0} f(z)\,dz \right|,$$

so we set $T^1 := T_i^0$. Then we can repeat the process inductively to T^1; here is the iterated image for T^1, working with $T^1 = T_2^0$.



This gives a sequence of nested triangles $T^0, T^1, \ldots$ such that

$$I_k := \left| \oint_{T^k} f(z)\,dz \right| \ge \frac{I}{4^k} > 0.$$

As another bound, we note that $\ell\left(T^k\right) = 2^{-k}\ell(T)$ by essentially geometry of midpoint triangles.

The idea, now, is to find a point contained in all of our triangles. Let V^k be the region enclosed by T^k (i.e., the convex hull). Thus, we have a descending sequence of nested closed sets

$$V^1 \supseteq V^2 \supseteq V^3 \supseteq \cdots.$$

Each of the V^k is closed and bounded and therefore compact, so it follows that the intersection in total is nonempty from the following lemma.

Lemma 4.66. Fix a descending chain

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

of nonempty compact subsets of $\mathbb{C}$. Then the intersection is nonempty.

Proof. Suppose for the sake of contradiction that

$$\bigcap_{i=0}^{\infty} V_i = \varnothing.$$

Then we can write

$$V_0 = V_0 \Big\backslash \bigcap_{i=0}^{\infty} V_i = \bigcup_{i=0}^{\infty} (V_0 \setminus V_i).$$

In particular, $V_0 \setminus V_i = V_0 \cap (\mathbb{C} \setminus V_i)$ is open in V_0, so the above provides an open cover of V_0. By compactness, this has a finite subcover $\{V_{i_k}\}_{k=1}^n$, so

$$V_0 = \bigcup_{k=1}^{n} (V_0 \setminus V_{i_k}) = V_0 \Big\backslash \bigcap_{k=1}^{n} V_{i_k},$$

so we see that

$$\varnothing = \bigcap_{k=1}^{n} V_{i_k} \supseteq \bigcap_{k=1}^{n} V_{\max_k i_k} = V_{\max_k i_k}$$

must be empty, which is a contradiction to the construction of the V_i. ■

Now, put z_0 in the intersection of our descending chain. Now, f is holomorphic and in particular complex differentiable at z_0, so Proposition 3.15 promises us a continuous function $h \colon \Omega \to \mathbb{C}$ continuous at z_0 such that

$$f(z) = f(z_0) + h(z)(z - z_0).$$

Quickly, we expand

$$\oint_{T^k} \big(h(z) - f'(z_0)\big)(z - z_0)\, dz = \oint_{T^k} f(z)\, dz - \oint_{T^k} f(z_0)\, dz - \oint_{T^k} f'(z_0)(z - z_0)\, dz.$$

Now, the constant function $z \mapsto f(z_0)$ has $f(z_0)z$ as a primitive, and $f'(z_0)(z - z_0)$ has $\frac{f'(z_0)}{2}(z - z_0)^2$ as a primitive, so Corollary 4.39 tells us that the two right-hand integrals vanish. Thus, we can estimate (by Proposition 4.35)

$$\begin{aligned} I_k &= \left| \oint_{T^k} \big(h(z) - f'(z_0)\big)(z - z_0)\, dz \right| \\ &\le \sup_{z \in V^k} \{|h(z) - f'(z_0)| \cdot |z - z_0|\} \cdot \ell\left(T^k\right) \\ &\le \sup_{z \in V^k} \{|h(z) - f'(z_0)|\} \cdot \sup_{z \in V^k} \{|z - z_0|\} \cdot \ell\left(T^k\right). \end{aligned}$$

Now, $\sup_{z \in V^k}\{|z - z_0|\}$ is less than the largest length in V^k, which we define to be $\operatorname{diam}\left(V^k\right)$. Re-expanding out to T, we see $\operatorname{diam}\left(V^k\right) = 2^{-k} \operatorname{diam}(V^0)$ and $\ell\left(T^k\right) \le \ell(T^0)$, so we get to bound

$$I^k \le 4^{-k} \sup_{z \in V^k} \{|h(z) - f'(z_0)|\} \cdot \operatorname{diam}(V) \cdot \ell(T).$$

We now take a moment to acknowledge that the point z_0 is the unique point in the intersection of the V^k because $\operatorname{diam}\left(V^k\right) = 2^{-k} \operatorname{diam}\left(V^0\right)$ goes to 0, zeroing in on z_0.

As such, we now take $z \to z_0$ and then $k \to \infty$. In particular, the continuity of h requires

$$4^k I_k \le \sup_{z \in V^k} \{|h(z) - f'(z_0)|\}$$

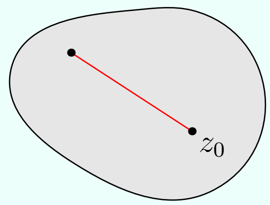
to go to 0 as $k \to \infty$. But now, $I \le 4^k I_k$, so $I = 0$ is forced, which is our final contradiction. ■

4.5.3 Not Just Triangles

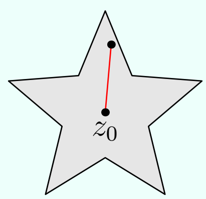
Triangles are a nice starting point for Theorem 4.65, but most sets we deal with will not be triangles. Here's a more general definition to help us.

Definition 4.67 (Star-like). A subset $X \subseteq \mathbb{C}$ is *star-like* with respect to $z_0 \in X$ if and only if each $w \in X$ has a line segment to z_0 contained in X.

Example 4.68. Any convex set X is star-like, for any point in its interior. To be explicit, fix any $z_0 \in X$. Then, for any $x \in X$, the line segment connecting z_0 and x lives in X, thus finishing. Here's the image.



Example 4.69. The star is star-like with respect to its center. Here is the image.



So here is our associated statement.

Theorem 4.70. Fix an open, connected, star-like subset $\Omega \subseteq \mathbb{C}$ with respect to z_0. Further, fix a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$. Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic,

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. The point is to construct a primitive for f by hand, similar to Theorem 4.44, using Theorem 4.65 instead of the listed condition. In particular, note that if we give f a primitive on Ω, then the conclusion will follow by Corollary 4.39.

We imitate the construction from Theorem 4.44. Indeed, we would like to integrate over a path to create our primitive, so we will use the star-like condition to get the desired path: for $w \in \Omega$, the star-like condition on Ω promises us the line segment $\gamma_w\colon [0,1] \to \Omega$ from z_0 to w, defined by

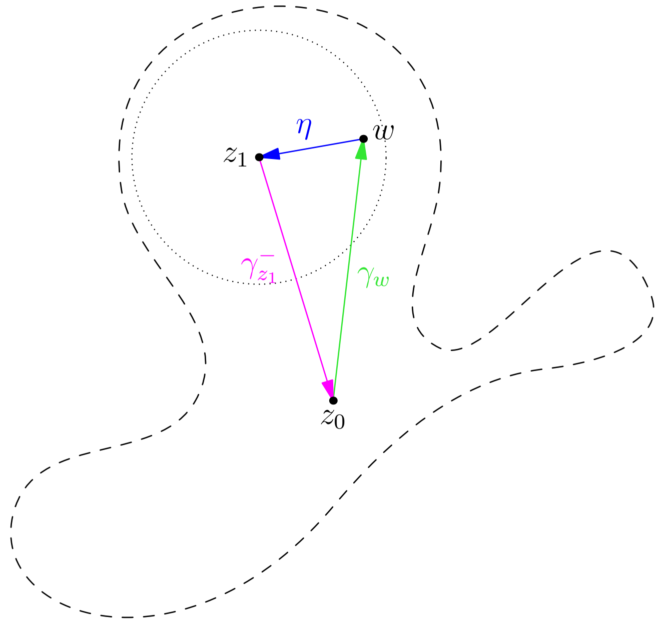
$$\gamma_w(t) := (1-t)z_0 + tw.$$

As such, we set

$$F(w) := \int_{\gamma_w} f(z)\,dz.$$

We now claim that F is our primitive, for which we have to show $F'(z_1) = f(z_1)$ for any $z_1 \in \Omega$.

For psychological reasons, we start by placing $z_1 \in \Omega$ inside some open ball $B(z_1, r) \subseteq \Omega$. We would like to control the value of F inside $B(z_1, r)$. Well, for any w in $B(z_1, r)$, we have the following image.



In words, we have the triangle

$$T_{z_0,w,z_1} := \gamma_w * \eta * \gamma_{z_1}^-$$

contained in Ω, where $\eta(t) := (1-t)w + tz_1$ is the line segment connecting w to z_1. In particular, $\gamma_w(1) = w = \eta(0)$ and $\eta(1) = z_1 = \gamma_{z_1}^-(1)$, so we may concatenate these segments into a triangle. Further, this triangle lives in Ω because $\operatorname{im}\gamma_w \subseteq \Omega$ and $\operatorname{im}\gamma_{z_1} \subseteq \Omega$ by hypothesis on Ω, and $\operatorname{im}\eta \subseteq B(z_1, r) \subseteq \Omega$ because $B(z_1, r)$ is convex.

Thus, by Theorem 4.65, we get to write

$$\begin{aligned} 0 &= \oint_{T_{z_0,w,z_1}} f(z)\,dz \\ &= \int_{\gamma_w} f(z)\,dz + \int_\eta f(z)\,dz + \int_{\gamma_{z_1}^-} f(z)\,dz \\ &= \int_{\gamma_w} f(z)\,dz + \int_\eta f(z)\,dz - \int_{\gamma_{z_1}} f(z)\,dz \\ &= F(w) - F(z_1) + \int_\eta f(z)\,dz. \end{aligned}$$

We are now ready to bound our difference quotient: by Proposition 4.35, we see

$$\begin{aligned} \left|\frac{F(z_1) - F(w)}{z_1 - w} - f(z_1)\right| &= \left|\frac{1}{z_1 - w}\int_\eta f(z)\,dz - f(z_1)\right| \\ &= \left|\int_0^1 \frac{f((1-t)w + tz_1)}{z_1 - w}\cdot(z_1 - w)\,dt - f(z_1)\right| \\ &= \left|\int_0^1 f((1-t)w + tz_1) - f(z_1)\,dt\right| \\ &= \sup_{t\in[0,1]}\left\{\left|f((1-t)w + tz_1) - f(z_1)\right|\right\}. \end{aligned}$$

Now, as $w \to z_1$, we see that $f((1-t)w + tz_1)$ will be forced to approach $f(z_1)$ by continuity of f, bounded uniformly by w, so the quantity approaches 0. More rigorously, for any $\varepsilon > 0$, choose $\delta < r$ so that $|z' - z_1| < \delta$ implies $|f(z') - f(z_1)| < \varepsilon$. Then any w with $|w - z_1| < \delta$ will have $|(1-t)w + tz_1 - z_1| < \delta$ as well, so

$$\sup_{t\in[0,1]}\left\{\left|f((1-t)w + tz_1) - f(z_1)\right|\right\} \le \varepsilon$$

by taking the supremum everywhere. Sending $\varepsilon \to 0$ gives the result. ■

4.6 March 18

This lecture was recorded.

4.6.1 Proving the Cauchy Integral Formula

Today we finish the proof of the Cauchy integral formula. Recall the statement.

Theorem 4.63 (Cauchy integral formula)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and some $z_0 \in \Omega$ with $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$. Further, fix the path $\gamma\colon [0,1] \to \Omega$ given by

$$\gamma(t) := z_0 + r\exp(2\pi i t).$$

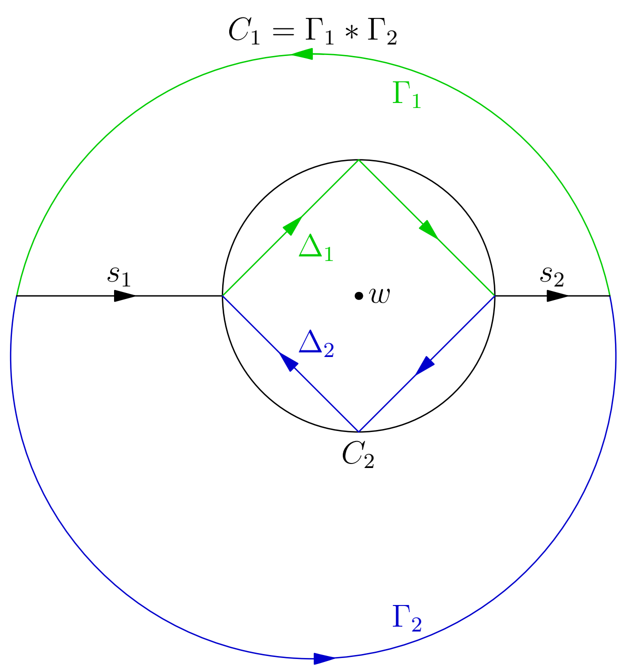
Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic, then any $w \in B(z_0, r)$ has

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \operatorname{Ind}_f(\gamma, w).$$

Proof. As needed, choose $w \in B(z_0, r)$, which is open, so we choose any $\varepsilon > 0$ such that $\overline{B(w,\varepsilon)} \subseteq B(z_0, r)$. As such, we set

$$C_1 := \partial B(z_0, r) = \operatorname{im}\gamma \qquad \text{and} \qquad C_2 = \partial B(w, \varepsilon).$$

Now, the main trick in the proof will be the following image, which will turn the integral around γ into a more controlled (and small!) square.



We will not spend the time to rigorously define what the various paths are, but we will list their properties.

- The concatenation $\Gamma_1 * \Gamma_2$ fully covers the circle C_1.
- The concatenation $\Delta_1 * \Delta_2$ creates a square around w whose vertices are $\{w-\varepsilon, w+i\varepsilon, w+\varepsilon, w-i\varepsilon\}$, in that order.

- The segments s_1 and s_2 are parallel to the real axis such that s_1 intersects C_1 ("on the left") and $w-\varepsilon$. Similarly, s_2 intersects C_1 ("on the right") and $w+\varepsilon$.
- The path Γ_1 starts where s_2 ends and ends where s_1 begins. Similarly, the path Γ_2 starts where s_1 begins and ends where s_2 ends.

Now, as promised, we move from an integral around γ to an integral around the square $\Delta_1 * \Delta_2$. In particular, we set $\widetilde{\gamma}\colon [0,1] \to \Omega$ to be a reparameterization of $(\Delta_1 * \Delta_2)^-$, and we will transfer the integral around γ into an integral around $\widetilde{\gamma}$.

For this, we use the work we did last class. Recall the following statement.

Theorem 4.70. Fix an open, connected, star-like subset $\Omega \subseteq \mathbb{C}$ with respect to z_0. Further, fix a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$. Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic,

$$\oint_\gamma f(z)\,dz = 0.$$

With this in mind, we set

$$\gamma_1 := \Gamma_1 * s_1 * \Delta_1 * s_2 \qquad \text{and} \qquad \gamma_2 := \Gamma_2 * s_2^- * \Delta_2 * s_1^-$$

to be closed paths, more or less representing the green and blue halves of our drawn contours. (These concatenations are well-defined and are closed by the chosen orientations of our paths.) In particular, applying our rules from Lemma 4.33 and Lemma 4.32, we see that

$$\begin{aligned}\oint_{\gamma_1} \frac{f(z)}{z-w}\,dz + \oint_{\gamma_2} \frac{f(z)}{z-w}\,dz &= \int_{\Gamma_1} \frac{f(z)}{z-w}\,dz + \int_{s_1} \frac{f(z)}{z-w}\,dz + \int_{\Delta_1} \frac{f(z)}{z-w}\,dz + \int_{s_2} \frac{f(z)}{z-w}\,dz \\ &\quad + \int_{\Gamma_2} \frac{f(z)}{z-w}\,dz - \int_{s_s} \frac{f(z)}{z-w}\,dz + \int_{\Delta_2} \frac{f(z)}{z-w}\,dz - \int_{s_1} \frac{f(z)}{z-w}\,dz \\ &= \int_{\Gamma_1 * \Gamma_2} \frac{f(z)}{z-w}\,dz + \int_{\Delta_1 * \Delta_2} \frac{f(z)}{z-w}\,dz \\ &= \oint_\gamma \frac{f(z)}{z-w}\,dz - \oint_{\widetilde{\gamma}} \frac{f(z)}{z-w}\,dz, \end{aligned} \tag{1}$$

where in the last step we have reparameterized (twice), as in Lemma 4.29. The negative sign in front of $\oint_{\widetilde{\gamma}}$ occurs because $\widetilde{\gamma}$ is a reparameterization of $(\Delta_1 * \Delta_2)^-$; pictorially, $\widetilde{\gamma}$ is counterclockwise.

We now finish by brute force. Note that the function $\frac{f(z)}{z-w}$ is a quotient of holomorphic functions on $\Omega \setminus \{w\}$ is holomorphic itself. Even though we cannot immediately apply Theorem 4.70 to $\Omega \setminus \{w\}$, we can apply it to the regions interior to γ_1 and γ_2; i.e., the top and bottom parts of $B(z_0, r) \setminus \overline{B(w,\varepsilon)}$, respectively. Both of these regions are star-like[5] (as witnessed by $w + i\varepsilon$ and $w - i\varepsilon$, respectively) because s_1 and s_2 are collinear and on opposite sides of our square, so Theorem 4.70 implies

$$\oint_{\gamma_1} \frac{f(z)}{z-w}\,dz = \oint_{\gamma_2} \frac{f(z)}{z-w}\,dz = 0 + 0 = 0.$$

As such, (1) tells us that

$$\oint_\gamma \frac{f(z)}{z-w}\,dz = \oint_{\widetilde{\gamma}} \frac{f(z)}{z-w}\,dz.$$

So we have indeed transformed our integral around γ into an integral around a square $\widetilde{\gamma}$. Observe that we can even make $\varepsilon > 0$ smaller and maintain the above equality.

[5] Technically, we should expand out the regions by a very small amount δ in order to make these regions also open and containing γ_1 and γ_2, but we will not bother to do this in any rigorous way.

We now run our computation of the integral around the square. We see

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)}{z-w} &= \frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)-f(w)}{z-w}\,dz+\frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(w)}{z-w}\,dz \\ &= \frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)-f(w)}{z-w}\,dz+f(w)\operatorname{Ind}(\widetilde{\gamma},w) \\ &= \frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)-f(w)}{z-w}\,dz+f(w), \qquad (2)\end{aligned}$$

where we have computed the winding number as in Lemma 4.54. Notably, our winding number is $+1$, perhaps by plugging into the definition via Lemma 4.46 because the normalized version of $\widetilde{\gamma}$ is just a circle, so the corresponding $\theta_{\widetilde{\gamma}}$ can be set to $\theta_0+2\pi t$ for some starting value θ_0. We will not make this more rigorous because look at it.

We now send $\varepsilon\to 0$, which will send $\frac{f(z)-f(w)}{z-w}\to f'(w)$ by definition of the derivative. More rigorously, for any $\varepsilon_0>0$, there exists $\varepsilon>0$ so that $|z-w|<\varepsilon$ implies

$$\left|\frac{f(z)-f(w)}{z-w}-f'(w)\right|<\varepsilon_0,$$

so Proposition 4.35 tells us that

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)-f(w)}{z-w}\,dz\right| &\le \frac{1}{2\pi}\cdot\sup_{t\in[0,1]}\left\{\left|\frac{f(\widetilde{\gamma}(z))-f(w)}{\widetilde{\gamma}(z)-w}\right|\right\}\cdot\ell(\widetilde{\gamma}) \\ &\le \frac{1}{2\pi}\cdot\big(|f'(w)|+\varepsilon_0\big)\cdot 2\pi\varepsilon,\end{aligned}$$

where in the last step we have bounded both the difference quotient and $\widetilde{\gamma}$ by the circumference of the circumscribed circle. Thus, sending $\varepsilon\to 0$ will force the entire integral to vanish, so we find from (2) that

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z-w}\,dz=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)}{z-w}\,dz=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{\widetilde{\gamma}}\frac{f(z)-f(w)}{z-w}\,dz+f(w)=f(w),$$

which is what we wanted. ■

4.6.2 Applications of the Cauchy Integral Formula

As a first application, we extend Corollary 4.64.

Corollary 4.71. Fix an open, connected subset $\Omega\subseteq\mathbb{C}$ and $f\colon\Omega\to\mathbb{C}$ some holomorphic function. Then f is analytic at any $z_0\in\Omega$. In fact, for any $r>0$ such that $\overline{B(z_0,r)}\subseteq\Omega$, the path

$$\gamma(t):=z_0+r\exp(2\pi it)$$

gives

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-w)^{n+1}}\,dz.$$

Proof. By Theorem 4.63, we know that

$$f(w)=\operatorname{Ind}_f(\gamma,w),$$

for any $w\in B(z_0,r)$. Now, applying Proposition 4.61, we see that

$$f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-z_0)^{n+1}}\right)(z-z_0)^n$$

for z in some open ball around z_0, which is our local power series expansion. Now, inductively throwing Proposition 3.44 at this power series, we see that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

which is what we wanted. ■

And here is another one.

Theorem 4.72 (Morera)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ such that $f\colon \Omega \to \mathbb{C}$ is continuous. Further, suppose that every closed, piecewise C^1 path $\gamma\colon [a,b] \to \Omega$ has

$$\oint_\gamma f(z)\,dz = 0.$$

Then f is holomorphic.

Proof. By Theorem 4.44 tells us that f has a primitive F on Ω. In particular, F is holomorphic on Ω (with $F' = f$) and therefore F is analytic by Corollary 4.71, so $f = F'$ is analytic by Lemma 3.51 and therefore holomorphic by Proposition 3.49. ■

Remark 4.73. I think a strengthening of Theorem 4.44 can show that we merely need to check

$$\oint_\gamma f(z)\,dz = 0$$

for C^1 paths γ.

4.6.3 Primitive Domains

To close our lecture, we build a little theory on domains.

Definition 4.74 (Domain)**.** A subset $\Omega \subseteq \mathbb{C}$ is a *domain* if and only if Ω is open and connected.

Definition 4.75 (Primitive domain)**.** A domain $\Omega \subseteq \mathbb{C}$ is a *primitive domain* if and only if every holomorphic function $f\colon \Omega \to \mathbb{C}$ admits a primitive.

Example 4.76. Star-like domains are primitive because we constructed a primitive for each holomorphic $f\colon \Omega \to \mathbb{C}$ by hand in the proof of Theorem 4.70. Alternatively, we can more directly just apply Theorem 4.70 and then Theorem 4.44 backwards to get our primitive.

Here is a quick reason why we might care about this definition.

Lemma 4.77. Fix a primitive domain $\Omega \subseteq \mathbb{C}$ and some holomorphic function $f\colon \Omega \to \mathbb{C}$. Then, given a closed, piecewise C^1 path $\gamma\colon [a,b] \to \mathbb{C}$, we have

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. Because Ω is a primitive domain, f admits a primitive. Then Corollary 4.39 finishes. ■

And here is the technical result we will need.

Lemma 4.78. Fix primitive domains $\Omega_1, \Omega_2 \subseteq \mathbb{C}$. Further, suppose that $\Omega_1 \cap \Omega_2$ is nonempty and connected. Then $\Omega_1 \cup \Omega_2$ is a primitive domain.

Proof. By Lemma 2.46, we see that $\Omega_1 \cup \Omega_2$ is connected, and because both these sets are open, we see that $\Omega_1 \cup \Omega_2$ is in fact a domain as well.

It remains to show that $\Omega_1 \cup \Omega_2$ is in fact a primitive domain. Well, fix any holomorphic function $f\colon (\Omega_1 \cup \Omega_2) \to \mathbb{C}$. For brevity, set $f_1 := f|_{\Omega_1}$ and $f_2 := f|_{\Omega_2}$ so that $f_1\colon \Omega_1 \to \mathbb{C}$ and $f_2\colon \Omega_2 \to \mathbb{C}$ are both holomorphic by restriction.

Thus, because Ω_1 and Ω_2 are both primitive domains, we are promised primitives F_1 and F_2 for f_1 and f_2 respectively. In particular,

$$F_1' = f_1 \qquad \text{and} \qquad F_2' = f_2.$$

It remains to stitch these together to create a single primitive for f. Well, $\Omega_1 \cap \Omega_2$ is also open and connected (as the intersection of open and connected sets) and hence a domain, and we note

$$(F_1 - F_2)'(z) = F_1'(z) - F_2'(z) = f(z) - f(z) = 0$$

for any $z \in \Omega_1 \cap \Omega_2$. In particular, $F_1 - F_2$ is constant on $\Omega_1 \cap \Omega_2$ by Lemma 2.46; note that here is where we use the condition that $\Omega_1 \cap \Omega_2$ is connected! So we set $(F_1 - F_2)(z) =: c$ for some $c \in \mathbb{C}$.

We now note that $F_2 + c$ will be a primitive for f on Ω_2 because

$$(F_2 + c)' = F_2' + c' = f.$$

With this in mind, we define $F\colon (\Omega_1 \cup \Omega_2) \to \mathbb{C}$ by

$$F(z) := \begin{cases} F_1(z) & z \in \Omega_1, \\ F_2(z) + c & z \in \Omega_2. \end{cases}$$

Note this is well-defined because $z \in \Omega_1 \cap \Omega_2$ has $F_1(z) = F_2(z) + c$. We can then check that

$$(F|_{\Omega_1})' = F_1' = f \qquad \text{and} \qquad (F|_{\Omega_2})' = (F_2 + c)' = f,$$

which is what we wanted. ■

After spring break, we prove some more consequences of the Cauchy integral formula.

4.7 March 28

Welcome back from spring break, everybody. Homework #7 has been released and is due on Sunday.

4.7.1 Liouville's Theorem

Today we are discussing consequences of the Cauchy integral formula. Here is the statement.

Theorem 4.63 (Cauchy integral formula)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and some $z_0 \in \Omega$ with $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$. Further, fix the path $\gamma\colon [0,1] \to \Omega$ given by

$$\gamma(t) := z_0 + r\exp(2\pi i t).$$

Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic, then any $w \in B(z_0, r)$ has

$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - w}\, dz = \operatorname{Ind}_f(\gamma, w).$$

Remark 4.79. There are two ways to read this: we could either try to evaluate f at w via the integral, or we could be handed an integral that looks something like the right-hand side and then compute by evaluating f at w.

As one immediate consequence, we showed that holomorphic functions are analytic.

Let's see another consequence.

Theorem 4.80 (Liouville's)**.** Fix an entire function $f\colon \mathbb{C} \to \mathbb{C}$. If f is bounded, then f is constant.

Proof. This isn't too hard. Because f is bounded, we are promised a real number $M \in \mathbb{R}^+$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$. Fix some $w \in \mathbb{C}$, and choose any r so that $r > |w|$. The idea is to take r very large in the Cauchy integral formula to show that $f(w) = f(0)$; for now, we innocently define $\gamma_r\colon [0,1] \to \mathbb{C}$ by

$$\gamma_r(0) := r\exp(2\pi i t)$$

as tracing the boundary of $B(0,r)$. In particular, our $w \in \mathbb{C}$ with $|w| < r$ (i.e., $w \in B(0,r)$) will have

$$|z-w| \geq |r-|w||$$

for any $z \in \operatorname{im}\gamma_r$.[6] We will show that $f(w) = f(0)$ by the Cauchy integral formula: by Theorem 4.63, we have

$$\begin{aligned}
|f(w)-f(0)| &= \left|\frac{1}{2\pi i}\oint_{\gamma_r}\left(\frac{f(z)}{z-w} - \frac{f(z)}{z}\right)dz\right| \\
&= \left|\frac{1}{2\pi i}\oint_{\gamma_r}\frac{wf(z)}{z(z-w)}\,dz\right| \\
&\leq \frac{1}{2\pi}\cdot \ell(\gamma_r)\cdot \sup_{z\in\operatorname{im}\gamma_r}\left\{\left|\frac{wf(z)}{z(z-w)}\right|\right\},
\end{aligned}$$

where we have applied Proposition 4.35 in the last step. Further, $\ell(\gamma_r) = 2\pi r$ because we are tracking out a circle. And lastly, we note that any $z \in \operatorname{im}\gamma_r$ will have

$$\left|\frac{wf(z)}{z(z-w)}\right| \leq \frac{|w|\cdot M}{r\cdot(r-|w|)},$$

so

$$|f(w)-f(0)| \leq \frac{1}{2\pi}\cdot 2\pi r\cdot\frac{|w|\cdot M}{r\cdot(r-|w|)} = \frac{|w|\cdot M}{r-|w|}.$$

Now, taking $r \to \infty$ will have

$$\frac{|w|\cdot M}{r-|w|} = 0,$$

so $f(w) = f(0)$ follows. Thus, f is indeed constant. ■

And now we can use Liouville's theorem for fun and profit.

Theorem 4.81 (Fundamental theorem of algebra)**.** Fix a polynomial $p(z) \in \mathbb{C}[z]$ of degree $n > 0$. Then p has a root in $\mathbb{C}$.

Proof. Let our polynomial be

$$p(z) = \sum_{k=0}^{n} a_k z^k.$$

[6] This is from the triangle inequality: note $|z-w| + |w| \geq |z| = r$ and $|w-z| + r \geq |w|$.

Note that $p(w) = 0$ if and only if $\frac{1}{a_n}p(w) = 0$, so for psychological reasons, we will replace $p(z)$ with $\frac{1}{a_n}p(z)$. In other words, we will simply assume that $a_n = 1$ and set

$$q(z) = \sum_{k=0}^{n-1} a_k z^k$$

so that $p(z) = z^n + q(z)$.

Now, suppose that p has no roots, and we will show that p is constant via Theorem 4.80; i.e., $p(z) \neq 0$ for any $z \in \mathbb{C}$. Then Proposition 3.6 tells us that $f(z) := \frac{1}{p(z)}$ is holomorphic no $\mathbb{C}$ (i.e., entire). We claim that f is bounded on $\mathbb{C}$. Well, by the triangle inequality again, we see

$$||z^n| - |q(z)|| \leq |p(z)|,$$

so

$$|f(z)| \leq \frac{1}{||z|^n - |q(z)||}.$$

But now, by sending $|z| \to \infty$, we may assume that $z \neq 0$ for $|z|$ sufficiently large, so

$$|f(z)| \leq \frac{1}{||z| - |q(z)/z^{n-1}||},$$

which goes to 0 as $|z| \to \infty$. As such, $f(z)$ is bounded and hence constant by Theorem 4.80, so $p(z) = \frac{1}{f(z)}$ is also bounded and hence constant. Note $f(z) \neq 0$ because $f(z) = \frac{1}{p(z)}$ everywhere. ■

Remark 4.82. This proof is somewhat non-constructive, in that we have no idea what the root is.

Remark 4.83. By inducting, we can show that p has exactly n roots, counted with multiplicity.

4.7.2 Poles and Zeroes Preview

Here is another result.

Theorem 4.84 (Riemann removable singularity)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$, and pick up some $z_0 \in \Omega$. If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then f extends to a holomorphic function on Ω.

Proof. We will construct $f(z_0)$ explicitly by starting with a function fully holomorphic on Ω, which we will then use to derive $f(z_0)$. In particular, we define $h: \Omega \to \mathbb{C}$ by

$$h(z) := \begin{cases} (z - z_0)^2 f(z) & z \neq z_0, \\ 0 & z = z_0. \end{cases}$$

Quickly, we claim that h is holomorphic on Ω. Because $h|_{\Omega \setminus \{z_0\}}(z) = (z-z_0)^2 f(z)$ is a product of holomorphic functions, we conclude that h is holomorphic on $\Omega \setminus \{z_0\}$. Thus, we merely have to check that h is holomorphic at z_0. In particular, we compute

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)(z - z_0)^2}{z - z_0} = \lim_{z \to z_0} f(z)(z - z_0) = 0,$$

where the last step is because f is bounded near z_0.

Now, h is holomorphic on Ω, so h is analytic on Ω by Corollary 4.71, so we are promised a local power series expansion at z_0: there are coefficients $\{a_k\}_{k\in\mathbb{N}} \subseteq \mathbb{C}$ with an $r > 0$ such that $z \in B(z_0, r)$ will have

$$h(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Quickly, we see that $a_0 = h(0) = 0$ and $a_1 = h'(0) = 0$ (by Corollary 4.71). Thus, we may write

$$f(z) = \frac{h(z)}{(z - z_0)^2} = \sum_{k=0}^{\infty} a_{k+2} (z - z_0)^k \tag{$*$}$$

for any $z \in B(z_0, r) \setminus \{z_0\}$. However, if we define $\widetilde{f} \colon \Omega \to \mathbb{C}$ by

$$\widetilde{f}(z) := \begin{cases} a_2 & z = z_0, \\ f(z) & z \neq z_0, \end{cases}$$

then $\widetilde{f}$ is holomorphic on $\Omega \setminus \{z_0\}$ by restriction and analytic at z_0 by $(*)$, so $\widetilde{f}$ is the holomorphic extension of f to Ω. ■

We close with one more statement.

Proposition 4.85. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a holomorphic function $f \colon \Omega \to \mathbb{C}$. Further, define $Z := f^{-1}(\{0\})$.

(a) If $z_0 \in Z$, then either z_0 is isolated, or there is some open neighborhood of z_0 in Z.

(b) If z_0 is isolated, then there is a unique integer n and holomorphic function $g \colon \Omega \to \mathbb{C}$ with $g(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^n g(z)$$

for $z \in \Omega$.

Proof. We will prove this next class. ■

THEME 5
SMOOTHING OVER

What we didn't do is make the construction at all usable in practice!
This time we will remedy this.

—Kiran S. Kedlaya, [Ked21]

5.1 March 30

Good morning everyone.

- Homework #7 is due on Sunday just before midnight.
- There will be office hours tomorrow from 2PM to 3:30PM, as usual.

5.1.1 More on Zeroes

We are talking about more consequences of the Cauchy integral formula. For example, last time we showed Liouville's theorem, the Fundamental theorem of algebra, and the Riemann removable singularity theorem. We are also about to show the following result.

Proposition 4.85. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a holomorphic function $f\colon \Omega \to \mathbb{C}$. Further, define $Z := f^{-1}(\{0\})$.

(a) If $z_0 \in Z$, then either z_0 is isolated, or there is some open neighborhood of z_0 in Z.

(b) If z_0 is isolated, then there is a unique integer n and holomorphic function $g\colon \Omega \to \mathbb{C}$ with $g(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^n g(z)$$

for $z \in \Omega$.

Proof. Fix some $z_0 \in Z$. Now, because f is holomorphic, f is analytic at z_0 (by Corollary 4.71), so we have some $r > 0$ such that $\overline{B(z_0, r)} \subseteq \Omega$ with

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad (*)$$

for any $z \in B(z_0, r)$. It is technically possible for $a_k = 0$ for all $k \in \mathbb{N}$. But now, f is zero on all of $B(z_0, r)$, which is one possibility for part (a). This is all we are going to say about this case.

Otherwise, let n be the minimum natural number such that $a_n \neq 0$. As such, we simply define $g\colon \Omega \to \mathbb{C}$ as

$$g(z) := \begin{cases} f(z)/(z-z_0)^n & z \neq z_0, \\ a_n & z = z_0. \end{cases}.$$

This function is at least holomorphic at all points away from z_0 as the quotient of two holomorphic functions (by Proposition 3.6), so we merely need to check that g is holomorphic at z_0. However, on $B(z_0, r)$, we see that $z \neq z_0$ will have

$$g(z) = \frac{f(z)}{(z-z_0)^n} = \sum_{k=n}^{\infty} a_k (z-z_0)^n$$

by $(*)$. But of course, this also works at $g(z_0) = a_n$, so we see that the above power series expansion works for all $z \in B(z_0, r)$. So g is in fact analytic at z_0 and hence holomorphic at z_0 by $(*)$.

We now show that z_0 is an isolated point of Z. Well, $g(z_0) \neq 0$ and g is continuous (in fact holomorphic), we are promised some $\varepsilon > 0$ such that

$$|g(z) - g(z_0)| > |g(z_0)|$$

for all $z \in B(z_0, \varepsilon)$, so in particular $g(z) \neq 0$ here. Thus, when we write

$$f(z) = (z-z_0)^n g(z),$$

the only time we can have $f(z) = 0$ for $z \in B(z_0, \varepsilon)$ is at $z = z_0$ because $z \neq z_0$ implies $(z-z_0)^n \neq 0$ and $g(z) \neq 0$.

Lastly, we get the uniqueness of the integer n follows from its minimality. ■

To use the above result, we show one of my personal favorite results from complex analysis.

Theorem 5.1 (Identity). Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ with two holomorphic functions $f_1, f_2\colon \Omega \to \mathbb{C}$. Further, set

$$Z := \{z \in \Omega : f_1(z) = f_2(z)\}.$$

If Z contains an accumulation point, then $f_2 = f_2$ on Ω.

Proof. For psychological reasons, we set $f(z) := f_1(z) - f_2(z)$ so that $z \in Z$ if and only if $f_1(z) = f_2(z)$ if and only if $f(z) = 0$. So $Z = f^{-1}(\{0\})$, and we are ripe to apply the previous result.

Now, we are granted an accumulation point $w \in Z$, so we have some sequence $\{z_k\}_{k\in\mathbb{N}} \subseteq Z \setminus \{w\}$ such that $z_k \to n$. In particular, w is not isolated: for every open neighborhood $B(w, \varepsilon)$ around w, the fact that $z_k \to n$ promises that $B(w, \varepsilon) \cap (Z \setminus \{w\}) \neq \varnothing$.

Thus, Proposition 4.85 kicks in, so there exists some $r > 0$ such that $f(z) = 0$ for all $B(w, r)$, so $B(w, r) \subseteq Z$. In other words, every accumulation point of Z is contained in the interior of Z, which we will denote Z°.

As such, we claim that Z° is closed. Quickly, note that $Z = f^{-1}(\{0\})$ is the pre-image of a closed set and hence closed by Lemma 2.92 because f is continuous. In particular, if α is any limit point of Z°, then α is an accumulation point of Z (because Z is closed), so $\alpha \in Z^\circ$.

Thus, Z° is indeed closed. But it is also open, so the connectivity of Ω forces $Z^\circ = \varnothing$ or $Z^\circ = \Omega$. But Z° is nonempty because we have an accumulation point, so $Z^\circ = \Omega$, so we are done. ■

Remark 5.2. This is really something special about holomorphic functions. For example, the function

$$f(z) = \begin{cases} e^{-1/z^2} & z > 0, \\ 0 & z \leq 0, \end{cases}$$

is real analytic everywhere, and it agrees with the zero function on $\mathbb{R}_{\leq 0}$, but of course f is nonzero.

We remark that the value of n in Proposition 4.85 is somewhat special.

Definition 5.3 (Multiplicity)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a holomorphic function $f\colon \Omega \to \mathbb{C}$. If we have some $z_0 \in \Omega$ such that $f(z_0) = 0$ and z_0 is isolated in $f^{-1}(\{0\})$, then by Proposition 4.85 there is a unique integer n and holomorphic function $g\colon \Omega \to \mathbb{C}$ with $g(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^n g(z).$$

This n is called the *multiplicity* of z_0 in f.

We actually know how to compute f because Proposition 4.85 is fully constructive: we simply expanded out the power series expansion of f at z_0 as

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

and then looked for the minimal n such that $a_n \neq 0$. However, we also know that these coefficients of the power series can be computed via the proof of Corollary 3.45 as

$$f^{(m)}(z_0) = m! a_m,$$

so we can alternatively look for the minimal n such that $f^{(n)}(z_0) \neq 0$.

Example 5.4. By Lemma 3.69, we computed

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k+1)!} z^{2k+1}.$$

We can check that $\sin 0 = 0$ while the linear term is nonzero, so we have multiplicity 1. Alternatively, we can compute the first derivative as

$$\sin'(0) = \cos(0) = 1 \neq 0.$$

5.1.2 More on Zeroes

We close by stating a theorem.

Theorem 5.5 (Maximum modulus principle)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a non-constant holomorphic function $f\colon \Omega \to \mathbb{C}$. For each $z \in \Omega$ and $r > 0$, there exists $w \in B(z, r) \cap \Omega$ such that

$$|f(w)| > |f(z)|.$$

Proof. We proceed by contraposition. Fix some $z_0 \in \Omega$ and $r > 0$ such that $w \in B(z_0, r) \cap \Omega$ has $|f(w)| \leq |f(z_0)|$. Note that making r smaller merely makes our search space smaller, so we may take r small enough so that $B(z_0, r) \subseteq \Omega$.

Further, note that if f is constant on $B(z_0, r)$, then f is constant on all of Ω because f will agree with a constant function on the set $B(z_0, r)$—which contains a limit point—forcing f to be constant on all of Ω by Theorem 5.1. Thus, it suffices to show that f is constant on $B(z_0, r)$.

As such, we think of f as a function on $B(z_0, r)$ such that $|f(z_0)| \geq |f(w)|$ for each $w \in \Omega$, and we want to show that f is constant. Very quickly, if $|f(z_0)| = 0$, then we get $f \equiv 0$ automatically, so we assume $f(z_0) \neq 0$. As such, we can replace f with $f(z)/f(z_0)$, which lets us assume that $f(z_0) = 1$.

The key point here, is to use Theorem 4.63 to note

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\, dz$$

for $\gamma_\varepsilon\colon [0,1] \to \mathbb{C}$ defined by $\gamma_\varepsilon(t) := z_0 + \varepsilon\exp(2\pi it)$, for any $\varepsilon \in (0,r)$. The main idea, then, is that the numerator $f(z)$ is in some sense "too small" to actually accumulate properly to $f(z_0) = 1$, especially if $f(z)$ ever deviates from exactly 1.

To be able to keep track of deviations in direction, we see that Lemma 4.46 promises us a function θ_ε with

$$(f \circ \gamma_\varepsilon)(t) = |f(\gamma(t))| \exp(2\pi i\theta_\varepsilon(t)).$$

Thus, we compute

$$\begin{aligned} 1 &= \frac{1}{2\pi i} \oint_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^1 \frac{|f(\gamma(t))| \exp(2\pi i\theta_\varepsilon(t))}{\exp(2\pi it)} \cdot 2\pi i \exp(2\pi it)\, dt \\ &= \int_0^1 |f(\gamma(t))| \exp(2\pi i\theta_\varepsilon(t))\, dt. \end{aligned}$$

In particular, extracting out the real part from the integral forces

$$1 = \int_0^1 |f(\gamma(t))| \cos(2\pi\theta_\varepsilon(t))\, dt.$$

Bounding the integral in $\mathbb{R}$, we see

$$\int_0^1 |f(\gamma(t))| \cos(2\pi\theta_\varepsilon(t))\, dt \le (1-0) \cdot \max_{t\in[0,1]} \left\{|f(\gamma(t))| \cos(2\pi\theta_\varepsilon(t))\right\} \le 1,$$

where equality is now holding only when $|f(\gamma(t))| \cos(2\pi\theta_\varepsilon(t)) = 1$ for all $t \in [0,1]$.[1] In particular, we need $|f(\gamma(t))| = 1$ exactly, and we also need $\theta_\varepsilon(t) \equiv 0 \pmod{2\pi}$ for all t, so in particular,

$$(f \circ \gamma_\varepsilon)(t) = |f(\gamma(t))| \exp(2\pi i\theta_\varepsilon(t)) = 1$$

always. Now, because any $z \in B(z_0, r) \setminus \{z_0\}$ can be written in polar form by

$$z - z_0 = \varepsilon \exp(i\theta)$$

for some $\varepsilon < r$ and some θ, we see that actually any $z \in B(z_0, r)$ will be forced to have $f(z) = 1$. This finishes the proof that f is constant. ■

Remark 5.6 (Nir). There are other ways to see this result. For example, it happens that holomorphic are open, so $f(B(z_0, r))$ must be open and in particular contains an open neighborhood around $f(z_0)$, and we can choose an output in this neighborhood smaller than $f(z_0)$ in magnitude.

The rough idea here is that f cannot obtain a maximum on an open set: we must always look to the boundary. More rigorously, we have the following statement.

Corollary 5.7. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a non-constant continuous function $f\colon \overline{\Omega} \to \mathbb{C}$ such that $f|_\Omega$ is holomorphic. Now, if $z_0 \in \overline{\Omega}$ such that $|f(z_0)|$ is maximal, then $z_0 \in \partial\Omega$.

Proof. If $z_0 \in \Omega$, then f is forced to be continuous on Ω by Theorem 5.5, which violates the hypothesis on f. Thus, we conclude $z_0 \in \overline{\Omega} \setminus \Omega$, which is $\partial\Omega$. ■

[1] The sharpness of these equalities really does need some continuity discussion. Roughly speaking, if we ever have a strict inequality $|f(\gamma(t_0))| \cos(2\pi\theta_\varepsilon(t_0)) < 1$, then we have strict inequality in some neighborhood around t_0, which we can track through to make the integral strictly less than 1. This argument is purely real analysis.

Corollary 5.8 (Schwarz's lemma)**.** Fix a holomorphic function $f\colon B(0,1)\to B(0,1)$ such that $f(0)=0$. Then actually $|f(z)|\le|z|$ for all $z\in B(0,1)$ and also $|f'(0)|\le 1$. Further, if $|f'(0)|=1$ or $|f(z)|=|z|$ for some nonzero $z\in B(0,1)$, then $f(z)=az$ for all $z\in\mathbb{C}$ for some fixed $a\in\mathbb{C}$.

Intuitively, holomorphic functions $B(0,1)\to B(0,1)$ must contract.

Proof. We will prove this next class. ■

5.2 April 1

Good morning everyone. It's April Fool's day.

- Homework #7 is still due on Sunday at 11:59PM.
- There are office hours today.

5.2.1 The Schwarz Lemma

We quickly review the following result.

Corollary 5.8 (Schwarz's lemma)**.** Fix a holomorphic function $f\colon B(0,1)\to B(0,1)$ such that $f(0)=0$. Then actually $|f(z)|\le|z|$ for all $z\in B(0,1)$ and also $|f'(0)|\le 1$. Further, if $|f'(0)|=1$ or $|f(z)|=|z|$ for some nonzero $z\in B(0,1)$, then $f(z)=az$ for all $z\in\mathbb{C}$ for some fixed $a\in\mathbb{C}$.

Proof. The main point is to use the Maximum modulus principle on a specially chosen holomorphic function. We define $g\colon B(0,1)\to\mathbb{C}$ as

$$g(z):=\begin{cases}f(z)/z & z\ne 0,\\ f'(0) & z=0.\end{cases}$$

As usual, we note that g is holomorphic: we are holomorphic at all $z\ne 0$ by restriction from $f(z)/z$, and we are in fact holomorphic at $z=0$ by doing a power series expansion there, by hand.

We now have two cases.

- Now, if g is constant, then $f(z)=az$ for each $z\in\mathbb{C}$, for some fixed $a\in\mathbb{C}$. We get $|f(z)|\le|z|$ because $|f(z)|\le 1$ forces $|a|\le 1$ (namely, by sending z to the boundary of $B(0,1)$).
- Otherwise, take g to be non-constant. To create a compact space, set $r\in(0,1)$ so that $\overline{B(0,r)}\subseteq B(0,1)$. Now, by compactness, we see that $|g|$ has a maximum on $\overline{B(0,r)}$, so Corollary 5.7 tells us that each of these r has a $w\in\partial B(0,r)$, so

  $$|g(z)|\le|g(w)|=\frac{|f(w)|}{|w|}\le\frac{1}{|w|}=\frac{1}{r}$$

  for all $z\in B(0,r)$. Now, sending $r\to 1$, we get the inequality $|g(z)|\le 1$ for all $z\in B(0,1)$, so $|f(z)|\le|z|$ follows.

The above casework finishes the first sentence of the proof.

We now show the second sentence. If $|f(z)|=|z|$ for some nonzero $z\in B(0,1)$, then g achieves 1 on its interior, which we know must be now be its maximum. So Theorem 5.5 forces g to be constant, giving the result. Otherwise, if $f'(0)=1$, then $g(0)=1$, so again g achieves its maximum in $B(0,1)$, so Theorem 5.5 still forces g to be constant. ■

Remark 5.9. The above result is approximately what lets us talk intelligently about automorphisms of $B(0,1)$.

5.2.2 Singularities

We will spend the rest of lecture today discussing singularities.

Definition 5.10 (Regular, singular)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$.

- A point $z_0 \in \overline{\Omega}$ is *regular* if and only if f is holomorphic at z_0.
- A point $z_0 \in \overline{\Omega}$ is a *singularity* otherwise.

Definition 5.11 (Isolated singularity)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$. A point $z_0 \in \overline{\Omega}$ is an *isolated singularity* if and only if we can find $r > 0$ with $B(z, r) \subseteq \mathbb{C}$ such that f is holomorphic on $B(z_0, r) \setminus \{z\}$.

- z_0 is *removable* if and only if f is bounded near z_0.
- z_0 is a *pole* if and only if f is not bounded near z_0, but z_0 is a removable singularity of $1/f(z)$.
- z_0 is an *essential singularity* if and only if z_0 is neither removable nor a pole.

Remark 5.12. Being a removable singularity means that we can extend f to be holomorphic at the point, by Theorem 4.84.

Here are some examples.

Example 5.13. The point $z_0 = 0$ is an isolated singularity of $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = \cos(z)/z^2$.

Example 5.14. The point $z_0 = 0$ is a removable singularity of $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = \sin(z)/z$, which we can check by bounding $\sin$ near 0.

Example 5.15. The function $e^{1/z}$ has an essential singularity at $z_0 = 0$.

The point of introducing these notions is to expand our study of holomorphic functions. We take the following definition.

Definition 5.16 (Meromorphic)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$. Then $f\colon \Omega \to \mathbb{C}$ is *meromorphic* if and only if all the singularities of f are isolated and poles.

The short version of where we are going is that meromorphic functions will also be very nice; for example, though they will not be literally power series at the singularity, they will be some power series with a finite negative tail, of sorts.

Anyway, we should probably prove something today.

Lemma 5.17. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$. If $z_0 \in \Omega$ is a pole of f, then

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

Intuitively, poles of f transfer to zeroes of $1/f$.

Proof. We expand out the definitions. By definition, z_0 is a removable singularity of $1/f(z)$, and because our singularity is removable, we are promised an open ball $B(z_0, r)$ so that f is nonzero at $B(z_0, r) \setminus \{z_0\}$, so we note that $1/f(z)$ will be holomorphic on this punctured ball.

Further, $1/f(z)$ is bounded near z_0, so Theorem 4.84 tells us that we can extend $1/f(z)$ to be holomorphic fully on $B(z_0, r)$, so we know that

$$w := frac1f(z_0) = \lim_{z\to z_0} \frac{1}{f(z)}$$

after extending $1/f$ appropriately. We want to show that $w = 0$. Well, suppose for the sake of contradiction that $w \neq 0$ so that we have

$$\lim_{z\to z_0} f(z) = \frac{1}{w}.$$

However, this contradicts the fact that f needs to not be bounded near z_0 because it does tell us that $(z - z_0)f(z) \to 0$ as $z \to z_0$. In particular, we are now invoking the fact that z_0 is a pole. ■

And here is the dual to this lemma.

Lemma 5.18. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ and $z_0 \in \Omega$ with a function $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ so that z_0 is an isolated singularity of f. Then z_0 is a pole of f if and only if

$$\lim_{z\to z_0} |f(z)| = \infty.$$

Proof. In the forward direction, z_0 being a pole forces

$$\lim_{z\to z_0} \frac{1}{|f(z)|} = 0$$

by Lemma 5.17. As such, we are forced to have

$$\lim_{z\to z_0} |f(z)| = \infty.$$

The backwards direction will require some effort. We need to show that z_0 is a removable singularity of $1/f$ and that f is not bounded near z_0. On one hand, we know

$$\lim_{z\to z_0} |f(z)| = \infty,$$

but then we can rearrange to

$$\lim_{z\to z_0} \frac{z - z_0}{f(z)} = 0,$$

so z_0 is indeed a removable singularity of $1/f$. On the other hand, suppose for the sake of contradiction that f is bounded near z_0; then Theorem 4.84 promises us that we can extend f to be holomorphic on Ω, and therefore we see

$$\lim_{z\to z_0} f(z)$$

exists. But then

$$\lim_{z\to z_0} \frac{1}{|f(z)|}$$

cannot be zero (it is either nonzero or not defined at all), which contradicts what we just showed. ■

5.2.3 Laurent Expansion

To deal with singularities, we have the following definition.

Definition 5.19 (Order)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f : \Omega \to \mathbb{C}$. Given a pole $z_0 \in \overline{\Omega}$ of f, we define the *order* of z_0 as a pole to equal the multiplicity of z_0 as a zero of $1/f(z)$.

Note that we are implicitly using Lemma 5.17.

We have the following lemma, which is intended to be analogous to the fact that holomorphic functions are analytic.

Lemma 5.20. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$, and suppose that z_0 is a pole of f with order $m > 0$. Then there exists any sufficiently small real number $r \in \mathbb{R}^+$ and a unique sequence $\{a_k\}_{k=-m}^{\infty} \subseteq \mathbb{C}$ such that $z \in B(z_0, r) \setminus \{z_0\}$ has

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k.$$

In particular, the order of our pole controls the length of our tail.

We will not prove the above result today, but we will give the parts names.

Definition 5.21 (Laurent expansion)**.** In the context of Lemma 5.20, the "power series" expansion

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k$$

is the *Laurent expansion* of f at z_0; here m is the order of the pole at z_0.

Definition 5.22 (Principal part)**.** In the context of Lemma 5.20, we call the negative tail

$$p_{f,z_0}(z) := \sum_{k=-m}^{-1} a_k (z - z_0)^k$$

the *principal part* of f at z_0.

Notably, the principal part is the "bad" part of our power series expansion.

Definition 5.23 (Residue)**.** In the context of Lemma 5.20, we call a_{-1} the *residue* of f at z_0, denoted $\operatorname{Res}_{z_0}(f)$.

Later on we will be able to compute residues via integrals.

5.3 April 4

Good morning everyone.

- Homework #8 is due on Friday at 11:59PM.
- Midterm #2 is on Friday, April 15th.
- Office hours on Thursday are in flux.

5.3.1 The Residue Theorem

Today we are talking about residues. We return to the following lemma.

Lemma 5.20. Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$, and suppose that z_0 is a pole of f with order $m > 0$. Then there exists any sufficiently small real number $r \in \mathbb{R}^+$ and a unique sequence $\{a_k\}_{k=-m}^{\infty} \subseteq \mathbb{C}$ such that $z \in B(z_0, r) \setminus \{z_0\}$ has

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k.$$

Proof. We symbol-shift. Because z_0 is a removable singularity for $1/f(z)$, Theorem 4.84 implies that we can extend $1/f(z)$ to z_0 in such a way that preserves being holomorphic. Further, Lemma 5.17 promises us that $1/f(z)$ goes to 0 at $z = z_0$. Namely, we can write

$$\frac{1}{f(z)} = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$ and m is the order of our pole, where we are using Proposition 4.85. Because $g(z_0) \neq 0$, we get a small neighborhood $B(z_0, \varepsilon)$ such that $g(z) \neq 0$ in this neighborhood (by continuity), so we can write

$$f(z) = (z - z_0)^{-m} \cdot \frac{1}{g(z)}$$

for $z \in B(0, \varepsilon)$. Now, setting $h(z) := 1/g(z)$, we see that $h(z)$ is holomorphic on $B(0, \varepsilon)$ by Proposition 3.6. Thus, h is holomorphic, so h is analytic at z_0 by Corollary 4.71, so we get a power series expansion

$$h(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in B(z_0, r)$, for some $r > 0$. Dividing out, we see that $z \in B(z_0, r) \setminus \{z_0\}$ will have

$$f(z) = (z - z_0)^{-m} \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=-m}^{\infty} a_{k+m} (z - z_0)^k,$$

which is what we wanted. ■

Now, here is our central definition today, which we introduced last class.

Definition 5.23 (Residue)**.** In the context of Lemma 5.20, we call a_{-1} the *residue* of f at z_0, denoted $\operatorname{Res}_{z_0}(f)$.

Here is the main result for today.

Theorem 5.24 (Residue)**.** Fix a primitive domain $\Omega \subseteq \mathbb{C}$ and some finite subset $S \subseteq \Omega$ such that we have a holomorphic function $f \colon \Omega \setminus S \to \mathbb{C}$, where S consists of the poles of f. Now, if $\gamma \colon [0, 1] \to \Omega$ is a closed, piecewise C^1 path such that $\operatorname{im} \gamma \cap S = \varnothing$, then

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{z_0 \in S} \operatorname{Res}_{z_0}(f) \operatorname{Ind}(\gamma, z_0).$$

Proof. We combine previous results. At a high level, we are going to fix f at all poles, and the process of "unfixing" the integrals will give rise to the residues. For each $z_0 \in S$, we take $p_{f,z_0}(z)$ to be the principal part of f at z_0, where $f(z) = \sum_{k=-m_w}^{\infty} a_{w,k}(z - z_0)^k$ is our Laurent expansion at z_0. The idea here is to kill all the "bad parts" of f: we set

$$g(z) := f(z) - \sum_{z_0 \in S} p_{f,z_0}(z).$$

We automatically know that g is holomorphic on $\Omega \setminus S$, and in fact, each $w \in S$ will have some power series expansion

$$g(z) = \sum_{k=-m_w}^{\infty} a_{w,k}(z - w)^k - \sum_{k=-m_w}^{-1} a_{w,k}(z - w)^k = \sum_{k=0}^{\infty} a_{w,k}(z - w)^k$$

in a neighborhood around w, so setting $g(w) := a_{w,0}$ makes g analytic and hence holomorphic at each $w \in S$. Thus, we can extend g to be holomorphic on all of Ω.

Now, because Ω is a primitive domain, so Lemma 4.77 tells us that

$$\oint_\gamma f(z)\,dz = \underbrace{\oint_\gamma g(z)\,dz}_{0} + \sum_{z_0\in S}\oint_\gamma p_{f,z_0}(z)\,dz = \sum_{z_0\in S}\oint_\gamma p_{f,z_0}(z)\,dz.$$

We now integrate by hand. Fix some $w \in S$, and we note that

$$\oint_\gamma p_{f,w}(z)\,dz = \sum_{k=-m_w}^{-1} a_k \oint_\gamma (z-z_0)^k\,dz.$$

Now, for $k \le -2$, we see that

$$\frac{d}{dz}\frac{(z-z_0)^{k+1}}{k+1} = (z-z_0)^k,$$

so the function $(z-z_0)^k$ has a primitive, so Corollary 4.39 promises us that

$$\oint_\gamma p_{f,w}(z)\,dz = \sum_{k=-m_w}^{-2} a_k \underbrace{\oint_\gamma (z-w)^k\,dz}_{0} + a_{-1}\oint_\gamma (z-w)^{-1}\,dz = \operatorname{Res}_w(f)\operatorname{Ind}(\gamma,w).$$

Thus, we conclude

$$\oint_\gamma f(z)\,dz = \sum_{z_0\in S} \operatorname{Res}_w(f)\operatorname{Ind}(\gamma,w),$$

which is what we wanted. ■

5.3.2 Computation with the Residue Theorem

The main point to Theorem 5.24 is that it helps us compute integrals, if only we could compute residues. So let's compute residues.

Lemma 5.25. Fix a domain Ω, and pick up a meromorphic function $f\colon \Omega\setminus S \to \mathbb{C}$ for some set S of the poles of f. Letting $z_0 \in S$ be a pole of order m, we get

$$\operatorname{Res}_{z_0}(f) = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right).$$

The point is that we can now compute residues in terms of derivatives, and we understand derivatives.

Proof. The main idea is to use the Laurent series expansion, turn it into a typical power series expansion, and then extract out the a_{-1} coefficient by hand. In particular, let our Laurent series expansion be

$$f(z) = \sum_{k=-m}^{\infty} a_k (z-z_0)^k$$

for z in some neighborhood $B(z_0,r)$ of z_0. Thus, we get

$$(z-z_0)^m f(z) = \sum_{k=0}^{\infty} a_{k-m}(z-z_0)^k$$

for each $z \in B(z_0,r)$. In particular,

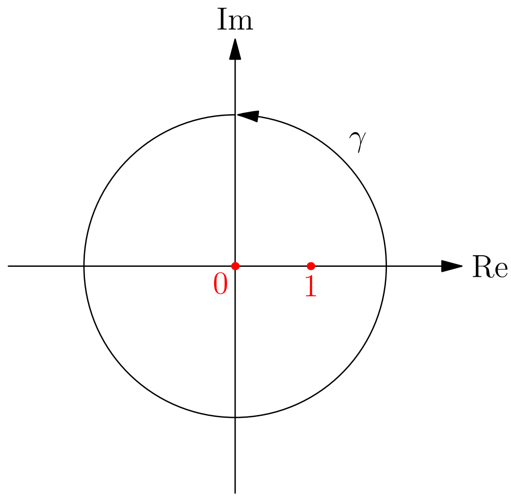
$$\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = \sum_{k=m-1}^{\infty} a_{k-m}\cdot k(k-1)\cdot\ldots\cdot(k-m+2)(z-z_0)^{k-m},$$

which is notably analytic at z_0 and hence holomorphic and hence continuous, so taking the limit at $z \to z_0$ recovers the value of a_{-1}. ■

Let's see some examples.

Exercise 5.26. We compute $\oint_{|z|=2} \frac{5z-2}{z(z-1)}\,dz$, where we are oriented counterclockwise around $\partial B(0,2)$.

Proof. Here is the image; we let our path be γ, and set $f(z) := \frac{5z-2}{z(z-1)}$.



In particular, we use Theorem 5.24 to get

$$\oint_\gamma f(z)\,dz = \sum_{w\in\{0,1\}} \operatorname{Res}_w(f)\operatorname{Ind}(\gamma,w).$$

Now, the poles of f are 0 and 1, and each have order 1 because $f(z)(z-w)$ is holomorphic in some neighborhood at w for each $w\in\{0,1\}$. Further, we see that $\operatorname{Ind}(\gamma,0)=\operatorname{Ind}(\gamma,1)=1$ from the image. It remains to compute the residues.

- At $z=0$, we see
$$\operatorname{Res}_0(f) = \lim_{z\to0}(z\cdot f(z)) = \lim_{z\to0}\frac{5z-2}{z-1} = 2.$$
- At $z=1$, we see
$$\operatorname{Res}_1(f) = \lim_{z\to1}((z-1)\cdot f(z)) = \lim_{z\to0}\frac{5z-2}{z} = 3.$$

In total, we see that

$$\oint_\gamma f(z)\,dz = \operatorname{Res}_0(f)\operatorname{Ind}(\gamma,0)+\operatorname{Res}_1(f)\operatorname{Ind}(\gamma,1) = 2\cdot1+3\cdot1 = \boxed{5},$$

so we are done. ■

5.4 April 6

Good morning everyone.

- Office hours tomorrow are still to be determined.
- Homework #8 is due on Friday at 11:59PM.
- It is Professor Morrow's birthday.

5.4.1 Homotopy

Today we enter the realm of algebraic topology. In particular, we are talking about homotopy because we want to talk about integration along "arbitrary" paths, but computing these can be potentially very annoying.

Theorem 4.8. For any path $\gamma\colon [a,b] \to \mathbb{C}$, there exists a sequence of piecewise C^1 paths $\{\gamma_k\}_{k\in\mathbb{N}}$ such that $\gamma_k \to \gamma$ uniformly.

As such, we have the following definition.

Definition 5.27 (Path integration)**.** Fix a domain $\Omega \subseteq \mathbb{C}$. Given a continuous function $f\colon \Omega \to \mathbb{C}$ and a path $\gamma\colon [0,1] \to \mathbb{C}$, let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of piecewise C^1 paths such that $\gamma_n \to \gamma$ uniformly. Then we define

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_{\gamma_n} f(z)\,dz.$$

Remark 5.28. Professor Morrow is not sure if this integral is well-defined.

Today we are going to talk about how we can vary paths and still be able to compute our integrals, provided that we are sufficiently careful. For example, we showed in Lemma 4.29 that we only care about paths up to equivalence, but it turns out that we can do better than this.

As such, we have the following definition.

Definition 5.29 (Homotopy)**.** Fix a domain Ω and two paths $\gamma, \eta\colon [0,1] \to \Omega$. Then a *homotopy* h between γ and η is a continuous map $h\colon [0,1]^2 \to \Omega$ such that

$$h(t,0) = \gamma(t) \qquad \text{and} \qquad h(t,1) = \eta(t).$$

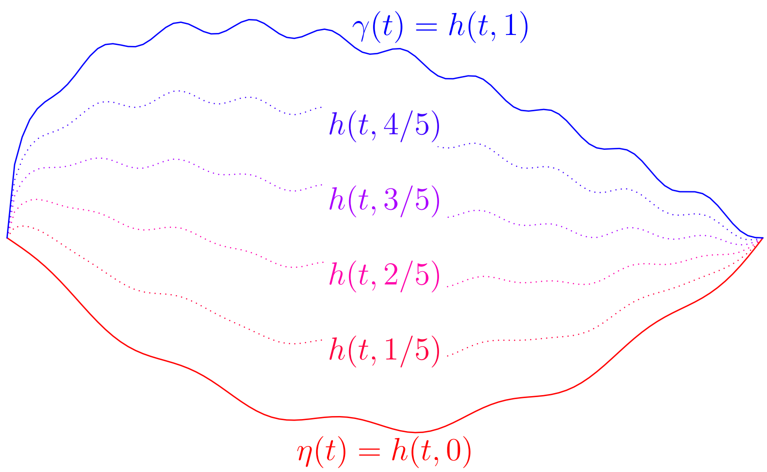
In this case, we say that γ and η are *homotopic*.

Definition 5.30 (Homotopic with fixed endpoints)**.** Fix a domain Ω and two paths $\gamma, \eta\colon [0,1] \to \Omega$. If $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$, and we have a homotopy $h\colon [0,1]^2 \to \Omega$ such that

$$h(0,t) = \gamma(0) = \eta(0) \qquad \text{and} \qquad h(1,t) = \gamma(1) = \eta(1)$$

for all t.

We now provide the obligatory picture of a homotopy, here with fixed endpoints.



Namely, the idea is that we can continuously move from one path to the other, and $h(-,s)$ is telling us how to do that.

Example 5.31. For $r \in \mathbb{R}^+$, we define $\gamma_r \colon [0,1] \to \mathbb{C}$ by

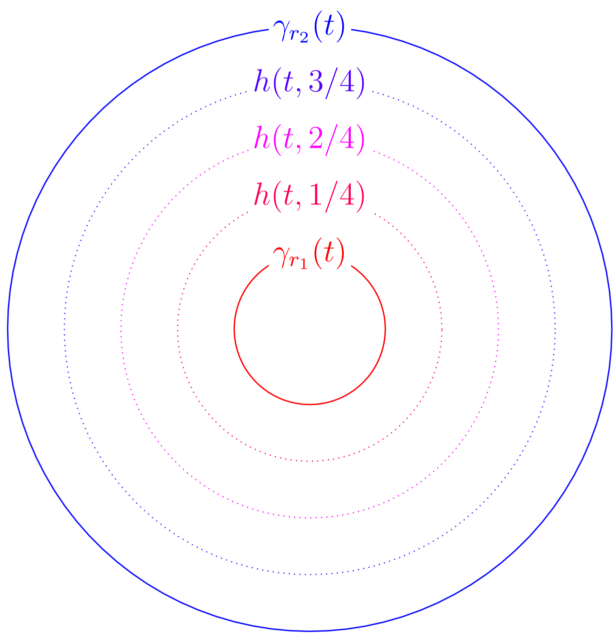
$$\gamma_r(t) := r\exp(2\pi i t).$$

We claim that the γ_r are all homotopic. Explicitly, given two radii $r_1, r_2 \in \mathbb{R}^+$, we can define our homotopy from γ_{r_1} to γ_{r_2} by

$$h(t,s) = (1-s)\gamma_{r_1}(t) + s\gamma_{r_2}(t) = \big((1-s)r_1 + sr_2\big)\exp(2\pi i t),$$

which we can check works.

Here is the image for the previous example.



Example 5.32. A coffee mug and a donut both have one hole and are therefore pretty much "homotopic" because we can imagine deforming one into the other.

5.4.2 Simply Connected Domains

It will turn out that homotopy provides the correct notion of equivalence. To see this, we have the following definition.

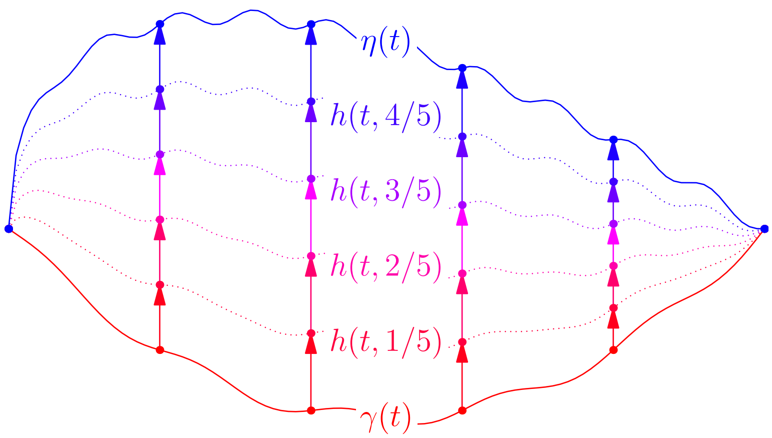
Definition 5.33 (Null homotopic)**.** Fix a domain Ω and a closed path $\gamma \colon [0,1] \to \mathbb{C}$. Further, let $\eta \colon [0,1] \to \mathbb{C}$ be defined by $\eta(t) := \gamma(0) = \gamma(1)$ for all t. Then γ is *null homotopic* if and only if γ is homotopic to the "constant path" η.

Definition 5.34 (Simply connected)**.** A domain Ω is *simply connected* if and only if every pair of paths γ and η with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$ are homotopic with fixed endpoints.

And here is our example

Lemma 5.35. Convex domains are simply connected.

Proof. The point is to draw line segments directly from one path to the other. Here is the image.



We now rigorize this. Pick up a convex domain Ω. Then, given two paths $\gamma, \eta\colon [0,1] \to \Omega$, we define $h\colon [0,1]^2 \to \Omega$ by

$$h(t,s) = (1-s)\gamma(t) + s\eta(t).$$

Notably, h is well-defined because Ω is convex: for any $s,t \in [0,1]$, we see $\gamma(t), \eta(t) \in \Omega$, so $(1-s)\gamma(t) + s\eta(t) \in \Omega$ by convexity.

Continuing our checks, h is continuous as a linear combination of continuous functions. Further, $h(t,0) = \gamma(t)$ and $h(t,1) = \eta(t)$, and in fact

$$h(0,s) = (1-s)\gamma(0) + s\eta(0) = \gamma(0) \qquad \text{and} \qquad h(1,s) = (1-s)\gamma(t) + s\eta(t) = \gamma(1),$$

so h does indeed witness the needed homotopy. ■

Remark 5.36. It is also true that star-like domains are simply connected. Roughly speaking, fix Ω a star-like domain so that we have some $z \in \Omega$ such that the line segment between z and any $w \in \Omega$ lives in Ω. The point is that we can contract any path to the constant path at z by drawing line segments in the same way as above. See sx1748540 for details.

Example 5.37. The open ball $B(z,r)$ for any $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$ is convex. Thus, $B(z,r)$ is simply connected.

5.4.3 Homotopic Independence of Integrals

We close class by proving this last result.

Theorem 5.38 (Homotopy independence)**.** Fix a domain Ω and a holomorphic function $f\colon \Omega \to \mathbb{C}$. Further, take two paths $\gamma, \eta\colon [0,1] \to \Omega$ with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$. If γ and η are homotopic with fixed endpoints, then

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz.$$

This is codifying the idea that homotopic paths (with fixed endpoints) should be essentially equivalent: they are giving the same integral.

Example 5.39. We already have reason to believe this theorem. The Cauchy integral formula told us that

$$f(w)=\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)}{z-w}\,dz$$

for any loop $\gamma_r(t):=w+r\exp(2\pi it)$. This makes sense because we showed that all circles γ_r are homotopic.

Anyway, here is our result.

Proof of Theorem 5.38. Fix our homotopy $h\colon[0,1]^2\to\mathbb{C}$ which fixes the endpoints. As an outline, we will show that two paths which are homotopic and "close together" in a suitable sense will have the same integral, which we can extend to the general case by some compactness argument.

For psychological reasons, we will get the compactness argument out of the way first. Set $K:=\operatorname{im}h$, which is compact because the image of a compact set under a continuous map. Thus, by compactness, there exists $\varepsilon>0$ such that $B(z,3\varepsilon)\subseteq\Omega$ for all $z\in K$, where we are using Lemma 2.118 with $\{\Omega\}$ as the open cover of K.

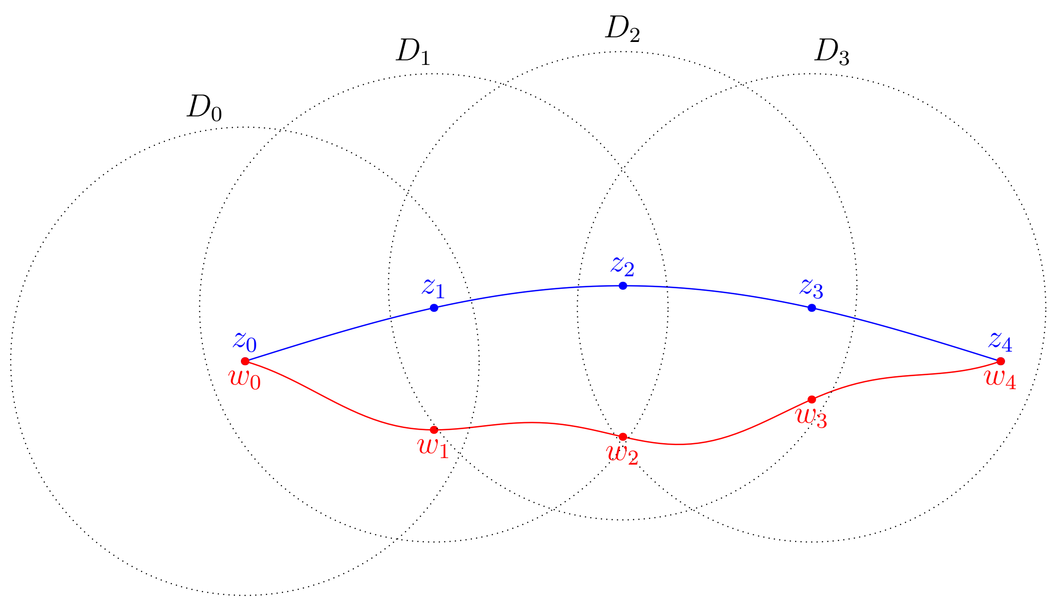
Now, view $h(t,s)$ as inducing a function taking $s\in[0,1]$ and outputting a function $h(-,s)\colon[0,1]\to\Omega$; i.e., $\gamma_s(t):=h(t,s)$. Notably, h being continuous implies that $\gamma_s\colon[0,1]\to\Omega$ is continuous (γ_s this is the composite $t\mapsto(t,s)\mapsto h(t,s)$), so because the codomain $t\in[0,1]$ is compact, the function γ_s is bounded.

As such, we can trigger Remark 2.132 to say that $s\mapsto\gamma_s$ is a continuous function on the compact set $[0,1]$ to the metric space of bounded functions $[0,1]\to\mathbb{C}$, so $s\mapsto\gamma_s$ is uniformly continuous, so there exists $\delta>0$ such that

$$|s_1-s_2|<\delta\implies\sup_{t\in[0,1]}\{|\gamma_{s_1}(t)-\gamma_{s_2}(t)|\}<\varepsilon.\qquad(*)$$

In particular, we are using the definition of the metric back in Remark 2.132.

Fix any two times $s_1<s_2$ with $|s_1-s_2|<\delta$. The homotopy h more or less restricts to a homotopy between $\alpha:=\gamma_{s_1}$ and $\beta:=\gamma_{s_2}$, but we now also have $(*)$, which tells us that α and β are ε-close together. The idea, is to use the closeness to place everything locally in a disk: we want to create an image that looks like the following.



In words, we want to choose disks $D_0,\ldots,D_n$ with points

$$z_0,\ldots,z_{n+1}\in\operatorname{im}\alpha\qquad\text{and}\qquad w_0,\ldots,w_{n+1}\in\operatorname{im}\beta$$

satisfying the following constraints.

- We want $\alpha(0) = \beta(0) = z_0 = w_0$ and $\alpha(1) = \beta(1) = z_{n+1} = w_{n+1}$. Notably, the endpoints of $\alpha = h(-, s_1)$ and $\beta = h(-, s_2)$ because h fixes the endpoints of γ and η.

- For technical reasons, we should have each D_i with center on $\operatorname{im}\alpha$ or $\operatorname{im}\beta$ and have radius at most 3ε. This ensures that the D_i are contained in Ω, by construction of ε. (Notably, $\operatorname{im}\alpha, \operatorname{im}\beta \subseteq \operatorname{im} h = K$.)

- We want consecutive "quadruplets" $z_k, z_{k+1}, w_k, w_{k+1} \in D_k$ for each $0 \le k \le n$.

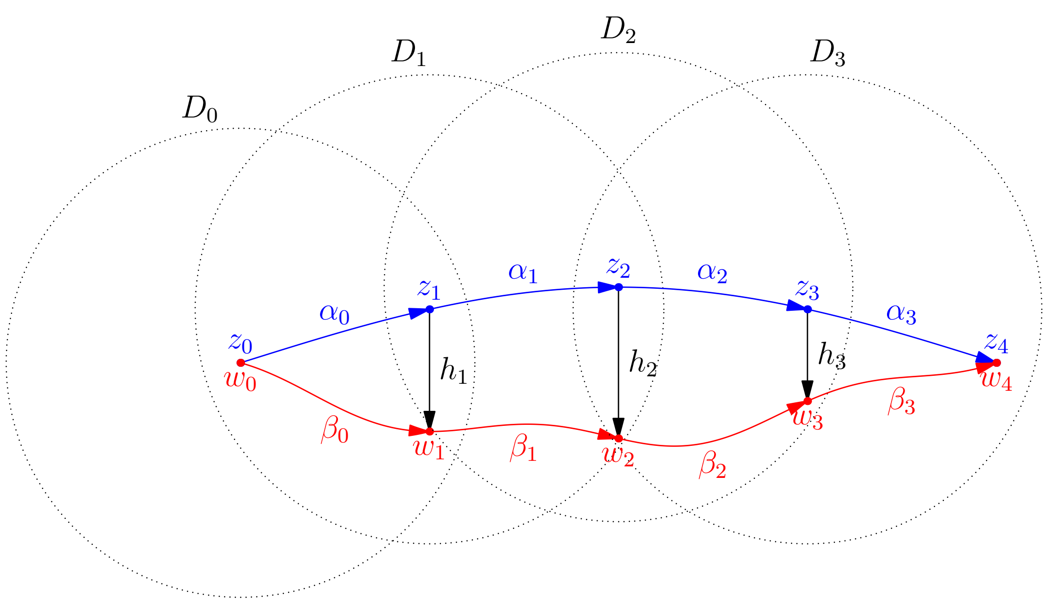
We will want a few other non-intuitive constraints that will pop out of our construction, but we will ignore these for now. Rigorously, we do the following.

- The path α is a continuous path with compact domain, so it is uniformly continuous, so there exists ε_0 such that implies
$$|t_1 - t_2| < \varepsilon_0 \implies |\alpha(t_1) - \alpha(t_2)| < \varepsilon.$$
In particular, choose some $n \in \mathbb{N}$ with $\frac{1}{n+1} < \varepsilon_0$ and then set $z_k := \alpha(k/(n+1))$ for each $k \in [0, n+1]$. Notably, $|z_{k+1} - z_k| < \varepsilon$ for each $k \in [1, n]$, by construction.
We also set $w_k := \beta(k/(n+1))$ for each $k \in [0, n+1]$. Note that we do indeed have $z_0 = \alpha(0) = \beta(0) = w_0$ and $z_{n+1} = \alpha(1) = \beta(1) = w_{n+1}$.

- As such, we set $D_k := B(z_k, 3\varepsilon)$ for each $k \in [0, n]$. Again, $z_k \in \operatorname{im}\alpha \subseteq \operatorname{im} h = K$ implies that $D_k \subseteq \Omega$ by construction of ε.

Now, for each $0 \le k \le n$, we see that $z_k \in D_k$ automatically, we have $|z_{k+1} - z_k| < \varepsilon$ by construction and
$$|z_k - w_k|, |z_{k+1} - w_{k+1}| < \varepsilon$$
by $(*)$, so $|z_k - w_{k+1}| < 2\varepsilon < 3\varepsilon$ as well. Thus, $z_k, z_{k+1}, w_k, w_{k+1} \in D_k$.

We now continue to decorate our diagram. For $k \in [0, n]$, let $\alpha_k = \alpha|_{[k/(n+1),(k+1)/(n+1)]}$ denote the part of α connecting z_k to z_{k+1}, let β_k denote the part of $\beta|_{[k/(n+1),(k+1)/(n+1)]}$ connecting w_k to w_{k+1}. Lastly, for $k \in [0, n+1]$, we define $h_k : [s_1, s_2] \to \Omega$ by
$$h_k(s) := h(s, k/(n+1))$$
so that $h_k(s_1) = \alpha(k/(n+1)) = z_k$ and $h_k(s_2) = \beta(k/(n+1)) = w_k$, making h_k a continuous path from z_k to w_k. Here is our image.



We would like to rigorize some aspects of the above diagram. In particular, for each $k \in [0,n]$, we claim that $h\left([s_1,s_2] \times \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]\right) \subseteq D_k$. To see this, pick up any $(s,t) \in [s_1,s_2] \times \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$. Then $|s_1 - s| \le |s_1 - s_2| < \delta$, so we see that

$$|\gamma_s(t) - \alpha(t)| = |\gamma_s(t) - \gamma_{s_1}(t)| < \varepsilon$$

by (∗). But now $t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$, so $\left|t - \frac{k}{n+1}\right| \le \frac{1}{n+1} < \varepsilon_0$, so

$$|\alpha(t) - z_k| = \left|\alpha(t) - \alpha\left(\frac{k}{n+1}\right)\right| < \varepsilon.$$

Combining, we see that

$$|\gamma_s(t) - z_k| < \varepsilon + \varepsilon < 3\varepsilon,$$

which is what we wanted. Thus, we make the following observations.

- $\operatorname{im}\alpha_k \subseteq D_k$ because $\alpha_k(t) = h(s_1,t)$ for $t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$.
- $\operatorname{im}\beta_k \subseteq D_k$ because $\beta_k(t) = h(s_2,t)$ for $t \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$.
- $\operatorname{im} h_k \subseteq D_k$ because $h_k(s) = h(s, k/(n+1))$ for $s \in [s_1,s_2]$.
- $\operatorname{im} h_{k+1} \subseteq D_k$ because $h_k(s) = h(s, (k+1)/(n+1))$ for $s \in [s_1,s_2]$.

Combining everything above, we see that we can write down the path

$$\alpha_k * h_k * \beta_k^- * h_{k+1}^-$$

as a closed path contained in D_k. Upon noting that D_k is a disk and hence convex and hence star-like, it follows from Theorem 4.70 that

$$\oint_{\alpha_k * h_k * \beta_k^- * h_{k+1}^-} f(z)\,dz = 0,$$

so

$$\int_{\alpha_k} f(z)\,dz - \int_{\beta_k} f(z)\,dz = \int_{h_{k+1}} f(z)\,dz - \int_{h_k} f(z)\,dz.$$

Summing over $k \in [0,n]$, we see that

$$\begin{aligned}\int_\alpha f(z)\,dz - \int_\beta f(z)\,dz &= \sum_{k=0}^n \left(\int_{\alpha_k} f(z)\,dz - \int_{\beta_k} f(z)\,dz\right)\\ &= \sum_{k=0}^n \left(\int_{h_{k+1}} f(z)\,dz - \int_{h_k} f(z)\,dz\right)\\ &= \int_{h_{n+1}} f(z)\,dz - \int_{h_0} f(z)\,dz\end{aligned}$$

by telescoping. However, h_{n+1} and h_0 are constant paths because h fixes the endpoints, so $h_{n+1}' = h_0' = 0$ everywhere, so the right-hand side above simply vanishes. So have verified that

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

This finishes this part of the proof.

We now finish the proof. Fix some N such that $\frac{1}{N} < \delta$ and set $s_k := k/N$ for $k \in [0, N+1]$. In particular, $|s_{k+1} - s_k| < \delta$ for each $k \in [0,N]$, so

$$\int_{\gamma_{s_k}} f(z)\,dz = \int_{\gamma_{s_{k+1}}} f(z)\,dz$$

for each $k \in [0, N]$, using the work above. Chaining these equalities together, we conclude that

$$\int_\gamma f(z)\,dz = \int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz = \int_\eta f(z)\,dz.$$

This is what we wanted. ■

5.5 April 8

Good morning everyone.

- Homework #8 is due tonight at 11:59PM.
- There are office hours today from 1PM to 2:30PM.
- Midterm #2 is next Friday. A review has been posted, with review problems and a practice midterm to come.

Remark 5.40 (Morrow)**.** Fun life tip: if you show up 10 minutes to jury duty, they will have enough jurors, and you will not get in trouble, so you will be excused.

5.5.1 Integrals in Simply Connected Domains

We continue our discussion of homotopy. We will not go over every single proof because they are somewhat laborious. Last time we showed the following.

Theorem 5.38 (Homotopy independence)**.** Fix a domain Ω and a holomorphic function $f\colon \Omega \to \mathbb{C}$. Further, take two paths $\gamma, \eta\colon [0,1] \to \Omega$ with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$. If γ and η are homotopic with fixed endpoints, then

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz.$$

Have some corollaries.

Corollary 5.41. Fix a simply connected domain Ω and a holomorphic function $f\colon \Omega \to \mathbb{C}$. Given two paths $\gamma, \eta\colon [0,1] \to \mathbb{C}$ with the same endpoints $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$, we have

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz.$$

Proof. Because Ω is simply connected, γ and η have a homotopy with fixed endpoints between them. ■

Corollary 5.42. Fix a domain Ω. If Ω is simply connected, then Ω is primitive.

Proof. The point is to use Theorem 4.44. As usual, pick up some holomorphic function $f\colon \Omega \to \mathbb{C}$ which we would like to give a primitive, and choose any closed piecewise C^1 path $\gamma\colon [0,1] \to \Omega$, so we want to show

$$\oint_\gamma f(z)\,dz \stackrel{?}{=} 0,$$

which by Theorem 4.44 will provide us with a primitive. Well, because Ω is simply connected and $\gamma(0) = \gamma(1) =: z_0$, we see that γ is homotopic with fixed endpoints to the path $c\colon [0,1] \to \mathbb{C}$ defined by

$$c(t) = z_0$$

for all $t \in [0,1]$. However, Corollary 5.41 now tells us that

$$\oint_\gamma f(z)\,dz = \oint_c f(z)\,dz = \int_0^1 f(c(t))c'(t)\,dt,$$

but this last integral is 0 because $c'(t) = 0$. ■

Example 5.43. We now know that the Residue theorem (Theorem 5.24) applies to simply connected domains.

5.5.2 A Better Cauchy Integral Formula

One of the main goals of homotopy is to be able to get a more general version of the Cauchy integral formula. Take the following definition.

Definition 5.44 (Homologous to zero)**.** Fix a domain Ω. Then a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$ is *homologous to* 0 if and only if $\operatorname{Ind}(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus \Omega$.

Roughly speaking, we are requiring that a path homologous to 0,

Theorem 5.45 (Cauchy integral formula)**.** Fix a domain Ω and a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$ which is homologous to 0. Then, given a holomorphic function $f\colon \Omega \to \mathbb{C}$, we have

$$\oint_\gamma f(z)\,dz = 0,$$

and for any $w \in \Omega \setminus \operatorname{im}\gamma$, we have

$$f(w)\operatorname{Ind}(\gamma, w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz.$$

Note that we are able to recover the first version of the Cauchy integral formula (namely, Theorem 4.63) by setting

$$\gamma(t) = z_0 + r\exp(2\pi i t),$$

where $\operatorname{im}\gamma = \overline{B(z_0, r)} \subseteq \Omega$. In this case, Theorem 4.70 was roughly speaking able to give us

$$\oint_\gamma f(z)\,dz = 0,$$

and Theorem 4.63 was able to give us

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz$$

for any γ inside the loop. This last part we generalize past the loop γ above to a more general closed, piecewise C^1 path homologous to 0, but we have to add in a winding number, lest we do something silly like $\gamma * \gamma$.

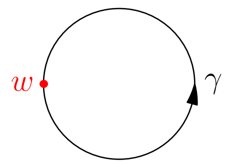
To prove Theorem 5.45, we will need the following result, but we will not prove it because it is somewhat technical.

Proposition 5.46. Fix a domain Ω and a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$. Given a $w \in \operatorname{im}\gamma$, then we can generate a closed, piecewise C^1 path η with $w \notin \operatorname{im}\eta$ while

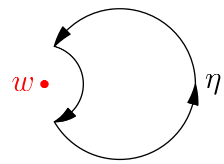
$$\oint_\gamma f(z)\,dz = \oint_\eta f(z)\,dz$$

for any holomorphic function $f\colon \Omega \to \mathbb{C}$.

Proof. The point is to do "surgery" on γ to avoid w. Here is the image of γ with some bad $w \in \operatorname{im}\gamma$.



Now, we explode w a little as follows to make our η, as follows.



By making the ball small enough, we can ensure that the entire ball lives in Ω, and this ball is simply connected, so the integrals over any f are the same by Corollary 5.41, roughly speaking. ■

Anyway, here is our proof.

Proof of Theorem 5.45. We proceed in steps. Replace Ω with some bounded domain containing γ, which we can do because $\operatorname{im}\gamma$ is compact and hence bounded. This won't affect the content of the conclusions; we merely have to replace f with its restriction.

1. We define some $\widetilde{F}$. Define $\widetilde{F}\colon (\mathbb{C} \setminus \operatorname{im}\gamma) \to \mathbb{C}$ by

$$\widetilde{F}(w) := \int_0^1 \frac{f(\gamma(t)) - f(w)}{\gamma(t) - w} \cdot \gamma'(t)\,dt.$$

   Notably, $\widetilde{F}$ is holomorphic on $\mathbb{C} \setminus \operatorname{im}\gamma$ by writing out a power series expansion at each point and then integrating the power series expansion by hand using some local absolute convergence result.

   Philosophically, the point is to show that $\widetilde{F} \equiv 0$, which will give the second desired equality

$$f(w)\operatorname{Ind}(\gamma, w) \stackrel{?}{=} \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - w}\,dz$$

   by rearranging. In particular, we will show that $\widetilde{F}$ can be extended to be entire and bounded (which by Theorem 4.80 forces $\widetilde{F}$ to be constant), and then we will show that $\widetilde{F}$ takes the value 0 somewhere.

2. We now extend to Ω. Give some $w \in \Omega$, we define

$$g(z, w) := \begin{cases} \frac{f(z) - f(w)}{z - w} & z \neq w, \\ f'(w) & z = w. \end{cases}$$

   We can check by hand (e.g., using a power series expansion) that $g(-, w)$ is holomorphic on Ω; notably, we are holomorphic at each $z \neq w$ for free, and then we can build a power series expansion at w by hand. As such, we extend $\widetilde{F}$ to $F\colon \Omega \to \mathbb{C}$ by

$$F(w) := \oint_\gamma g(z, w)\,dz.$$

Notably, F does indeed restrict down to

$$F(w)=\int_0^1 \frac{f(\gamma(t))-f(w)}{\gamma(t)-w}\cdot\gamma'(t)\,dt=\widetilde{F}(w)$$

for $w\in\mathbb{C}\setminus\operatorname{im}\gamma$.

3. We check that F is holomorphic on Ω. Our only problem is to check points $w\notin\operatorname{im}\gamma$. By Proposition 5.46, there exists η with $w\notin\operatorname{im}\eta$ such that

$$F(w)=\oint_\gamma g(z,w)\,dz=\oint_\eta g(z,w)\,dz,$$

but now this last integral is manifestly holomorphic at w because $w\notin\operatorname{im}\eta$, where here we are appealing to the previous steps to note that

$$F(w)=\oint_\eta f(z)\,dz=\int_0^1\frac{f(\eta(t))-f(w)}{\eta(t)-w}\,dt$$

is holomorphic on $\mathbb{C}\setminus\operatorname{im}\eta$ and in particular at $w\notin\operatorname{im}\eta$.

4. We check that F is entire. Well, we have shown that F is holomorphic on $\operatorname{im}\gamma$, and we started with $\widetilde{F}$ which is holomorphic on $\mathbb{C}\setminus\operatorname{im}\gamma$, so we can glue these together to get an actual entire function.

5. We show the integral formulae. Note that F is continuous and hence uniformly continuous on the compact set $\operatorname{im}\gamma$, so F is bounded there. On the other hand, we see that taking $w\notin\mathbb{C}\setminus\operatorname{im}\gamma$ gives

$$\begin{aligned}F(w)&=\int_0^1\frac{f(\gamma(t))-f(w)}{\gamma(t)-w}\cdot\gamma'(t)\,dt\\&=\oint_\gamma\frac{f(z)}{z-w}\,dz-f(w)\oint_\gamma\frac{1}{z-w}\,dz\\&=\oint_\gamma\frac{f(z)}{z-w}\,dz-2\pi i\cdot f(w)\operatorname{Ind}(\gamma,w).\end{aligned}\qquad(*)$$

Notably, for $w\notin\mathbb{C}\setminus\Omega$, the term $\operatorname{Ind}(\gamma,w)$ will vanish because γ is homologous to 0 (!). Because Ω is bounded, fix $R\in\mathbb{R}^+$ with $\Omega\subseteq B(0,R)$, we can just say that w with $|w|>R$ will have

$$F(w)=\oint_\gamma\frac{f(z)}{z-w}\,dz.$$

Now, $t\mapsto f(\gamma(t))$ is a continuous function $[0,1]\to\mathbb{R}$ on a compact set and hence has a maximum M. As such, we use Proposition 4.35 to write

$$|F(w)|=\left|\oint_\gamma\frac{f(z)}{z-w}\,dz\right|\le\ell(\gamma)\cdot\max_{t\in[0,1]}\left\{\left|\frac{f(\gamma(t))}{\gamma(t)-w}\right|\right\}\le\frac{M\ell(\gamma)}{|w|-R}$$

for $|w|>R$. Now, sending $|w|\to\infty$ causes $|F(w)|\to 0$.

To finish, being entire implies that F is bounded on the compact set $\overline{B(0,R+1)}$. Further, we have bounded

$$|F(w)|\le\frac{M\ell(\gamma)}{R+1-R}$$

for $|w|\ge R+1$, so F is a bounded, entire function and hence constant by Theorem 4.80. However, $|F(w)|\to 0$ as $|w|\to\infty$, so we must have $F\equiv 0$. So we conclude that any $w\notin\operatorname{im}\gamma$ will have

$$f(w)\operatorname{Ind}(\gamma,w)=\frac{1}{2\pi i}\oint_\gamma\frac{f(z)}{z-w}\,dz$$

by rearranging $F\equiv 0$ with $(*)$.

6. It remains to show that $\oint_\gamma f(z)\,dz = 0$. Well, given $w \in \Omega \setminus \operatorname{im}\gamma$, we define
$$g_w(z) = (z-w)f(z).$$
Then we compute
$$\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{g_w(z)}{z-w}\,dz \stackrel{*}{=} g_w(w)\operatorname{Ind}(\gamma,w) = 0$$
by using the integral formula in $\stackrel{*}{=}$. ■

To close out class, have a corollary, where we impose conditions on Ω instead of γ.

Corollary 5.47. Fix a simply connected domain Ω and a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$. Then, given a holomorphic function $f\colon \Omega \to \mathbb{C}$, we have
$$\oint_\gamma f(z)\,dz = 0,$$
and for any $w \in \Omega \setminus \operatorname{im}\gamma$, we have
$$f(w)\operatorname{Ind}(\gamma,w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz.$$

Proof. The main point is to show that γ is in fact homologous to 0, from which the result will follow directly from Theorem 5.45.

As such, pick up $w \in \mathbb{C} \setminus \Omega$, and we show that $\operatorname{Ind}(\gamma,w) = 0$. Because $w \in \mathbb{C} \setminus \Omega$, the function $f(z) := \frac{1}{z-w}$ is holomorphic on Ω as the quotient of nonzero holomorphic functions. Now, Corollary 5.42 promises that f has a primitive, so Corollary 4.39 forces
$$\operatorname{Ind}(\gamma,w) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-w}\,dz = \frac{1}{2\pi i}\oint_\gamma f(z)\,dz = 0,$$
which is what we wanted. ■

5.6 April 11

Good morning, everyone.

- Midterm #2 is on Friday. Both practice problems and a practice midterm were released.
- There are extra office hours.
- There is a review session on Wednesday.

5.6.1 Rouché's Theorem

We proved the following result on the homework.

Theorem 5.48 (Argument principle)**.** Fix a domain Ω and a meromorphic function $f: \Omega \to \mathbb{C}$, and pick up $z_0 \in \Omega$ and $r > 0$ such that $B(z_0, r) \subseteq \Omega$ and f has no zeroes nor poles on $\partial B(z_0, r)$. Further, set the following.

- N_f is the number of zeroes of f, counted with multiplicity, in $B(z_0, r)$.
- P_f is the number of poles of f, counted with multiplicity, in $B(z_0, r)$.

Then

$$N_f - P_f = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz = \operatorname{Int}(f \circ \gamma, 0),$$

where $\gamma: [0,1] \to \mathbb{C}$ is $\gamma(t) := z_0 + r\exp(2\pi i t)$.

Proof. The point is to use the Residue theorem on f'/f. We can check that f'/f will only have poles when either $f(z)$ has a pole or zero. Then, at a point $w \in \Omega$, we can write down

$$f(z) = (z - w)^n g(z)$$

for some integer n and for some holomorphic function $g: \Omega \to \mathbb{C}$ such that $g(w) \neq 0$. Then we can see that

$$\frac{f'(z)}{f(z)} = \frac{n}{z - w} + \frac{g'(z)}{g(z)}$$

by taking the derivative by hand, so we can see that

$$\operatorname{Res}_w(f'/f) = n.$$

Thus, $\operatorname{Res}_w(f'/f)$ counts zeroes with multiplicity positively and counts poles with multiplicity negatively. Summing over these residues in $B(z_0, r)$ (via Theorem 5.24) gives the result. ■

Now, here is the statement we are going to prove today.

Theorem 5.49 (Rouché's)**.** Fix a domain Ω and two holomorphic functions $f, g: \Omega \to \mathbb{C}$. Further, suppose that we have $z_0 \in \Omega$ and $r > 0$ such that $\overline{B(z_0, r)} \subseteq \Omega$ and

$$|g(z)| < |f(z)|$$

for each $z \in \partial B(z_0, r)$. Then f and $f + g$ have the same number of zeroes, counted with multiplicity, contained in the ball $B(z_0, r)$.

Remark 5.50. As in Theorem 4.63, the main point is that we can talk about the behavior of f by only weak information at the boundary. In particular, perturbations by "small" functions g are unable to alter how f works in practice.

Warning 5.51. The proof in the Eterović notes is incorrect.

The point of Theorem 5.49 is to be able to count and determine the location of zeroes of some holomorphic function f by relating f to a simpler function.

Exercise 5.52. We show that the roots of the polynomial $p(z) = z^4 + 5z + 2$ all lie in $B(0, 2)$.

Proof. To be able to use Rouché's theorem, we need to choose some f and g. Because g should be some "small" perturbation to f, we take $f(z) := z^4$ and $g(z) := 5z + 2$ so that $p(z) = f(z) + g(z)$. Now, for $z \in \partial B(0, 2)$, we see that

$$|g(z)| = |5z + 2| \le 5|z| + 2 = 5 \cdot 2 + 2 = 12 < 16 = 2^4 \le |z|^4 = |f(z)|.$$

Thus, Theorem 5.49 tells us that f and $f + g$ have the same number of zeroes in $B(0, 2)$, but f has four zeroes in $B(0, 2)$ when counted with multiplicity (namely, four zeroes at $z = 0$), so we can say the same for $p = f + g$. This finishes. ■

Now, let's prove Theorem 5.49.

Proof of Theorem 5.49. Note that f has no zeroes on $\partial B(z_0, r)$ because f is strictly larger than $|g(z)| \ge 0$ for each $z \in \partial B(z_0, r)$. Similarly, $|f(z) + g(z)| \ge |f(z)| - |g(z)| > 0$ for $z \in \partial B(z_0, r)$ by assumption, so $f + g$ also has no zeroes on this boundary. As such, we define

$$h(z) := \frac{f(z) + g(z)}{f(z)}.$$

Further, set $\gamma\colon [0, 1] \to \Omega$ by $\gamma(t) := z_0 + r\exp(2\pi it)$ to trace out $\partial B(z_0, r)$.

Continuing, note that the zeroes of h will only occur at zeroes of $f(z) + g(z)$, and the poles of $h(z)$ will occur only at poles of $f(z)$. Now, h has neither zero nor pole on $\partial B(z_0, r)$, so Theorem 5.48 tells us that

$$N_h - P_h = \operatorname{Ind}(h \circ \gamma, 0).$$

Notably, $N_h - P_h$ is the number of zeroes of $f + g$ minus the number of zeroes of f, even if there is some cancellation with having a zero in the same place. Thus, we would like to show that the above integral vanishes.

Well, for each $z \in \partial B(z_0, r)$, we see

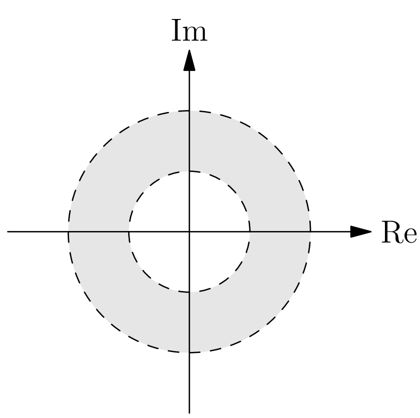
$$|h(z) - 1| = \left|\frac{f(z) + g(z)}{f(z)} - 1\right| = \left|\frac{g(z)}{f(z)}\right| < 1.$$

Thus, $\operatorname{im}(h \circ \gamma) \subseteq B(1, 1)$ and in particular is nonzero everywhere. In particular, $h \circ \gamma$ does not wind around 0 at all, so $\operatorname{Ind}(h \circ \gamma) = 0$. ■

And here is another example.

Exercise 5.53. We compute the number of zeroes of $p(z) := z^5 + 3z^2 + 1$ for z in the "annulus" $1 < |z| < 2$.

Proof. Here is our image.



The point is to find the zeroes in $B(0, 2)$ and the zeroes in $B(0, 1)$ and then subtract. As such, we do two computations.

- For $z \in \partial B(0,1)$, we have $|z| = 1$, so we note $g(z) := z^5 + 1$ and $f(z) := 3z^2$ give
$$|g(z)| = \left|z^5 + 1\right| \le 2 < 3 = 3|z|^2 = |f(z)|,$$
so we conclude that $p = f + g$ has two zeroes in $B(0,1)$.
- For $z \in \partial B(0,2)$, we have $|z| = 2$, so we note $g(z) := 3z^2 + 1$ and $f(z) := z^5$ give
$$|g(z)| = \left|3z^2 + 1\right| \le 3 \cdot 4 + 1 = 13 \le 32 = |z|^5 = |f(z)|,$$
so we conclude that $p = f + g$ has all five zeroes in $B(0,2)$.

Subtracting, it follows that there are three zeroes in $B(0,2) \setminus B(0,1)$. To claim this as our answer, we check that there is no zero on $\partial B(0,1)$. Well, if $|z| = 1$, then we compute
$$\left|z^5 + 3z^2 + 1\right| \ge \left|3z^2\right| - \left|z^5\right| - |1| = 3 - 1 - 1 > 0,$$
so there are no zeroes here. Thus, there are indeed $\boxed{3}$ total zeroes in the annulus. ■

5.6.2 The Open Mapping Theorem

We close class with the following nice consequence of Theorem 5.49.

> **Theorem 5.54** (Open mapping). Fix a domain Ω and a non-constant holomorphic function $f \colon \Omega \to \mathbb{C}$. For open subsets $U \subseteq \Omega$, the set $f(U)$ is also open.

This is very surprising! For example, this is very much not true in $\mathbb{R}$: the function $f(x) := \sin x$ sends the open set $\mathbb{R}$ to $[-1,1]$, which is closed. In general, continuous and even differentiable functions really not need be open—open is a very different notion.

Proof. Fix $w_0 \in f(U)$ with $z_0 \in U$ such that $f(z_0) = w_0$, and we need to put a neighborhood around w_0 in $f(U)$. To help us our, we define $g \colon \Omega \to \mathbb{C}$ by
$$g(z) := f(z) - w_0$$
so that $g(z_0) = 0$. Now, g is a non-constant holomorphic function, so Theorem 5.1 tells us that g cannot have zeroes accumulating to z_0 (lest g be equivalent to 0), so there is some $r > 0$ such that g does not vanish on
$$\overline{B(z_0,r)} \setminus \{z_0\}.$$
Further, by making r small enough, we can also assume that $\overline{B(z_0,r)} \setminus \{z_0\} \subseteq U$. Now, $\partial B(z_0,r)$ is compact, so we can find $\delta > 0$ such that
$$|g(z)| \ge \delta$$
for all $z \in \partial B(z_0,r)$ because continuous functions have achieved minimums, and g never achieves 0. This δ will give our neighborhood.

We are now almost ready to apply Rouché's theorem. In particular, we would like to show that $B(z_0,\delta) \subseteq f(U)$. Well, pick up some $w \in B(w_0,\delta)$, and we set $h_w \colon \Omega \to \mathbb{C}$ by
$$h_w(z) := g(z) + w_0 - w.$$
In particular, we can compute that
$$|w_0 - w| < \delta < |g(z)|$$
for all $z \in \partial B(z_0,r)$, so Theorem 5.49 promises us that $h_w(z) = g(z) + w_0 - w$ has the same number of zeroes as g on $B(w_0,\delta)$. However, by construction of r, we see that g has a zero in $B(z_0,r)$, so h does as well, so there exists $z \in B(z_0,r) \subseteq U$ such that $h_w(z) = 0$ and hence $f(z) = w$, giving $w \in f(U)$. This finishes. ■

5.7 April 13

Good morning, everyone.

- There are office hours today from 11AM–12PM and 1PM–2:30PM. There are also office hours tomorrow from 10PM–12PM and 2PM–4PM.
- The midterm is still on Friday.

5.7.1 Integral Commentary

It's a review session today. We will be computing a lot of integrals for the midterm, for which we have many techniques. Here are some guidelines for finding the quickest computation for

$$\oint_\gamma f(z)\,dz.$$

1. Is f holomorphic or meromorphic almost everywhere? If not, we basically have to parameterize γ and proceeding with the definition. For example, integrals such as

   $$\oint_\gamma \overline{z}\,dz \quad \text{or} \quad \oint_\gamma |z|\,dz \quad \text{or} \quad \oint_\gamma \operatorname{Re} z\,dz$$

   all fall under this category.

2. If f is close to holomorphic, look at the integral. We might try to pattern-match with

   $$f^{(n)}(w) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{n+1}}\,dz,$$

   where n is some nonnegative integer. In life, sometimes this fails, and we still have to parameterize.

3. If f is meromorphic, we should use the Residue theorem, which states

   $$\oint_\gamma f(z)\,dz = \sum_{\text{poles } z_0} \operatorname{Ind}(\gamma, z_0)\operatorname{Res}_{z_0}(f),$$

   and we can compute the winding numbers and the residues by hand.

4. If f is not quite holomorphic or meromorphic but has an essential singularity, we can reparametrize the path to make the function f meromorphic. Alternatively, we can use a power series expansion and attempt to switch the sum with the integral, using the residue theorem by hand. For example, we claim

   $$\oint_{|z|=1} z^2\sin(1/z)\,dz = -\oint_\eta z^{-4}\sin z\,dz$$

   by sending the path γ to $\eta := 1/\gamma$. Indeed, running this through, we compute $\eta'(t) = -\gamma'(t)/\gamma(t)^2$, so

   $$\oint_\gamma z^2\sin(1/z)\,dz = \oint_0^1 \gamma(t)^2\sin(1/\gamma(t))\gamma'(t)\,dz = -\oint_0^1 \eta(t)^{-4}\sin(\eta(t))\eta'(t)\,dz = \oint_\eta z^{-4}\sin z\,dz$$

   Now, we can just compute this directly via the Residue theorem.

5.7.2 Review

Here are some questions from class.

- There might be a more general version of Corollary 5.47 allowing for derivatives of f.
- Technically speaking, the Cauchy integral formula is a subset of the Residue theorem.
- We will not need homotopy on the exam.

Let's see some practice problems.

Exercise 5.55. Fix a polynomial $f(z) \in \mathbb{C}[z]$ of degree $d \geq 2$. Taking $R > 0$ such that f does not vanish for all $|z| \geq R$, we show that

$$\oint_{|z|=R} \frac{dz}{f(z)} = 0.$$

Proof. The fact that $f(z)$ does not vanish for $|z| \geq R$ promises us that $1/f(z)$ is holomorphic for $|z| \geq R$. The point, now, is to use the Residue theorem to bound the integral. Explicitly, pick up some $r \geq R$, and we set γ_r to the counter-clockwise path around $|z| = r$ so that

$$\oint_{\gamma_r} \frac{dz}{f(z)} = 2\pi i \sum_{\substack{z_0 \text{ zero of } f \\ |z_0| \leq r}} \underbrace{\operatorname{Ind}(\gamma_r, z_0)}_{1} \cdot \operatorname{Res}_{z_0}(f) = 2\pi i \sum_{\substack{z_0 \text{ zero of } f \\ |z_0| \leq R}} \underbrace{\operatorname{Ind}(\gamma_R, z_0)}_{1} \cdot \operatorname{Res}_{z_0}(f) = \oint_{\gamma_R} \frac{dz}{f(z)}$$

because all poles of $1/f(z)$ are zeroes of $f(z)$, and those all live in the region with $|z| \leq R \leq r$. As such, the estimation lemma tells us that

$$\left| \oint_{|z|=R} \frac{dz}{f(z)} \right| = \left| \oint_{|z|=r} \frac{dz}{f(z)} \right| \leq 2\pi r \cdot \max_{|z|=r} \left\{ \frac{1}{|f(z)|} \right\}.$$

To bound the size of $f(z)$, we set

$$f(z) = \sum_{k=0}^{d} a_k z^k$$

so that

$$\left| \frac{1}{f(z)} \right| \leq \frac{1}{|a_d| \cdot |z|^d - |a_{d-1}| \cdot |z|^{d-1} - \cdots - |a_0|},$$

so

$$\left| \oint_{|z|=R} \frac{dz}{f(z)} \right| \leq 2\pi r \cdot \frac{1}{|a_d| \cdot r^d - |a_{d-1}| \cdot r^{d-1} - \cdots - |a_0|},$$

which goes to 0 as $r \to \infty$ because $d \geq 2$. This finishes. ■

Exercise 5.56. Fix a polynomial $p(z) \in \mathbb{C}[z]$ of degree n. Suppose that we have some M such that $|p(z)| \leq M$ for $|z| < 1$. Then we show that $|p(z)| \leq M|z|^n$ for all z with $|z| \geq 1$.

Proof. The main point is that we know how p behaves on $B(0,1)$ as a bound, so we are going to want to use the Maximum modulus principle. As such, we set $f(z) := p(z)/z^n$ and $g(z) := f(1/z)$; notably, a computation shows that g is holomorphic (it's the "reversed" version of p), so we see that we already have a bound on the behavior of large values of p from this.

So now we push harder. By the Maximum modulus principle, the maximum of $|g(z)|$ on $\overline{B(0,1)}$ will be achieved on $\partial B(0,1)$. But now, the values of g agree with the values of p on $\partial B(0,1)$ (because $z \mapsto 1/z$ is a bijection $\partial B(0,1) \to \partial B(0,1)$), and we know that the values of p are upper-bounded by M on $\partial B(0,1)$. As such, we know that

$$|g(z)| \le M$$

on $\overline{B(0,1)}$, which rearranges to showing $|z^n p(1/z)| \le M$ for all $z \in \overline{B(0,1)} \setminus \{0\}$ and so $|p(z)| \ge M \cdot |z|^n$ for all z with $|z| \ge 1$. ■

THEME 6

EXTRA TOPICS

You take the red pill, you stay in wonderland, and I show you how deep the rabbit hole goes.

—Morpheus, [WW99]

6.1 April 18

Last class there was a midterm. Today we mourn.

- Homework #9 will be posted later today, due Sunday at 11:59PM.
- Homework #10 will be the last homework.
- Midterm #2 will be returned on Wednesday.

6.1.1 Applications of Rouché's Theorem

We begin by recalling the statement, as follows.

Theorem 5.49 (Rouché's)**.** Fix a domain Ω and two holomorphic functions $f, g\colon \Omega \to \mathbb{C}$. Further, suppose that we have $z_0 \in \Omega$ and $r > 0$ such that $\overline{B(z_0, r)} \subseteq \Omega$ and

$$|g(z)| < |f(z)|$$

for each $z \in \partial B(z_0, r)$. Then f and $f + g$ have the same number of zeroes, counted with multiplicity, contained in the ball $B(z_0, r)$.

The main use of Theorem 5.49 is to determine where there are zeroes of a given holomorphic function. We also showed Theorem 5.54; on the homework, we will prove the Fundamental theorem of algebra.

Before continuing, we give another example.

Exercise 6.1. We compute the number of roots of $h(z) = 6z^3 + \exp(z) + 1$ in $B(0, 1)$.

Proof. Note that f is holomorphic, so although this is not a polynomial, we can still use Theorem 5.49. Indeed, our largest term seems to be $f(z) := 6z^3$ and $g(z) := \exp(z) + 1$ so that, for $z \in \partial B(0, 1)$,

$$|\exp(z) + 1| \le |\exp(z)| + 1 \overset{*}{\le} \exp(|z|) + 1 \le e + 1 < 6 = 6 \cdot \left|z^3\right| = |f(z)|,$$

where $\overset{*}{\le}$ holds by expanding out $\exp$ as a series. It follows that f and $h = f + g$ have the same number of zeroes, so h has $\boxed{3}$ zeroes in $B(0, 1)$. ■

Anyway, let's prove something today.

Proposition 6.2. Fix a domain Ω and a non-constant holomorphic function $f \colon \Omega \to \mathbb{C}$. Given $z_0 \in \Omega$, then $f'(z_0) \ne 0$ if and only if $f|_{B(z_0,r)}$ is injective for some $r > 0$.

Intuitively, we are saying that having derivative zero means that f is locally injective.

Example 6.3. The function $f(z) = z^2$ is not injective on $B(0, r)$ for any $r > 0$.

Anyway, let's prove this.

Proof of Proposition 6.2. We show the directions independently.

- We start by taking $f'(z_0) \ne 0$; we imitate the proof of Theorem 5.54. Let $w_0 := f(z_0)$ and define

  $$g(z) := f(z) - w_0$$

  so that $g(z_0) = 0$. Additionally, because f is non-constant, g is also non-constant and in particular not zero everywhere, so Theorem 5.1 forces z_0 to be an isolated zero of g. As such, there is some $r_0 > 0$ such that g does not vanish on

  $$\overline{B(z_0, r_0)} \setminus \{z_0\}.$$

  We now bring in the condition $f'(z_0) \ne 0$: because $f'(z_0) \ne 0$, we see $g'(z_0) \ne 0$, so z_0 is a zero of $g'(z_0)$ of multiplicity 1—indeed, if we had $g(z) = (z - z_0)^2 h(z)$, then $g'(z) = (z - z_0)\big(2h(z) + (z - z_0)h'(z)\big)$, so $g'(z_0) = 0$. It follows that g has one zero in $B(z_0, r_0)$, at $z = z_0$, even when counted with multiplicity.

  We now continue as in Theorem 5.54. Because $\partial B(z_0, r_0)$ is closed and bounded and hence compact, there exists $\delta > 0$ so that

  $$|g(z)| \ge \delta$$

  for all $z \in \partial B(z_0, r_0/2)$ by giving $|g|$ a minimum; we can set $\delta > 0$ because g does not vanish on $\partial B(z_0, r_0/2)$.

  Now, to apply Theorem 5.49, we pick up some $w \in B(w_0, \delta)$, and we would like to show

  $$h_w(z) = g(z) + w_0 - w$$

  has exactly one root in $B(w_0, \delta)$; this will be enough because it shows g is injective on $g^{-1}(B(w_0, \delta))$, from which we can extract an open neighborhood around z_0. Well, we compute

  $$|w_0 - w| < \delta \le |g(z)|,$$

  for $z \in \partial B(z_0, r_0/2)$, so h_w and g have the same number of roots on $B(w_0, \delta)$ by Theorem 5.49, which in particular is exactly one by our discussion above.

- Now, suppose that $f'(z_0) = 0$, and we show that f is not injective on some any neighborhood around z_0; as such, fix any $r > 0$, and we show f is not injective on $B(z_0, r)$. Because f is holomorphic, it is analytic, so by taking r small enough (which will not harm our conclusion because our injectivity is local), we have

  $$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for $z \in B(z_0, r)$. Because $f'(z_0) = 0$, we have $a_1 = 0$ above. However, f' is holomorphic and non-constant (because f is holomorphic and non-constant, so some $a_k \neq 0$ for $k > 1$ above due to $a_1 = 0$), so Theorem 5.1 forces z_0 to be an isolated zero of f'. In particular, we may take r even smaller so that f' does not vanish on

$$\overline{B(z_0, r)} \setminus \{z_0\}.$$

Running through the argument in the previous point once more, we are told that, for some w in the image of f under $B(z_0, r)$ not equal to $f(w)$, we have that

$$f(z) = w$$

has at least two roots in $B(z_0, r)$, counted with multiplicity.

We now push this further. If f were in fact injective on $B(z_0, r)$, then $f(z) - w$ has a double root at some $z = z_1 \in B(z_0, r)$, but then $f'(z_1) = 0$ would follow, which contradicts our construction of r because f' does not vanish on $\overline{B(z_0, r)} \setminus \{z_0\}$. ■

Remark 6.4. We can measure the failure of the locally injective by staring carefully at the argument at the end: if $f(z) - f(z_0)$ has a root of multiplicity m at $z = z_0$, then f is m-to-1 in some neighborhood around z_0.

Non-Example 6.5. In real analysis, this statement is not true. For example, $f(z) := z^3$ is bijective on $\mathbb{R}$ while $f'(0) = 0$. The issue here is that working in $\mathbb{R}$ is hiding the "rotation" that f is doing.

6.1.2 The Inverse Function Theorem

We close class with the following result.

Theorem 6.6 (Inverse function). Fix a domain Ω and an injective, holomorphic function $f : \Omega \to \mathbb{C}$. If $g : \operatorname{im} f \to \Omega$ is the right inverse of f (i.e., $f(g(z)) = z$ for all $z \in \operatorname{im} f$), then g is holomorphic, and

$$g(w) = \frac{1}{f'(g(w))}$$

for all $w \in \operatorname{im}$.

Proof. We proceed in steps.

1. We show that g is continuous. Well, take $U \subseteq \Omega$, and we need to show $g^{-1}(U) \subseteq \operatorname{im} f$ is open. For this, we simply write down the computation

   $$g^{-1}(U) = g^{-1}\left(f^{-1}(f(U))\right) = (f \circ g)^{-1}(f(U)) = \operatorname{id}_{f(\Omega)}(f(U)).$$

   Notably, we are using the fact that f surjects onto $U \subseteq \Omega$ to say that $U = f^{-1}(f(U))$. Now, $f(U)$ is open by Theorem 5.54, so we are done.

2. We now compute the derivative of g by hand. Note that f is injective, so f' is locally injective everywhere, so $f'(z) \neq 0$ for all $z \in \Omega$ by Proposition 6.2.

   Now, fix $z_0 \in \Omega$ and $w_0 := f(z_0)$, which implies hat $g(w_0) = z_0$ is forced by the injectivity of f. Note that any $w \in \operatorname{im} f$ will have some unique $z \in \Omega$ with $f(z) = w$ by injectivity. As such, the continuity of f and g implies that a sequence $\{w_n\}_{n \in \mathbb{N}} \subseteq \operatorname{im} f$ has some unique pullbacks $z_n := g(w_n)$ so that $f(z_n) \to w_0$ if and only if $z_n \to z$. Thus, we can compute

   $$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)} = \frac{1}{f'(g(w_0))},$$

   which is what we wanted. ■

Remark 6.7. This result is somewhat surprising: a priori, we should only expect our inverse to be some set-theoretic construction, but in our case this happens to be holomorphic.

6.2 April 20

Welcome back everyone.

- Homework #9 is due on Sunday, at 11:59PM.
- The average on the midterm was 76.4, which is a few points lower than desired.

6.2.1 Defining Laurent Series

Today we are talking about Laurent series in their full power. This will allow us to add some power to our Residue theorem.

Quote 6.8. It is not lost on me what today is.

Anyway, we begin with the following definition.

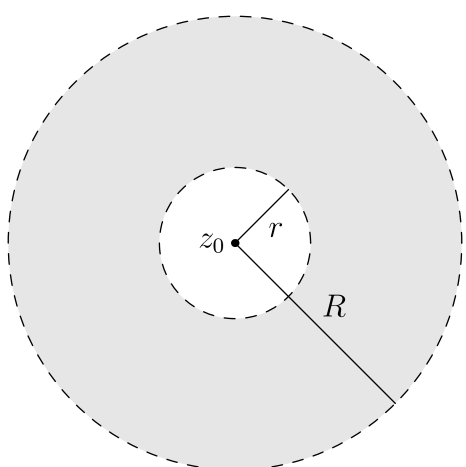
Definition 6.9 (Open annulus)**.** The *open annulus* centered z_0 is

$$A(z_0, r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

Remark 6.10. We can also write $A(z_0, r, R) = B(z_0, R) \setminus \overline{B(z_0, r)}$, so this is an open set.

Remark 6.11. We permit $r = 0$, which makes the annulus a punctured ball.

Here is the image.



Now, we pick up the following definition of a Laurent series, generalizing our previous one.

Definition 6.12 (Laurent series)**.** A *Laurent series* is a (formal) expression

$$L(z) \coloneqq \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $\{c_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$. This *converges* if and only if the individual series

$$\sum_{n=0}^{\infty} c_n z^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_{-n} z^{-n}$$

both converge.

An alternate way to state this convergence is to set

$$S_+(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n \qquad \text{and} \qquad S_-(z) \coloneqq \sum_{n=0}^{\infty} c_{-n} z^{-n}.$$

As such, we let $R_+ > 0$ be the radius of convergence of S_+ and R_- the radius of convergence of S_-, which means that both of these series will converge if and only if

$$\frac{1}{R_-} < |z| < R_+,$$

which creates an "annulus" of convergence.

Remark 6.13. In the cases we discussed previously, we had the Laurent series have a finite tail, which made $R_- = +\infty$ and hence we were able to deal with the annulus/punctured ball

$$0 < |z| < R_+.$$

We will also want a shifting.

Definition 6.14 (Laurent series)**.** A *Laurent series centered at z_0* is a (formal) expression

$$L(z) \coloneqq \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$

6.2.2 Making Laurent Series

The reason we allowed infinite tails is to give us more power with series expansions, expanding from mere meromorphic functions.

Theorem 6.15. Fix an open annulus $A(z_0, r, R)$ and a domain Ω containing $\overline{A(z_0, r, R)}$. Given a holomorphic function $f\colon \Omega \to \mathbb{C}$, we can construct

$$c_n \coloneqq \frac{1}{2\pi i} \oint_{\gamma_s} \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

where $\gamma_s\colon [0,1] \to \Omega$ is $\gamma_s(t) \coloneqq z_0 + s\exp(2\pi i t)$ for $s \in [r, R]$. Then we claim

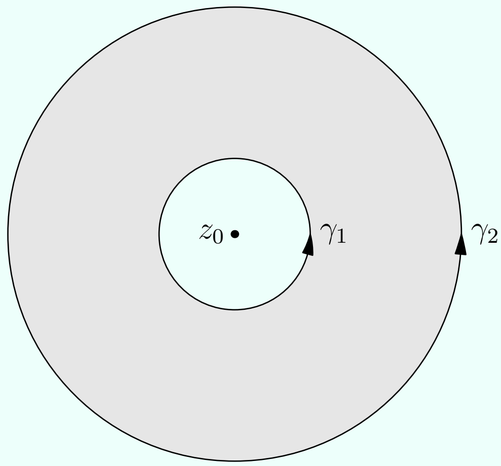
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

for $z \in A(z_0, r, R)$.

Before proving this, we need to strengthen our version of Cauchy's integral formula.

Definition 6.16 (Cycles)**.** Fix a domain Ω. A *cycle* Γ in Ω is a formal $\mathbb{C}$-linear combination of closed piecewise C^1 paths homologous to 0. We will write $\operatorname{im}\Gamma$ to be the union of the individual paths making up Γ.

Example 6.17. Consider the following annulus.



Then we can set, for example, $\Gamma := \gamma_1 + \gamma_2$.

These cycles are essentially bookkeeping devices to go around multiple paths. In particular, we have the following definitions.

Definition 6.18 (Cycle integration)**.** Fix a domain Ω and a holomorphic function $f\colon \Omega \to \mathbb{C}$. Then, given a cycle $\Gamma = \sum_{i=1}^n a_i\gamma_i$, we define

$$\oint_\Gamma f(z)\,dz := \sum_{i=1}^n a_i \oint_{\gamma_i} f(z)\,dz.$$

Definition 6.19 (Winding number, cycles)**.** Fix a domain Ω and a cycle Γ. Then we define the *winding number* of Γ around $w \in \mathbb{C}$ to be

$$\operatorname{Ind}(\Gamma, w) := \sum_{i=1}^n a_i \operatorname{Ind}(\gamma_i, w).$$

Definition 6.20 (Inside)**.** The *inside* of a cycle Γ consists of all the points $w \in \Omega \setminus \operatorname{im}\Gamma$ with nonzero winding number.

Example 6.21. Work in the context of Example 6.17.

- If we set $\Gamma := \gamma_1 + \gamma_2$, then the interior will just be everything inside γ_2.
- If we set $\Gamma := \gamma_1 - \gamma_2$, then the interior will just be everything inside γ_2 but outside γ_1: everything inside both γ_1 and γ_2 will have the winding number be $1 - 1 = 0$ and cancel out!

In particular, our cycle is letting us pick out the annulus itself.

Now, here is our stronger version of the Cauchy integral formula.

Theorem 6.22 (Cauchy integral formula)**.** Fix a domain Ω with a cycle Γ. Then, given a holomorphic function $f\colon\Omega\to\mathbb{C}$, we have

$$\oint_\Gamma f(z)\,dz=0,$$

and for any $w\in\Omega\setminus\operatorname{im}\gamma$, we have

$$f(w)\operatorname{Ind}(\Gamma,w)=\frac{1}{2\pi i}\oint_\Gamma\frac{f(z)}{z-w}\,dz.$$

Proof. Simply split all the integrals into their formal sum over Γ, apply Theorem 5.45, and then sum back to values over Γ. ■

We are now ready to prove our theorem.

Proof of Theorem 6.15. By shifting, we take z_0 to be 0. Now, we can recover $A(z_0,r,R)$ by setting $\Gamma:=\gamma_R-\gamma_r$ so that the inside of Γ is $A(z_0,r,R)$, as discussed in Example 6.21. As such, Theorem 6.22 tells us that all $w\in A(z_0,r,R)$ have

$$f(w)\operatorname{Ind}(\gamma_R,w)-f(w)\operatorname{Ind}(\gamma_r,w)=f(w)\operatorname{Ind}(\Gamma,w)=\oint_\Gamma f(z)\,dz=\oint_{\gamma_R}f(z)\,dz-\oint_{\gamma_r}f(z)\,dz.$$

Now, $\operatorname{Ind}(\gamma_R,w)=1$ and $\operatorname{Ind}(\gamma_r,w)=0$, so we get

$$f(w)=\frac{1}{2\pi i}\oint_{\gamma_R}f(z)\,dz-\frac{1}{2\pi i}\oint_{\gamma_r}f(z)\,dz.$$

We will compute the integrals separately. Indeed, we notice that any $z\in\mathbb{C}$ with $|z|>|w|$ will have

$$\frac{1}{z-w}=\frac{1/z}{1-(w/z)}\sum_{k=0}^\infty\frac{w^k}{z^{k+1}},$$

which by the Weierstrass M-test will converge uniformly when $|z|>|w|+\varepsilon$ for any $\varepsilon>0$. In particular, $R>|w|$, so we may write

$$\frac{1}{2\pi i}\oint_{\gamma_R}\frac{f(z)}{z-w}\,dz=\sum_{k=0}^\infty\underbrace{\left(\frac{1}{2\pi i}\oint_{\gamma_R}\frac{f(z)}{z^{k+1}}\,dz\right)}_{c_k}w^k$$

by interchanging the sum and integral. Similarly, $|w|>|z|$ implies

$$\frac{1}{z-w}=\frac{-1/w}{1-(z/w)}=-\sum_{k=0}^\infty\frac{z^k}{w^{k+1}}=-\sum_{k=-1}^{-\infty}\frac{w^k}{z^{k+1}}.$$

This still absolutely converges and hence uniformly converges for $|w|>|z|$, so taking $|z|=r$, we can get

$$\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)}{z-w}\,dz=-\sum_{k=-1}^{-\infty}\underbrace{\left(\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)}{z^{k+1}}\,dz\right)}_{c_k}w^k.$$

Subtracting our two integrals gets the desired result. ■

6.3 April 22

Good morning, everyone.

- Homework #9 is due on Sunday at 11:59PM. One of the questions has since been corrected.
- There are (extended) office hours today from 12:30PM–3PM because we did not have office hours yesterday.
- Homework #10 will be released later today. This will be our last homework.

6.3.1 Residue Theorem Two, Electric Boogaloo

Today we are talking about the more general Residue theorem. Last time we showed that all holomorphic functions have a Laurent series over an annulus. Here is a corollary, which will be our jumping-off point.

Corollary 6.23. Fix a domain Ω and a holomorphic function $f\colon \Omega \to \mathbb{C}$. If $z_0 \in \overline{\Omega}$ is an isolated singularity of f, then f has a Laurent series expansion at z_0 in the punctured ball $B(z_0, r) \setminus \{z_0\}$ for some (small) $r > 0$.

Proof. When z_0 is a pole, we were able to make our Laurent series with finite tail, and we were able to control the size of the tail.

Anyway, by shifting we may assume that $z_0 = 0$. Now, fix any r so that $B(z_0, r) \setminus \{z_0\} \subseteq \Omega$, and Theorem 6.15 promises us that any $r' > 0$ will give $\overline{A(0, r', r)} \subseteq \Omega$ with

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right) z^n,$$

where $s := r$. Now, the coefficient depends on r but not on r', so we may send r' to 0 to say that this series holds on $B(z_0, r) \setminus \{z_0\}$. This finishes. ■

And here are our generalized versions of residue and principal part.

Definition 6.24 (Residue)**.** Fix a domain Ω and some isolated set $S \subseteq \Omega$ so that $f\colon (\Omega \setminus S) \to \mathbb{C}$ can be a holomorphic function with isolated singularities S. Then, writing our Laurent series for f as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

we define the *residue* as $\operatorname{Res}_{z_0}(f) := c_{-1}$.

Definition 6.25 (Residue)**.** Fix a domain Ω and some isolated set $S \subseteq \Omega$ so that $f\colon (\Omega \setminus S) \to \mathbb{C}$ can be a holomorphic function with isolated singularities S. Then, writing our Laurent series for f as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

we define the *principal part* as $P_{f,z_0}(z) := \sum_{n=-1}^{-\infty} c_n (z - z_0)^n$.

And so here is our theorem.

Theorem 6.26 (Residue)**.** Fix a domain Ω and a finite set $S \subseteq \Omega$ so that $f\colon (\Omega \setminus S) \to \mathbb{C}$ can be a holomorphic function with isolated singularities S. Given a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$ such that $\operatorname{im}\gamma \cap S = \varnothing$ with inside contained in Ω (i.e., homologous to 0 in Ω), we have

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \in S} \operatorname{Res}_{z_0}(f)\operatorname{Ind}(\gamma, z_0).$$

Proof. We imitate the proof of Theorem 5.24. For each $z_0 \in S$, let P_{f,z_0} denote the principal part of f at z_0. In particular, for each $z_0 \in \mathbb{C}$, we see that $f - P_{f,z_0}$ is holomorphic at z_0 and that P_{f,z_0} is holomorphic at all points aside z_0 (and therefore won't affect differentiability away from z_0). Thus,

$$f - \sum_{z_0 \in S} P_{f,z_0}$$

is holomorphic on Ω. In particular, because $\operatorname{im}\gamma \cap S$ and that γ is homologous to 0 in Ω, we may bop this with Theorem 5.45 to see

$$\oint_\gamma \left(f(z) - \sum_{z_0 \in S} P_{f,z_0}(z) \right) dz = 0.$$

Rearranging, we see

$$\oint_\gamma f(z)\,dz = \sum_{z_0 \in S} \oint_\gamma P_{f,z_0}(z)\,dz.$$

Now, in our proof of Theorem 6.15, we showed that the series for P_{f,z_0} converges uniformly on $\operatorname{im}\gamma$ because $\operatorname{im}\gamma$ is a compact set away from z_0 (namely, they were integrals of some geometric series, which have a perfectly fine radius of convergence). Thus, fixing some particular P_{f,z_0}, we compute

$$\oint_\gamma P_{f,z_0}(z)\,dz = \oint_\gamma \sum_{k=-1}^{-\infty} c_{k,z_0}(z-z_0)^k\,dz = \sum_{k=-1}^{-\infty} c_{k,z_0} \oint_\gamma (z-z_0)^k\,dz.$$

Now, for each $k < -1$, the function $(z-z_0)^k$ has a primitive (namely, $\frac{1}{k+1}(z-z_0)^{k+1}$), so Corollary 4.39 tells us that the integral vanishes. Otherwise, at $k = -1$, we see that

$$\oint_\gamma P_{f,z_0}(z)\,dz = c_{-1,z_0} \oint_\gamma \frac{1}{z-z_0}\,dz = 2\pi i \operatorname{Res}_{z_0}(f)\operatorname{Ind}(\gamma, z_0).$$

Thus,

$$\oint_\gamma f(z)\,dz = \sum_{z_0 \in S} \oint_\gamma P_{f,z_0}(z)\,dz = 2\pi i \sum_{z_0 \in S} \operatorname{Res}_{z_0}(f)\operatorname{Ind}(\gamma, z_0),$$

which is what we wanted. ■

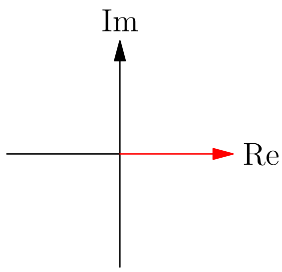
6.3.2 Example Contour Integral

So now we get to compute all the integrals we could want.

Exercise 6.27. We compute

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx.$$

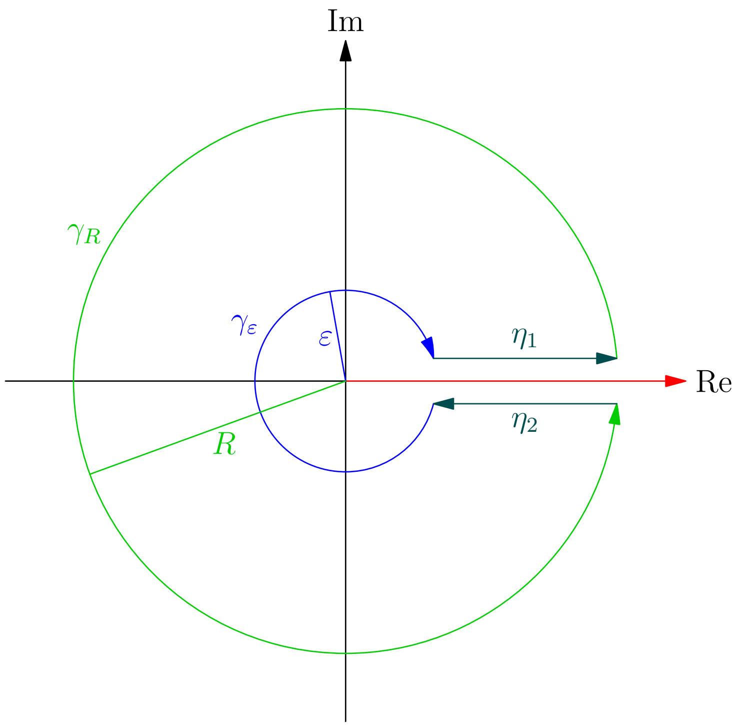
Proof. We use a keyhole contour. To begin, we fix the branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, and we will work with the following image, where the red is our ray of death.



As such, we set $f(z) := \sqrt{z}/\left(z^2+1\right)$ and write

$$z := r\exp(i\theta)$$

where now $\theta \in (0, 2\pi)$ avoids the ray of death. We now draw the following contour.



Let γ be the full contour. To be explicit, γ_ε and γ_R are two arcs, oriented as drawn, with radii ε and R respectively. Then "cut out" from these are the horizontal paths η_1 and η_2 to connect them. We will send $\varepsilon \to 0$ and $R \to \infty$ so that the figure essentially becomes two copies of the real line, moving in opposite directions. As such, we compute the integrals making up γ one at a time.

- As $R \to \infty$, the integral along γ_R becomes a circle. So we bound $|f(z)| \le \frac{\sqrt{R}}{R^2+1}$ so that

  $$\left|\oint_{\gamma_R} f(z)\,dz\right| \le 2\pi R \cdot \frac{\sqrt{R}}{R^2+1},$$

  which goes to 0 for R large.

- As $\varepsilon \to 0$, the integral along γ_ε becomes a circle. So we do the same bound to see that

  $$\left|\oint_{\gamma_\varepsilon} f(z)\,dz\right| \le 2\pi\varepsilon \cdot \frac{\sqrt{\varepsilon}}{\varepsilon^2+1},$$

  which still goes to 0 for ε small.

- It remains to compute the integral the integrals over η_1 and η_2. Because η_1 and η_2 have constant imaginary parts, we let this imaginary part be $\pm\delta$, which goes to 0 for R large and ε small. As such

$$\lim_{\delta\to 0}\oint_{\eta_1} f(z)\,dz = \oint_{\varepsilon}^{R} \frac{\sqrt{x}}{x^2+1}\,dz$$

  and

$$\lim_{\delta\to 0}\oint_{\eta_2} f(z)\,dz = \oint_{R}^{\varepsilon} \frac{-\sqrt{x}}{x^2+1}\,dz.$$

  In particular, we have a $-$ sign here because η_2 lives on the other side of our ray of death/branch cut. Thus, the sum of the two integrals over the $\eta_\bullet$s is simply

$$2\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$$

  as δ goes to 0.

- It remains to compute the integral of f over the entire contour. We use Theorem 6.26; note that f only has poles at $\pm i$, and the square root portion can be defined to be holomorphic, given our branch cut. Thus, we compute

$$\operatorname{Res}_i(f) = \frac{1}{2}\exp(7\pi i/4) \qquad \text{and} \qquad \operatorname{Res}_{-i}(f) = -\frac{1}{2}\exp(i\pi/4).$$

  So in total, our integral comes out to

$$\oint_\gamma f(z)\,dz = 2\pi i\big(\operatorname{Res}_i(f) + \operatorname{Res}_{-i}(f)\big) = \pi\sqrt{2}$$

  because our only singularities are at $\pm i$, where we are using Theorem 6.26.

Synthesizing, we find that

$$\oint_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{1}{2}\lim_{\substack{R\to 0\\ \varepsilon\to 0}}\oint_\gamma f(z)\,dz = \frac{\pi}{\sqrt{2}}.$$

This finishes. ■

6.4 April 25

Good morning, everyone.

- Homework #10 is due on Friday, at 11:59PM.
- Course evaluations exist.
- Today will be our last "material for the final."
- On Wednesday, we'll talk about complex dynamics. There is a talk (for general audience) on complex dynamics on Thursday at 4:10PM, in Evans 60.

6.4.1 Möbius Transformations

Today we are talking about Möbius transformations. Here is our definition.

Definition 6.28 (Möbius tranformation)**.** Fix a domain Ω. A *Möbius transformation* is a function $f\colon \Omega \to \mathbb{C}$ of the form

$$f(z) \coloneqq \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

The point is that Möbius transformations are more or less matrices in $\mathrm{GL}_2(\mathbb{C})$, the group of 2×2 matrices with complex coefficients. Namely,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z \coloneqq \frac{az+b}{cz+d}$$

provides a group action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}$.

Example 6.29. When $c = 0$, then $f(z) = \frac{az+b}{d}$ with $ad \neq 0$, so f is non-constant and entire.

Example 6.30. When $c \neq 0$, then $f(z) = \frac{az+b}{cz+d}$ will have a pole at $z = -d/c$. Notably, $a \cdot -d/c + b \neq 0$ because $ad - bc \neq 0$, so this singularity is indeed not removable.

6.4.2 Generating Möbius Transformations

There are, roughly speaking, three types of Möbius transformations.

Definition 6.31 (Möbius transformation, types)**.** Here are some examples of Möbius transformations.

- The Möbius transformations
$$T_b(z) \coloneqq \frac{1z+b}{0z+1} = z + b$$
are called the *translations*.
- The Möbius transformations
$$D_a(z) \coloneqq \frac{az+0}{0z+1} = ax$$
are called the *dilations*.
- The Möbius transformation
$$I(z) \coloneqq \frac{0z+1}{1z+0} = \frac{1}{z}$$
is called the *inversion*.

It will turn out that these generate all of our Möbius transformations.

Here are some computational lemmas to rigorize our notion of "generate."

Lemma 6.32. Let f and g be Möbius transformations.

- $f \circ g$ is also a Möbius transformation, with composition give as multiplication of matrices.
- f is bijective, and its inverse is
$$f^{-1}(z) \coloneqq \frac{dz-b}{-cz+a}.$$

Proof. The first is a direct computation. The second comes down to noting that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

but the factor of $1/(ad-bc)$ does nothing. ■

Remark 6.33. The above computations turn our set of Möbius transformations into a group under composition.

And here is our result.

Proposition 6.34. Every Möbius transformation can be written as a composition of translations, dilations, and inversions.

Proof. We proceed by hand. Fix

$$f(z) := \frac{az+b}{cz+d}$$

a Möbius transformation.

- If $c \neq 0$, then proceed as

$$z \overset{D_c}{\longmapsto} cz \overset{T_d}{\longmapsto} cz+d \overset{I}{\longmapsto} \frac{1}{cz+d} \overset{D_{(bc-ad)/c}}{\longmapsto} \frac{(bc-ad)/c}{cz+d}.$$

  From here, we can apply $T_{a/c}$ to get

$$\frac{a}{c} + \frac{(bc-ad)/c}{cz+d} = \frac{1}{c}\left(\frac{a(cz+d)+bc-ad}{cz+d}\right) = \frac{az+b}{cz+d}.$$

  So in total, we have

$$f = T_{a/c} \circ D_{(bc-ad)/c} \circ I \circ T_d \circ D_c.$$

- If $c = 0$, then proceed as

$$z \overset{TD_{a/d}}{\longmapsto} \frac{a}{d}z \overset{D_{a/d}}{\longmapsto} \frac{a}{d}z + \frac{b}{d},$$

  which checks $f = T_{b/d} \circ D_{a/d}$.

These cases finish the proof. ■

Exercise 6.35. We verify Proposition 6.34 for

$$f(z) := \frac{iz+0}{z-i}.$$

Proof. Following the algorithm, we get

$$z \longmapsto z \longmapsto z-i \longmapsto \frac{1}{z-i} \longmapsto \frac{-1}{z-i} \longmapsto \frac{-1}{z-i} + i.$$

We can check that $\frac{-1}{z-i} + i = \frac{iz}{z-i} = f(z)$, which finishes. ■

6.4.3 Classifying Automorphisms of $B(0,1)$

The point of Möbius transformations is to be able to describe certain very nice maps. Here is our definition.

Definition 6.36 (Biholomorphic)**.** Fix domains Ω_1, Ω_2. A function $f\colon \Omega_1 \to \Omega_2$ is *biholomorphic* if and only if f is bijective and holomorphic.

Note that, by Theorem 6.6, we know that the inverse function f^{-1} is holomorphic.

In the case of $\Omega_1 = \Omega_2$, we get a well-defined composition and hence group structure.

Definition 6.37 (Automorphism)**.** Fix a domain Ω. Then the *automorphism group* of Ω is

$$\operatorname{Aut}(\Omega) := \{\text{biholomorphic maps } f\colon \Omega \to \Omega\}.$$

Automorphisms (and more generally biholomorphic maps) are good to consider because they are in some sense the natural symmetries of a complex space, so we often want to "mod out" by them in some suitable sense.

Anyway, here is our theorem.

Theorem 6.38. The group $\operatorname{Aut}(B(0,1))$ is equal to

$$\left\{ f(z) := \frac{az+b}{cz+d} : |a|^2 - |b|^2 = 1 \qquad \text{and} \qquad c = \overline{b}, d = \overline{a} \right\}.$$

Proof. We show our inclusions separately.

- Let $f\colon B(0,1) \to B(0,1)$ be an automorphism; we will show that f is a Möbius transformation of the required type. Fix some $z \in B(0,1)$ and $w := f(z)$. There are three steps.

  1. Suppose that $f(0) = 0$. Then we may apply the Schwarz lemma: Corollary 5.8 with f^{-1}, which tells us that
  $$|z| = \left|f^{-1}(w)\right| \le |w| = |f(z)|.$$
  Applying Corollary 5.8 this time to f tells us that
  $$|w| = |f(z)| \le |z|.$$
  In particular, $|f(z)| = |z|$, so Corollary 5.8 one more time (namely, the second sentence) tells us that $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$; note that $|f(z)| = |z|$ forces $|\alpha| = 1$.
  Now, setting $\alpha = r\exp(i\theta)$, we see $r = 1$ is forced, so we take $a := \exp(i\theta/2)$ and $b := 0$ and $c := 0$ and $d := \exp(-i\theta/2)$ to get
  $$f(z) = \alpha z = \exp(i\theta)z = \frac{\exp(i\theta/2)z + 0}{0 + \exp(-i\theta/2)},$$
  which finishes.
  2. Suppose that $c := f(0) \neq 0$ with $|c| < 1$. Notably, $|c| > 0$ as well. Now, set
  $$g(z) := \frac{z-c}{1-\overline{c}z} = \frac{z-c}{-\overline{c}+1}.$$
  Now, the only pole here is at $1/\overline{c}$, which has magnitude larger than 1 and hence does not live in the ball, so g is holomorphic on $B(0,1)$. We claim that g is an automorphism in $B(0,1)$, for which we need to show that $g\colon B(0,1) \to B(0,1)$ is a bijection.

- In one direction, suppose $z \in B(0,1)$. Then

$$|z|^2 - |cz|^2 = |z|^2\left(1 - |c|^2\right) < 1 - |c|^2,$$

which rearranges to $|z|^2 + |c|^2 < 1 + |cz|^2$, which gives

$$|g(z)|^2 = \frac{|z|^2 + |c|^2 - \overline{c}z - c\overline{z}}{1 + |cz|^2 - \overline{c}z - z\overline{c}} < 1$$

by using our bound above.

- In the other direction, we note that the inverse of g is

$$g^{-1}(z) = \frac{z - \overline{c}}{-c + 1}$$

from Lemma 6.32, which has the same form as g, so we appeal to the previous case.

3. To finish, consider $g \circ f$. This is certainly an automorphism because compositions of automorphisms give another automorphism. But

$$(g \circ f)(0) = g(f(0)) = g(c) = 0,$$

so we conclude from our first step that $g \circ f$ is a dilation of the form $z \mapsto \exp(i\theta)z$. In particular, we get to write

$$f(z) = g^{-1}(\exp(i\theta)z) = \frac{\exp(i\theta)z + c}{\overline{c}\exp(i\theta)z + 1}$$

from the above computation. As such, we set $d := 1/\left(1 - |c|^2\right)$ and $a := \sqrt{d}\exp(i\theta/2)$ and $b := c\sqrt{d}\exp(i\theta)$. Then we can check by hand that

$$f(z) = \frac{az + b}{cz + d}$$

and $|a|^2 - |b|^2 = d\left(1 - |c|^2\right) = 1$. This finishes.

- We omit the proof that all the given Möbius transformations are in fact automorphisms. The proof is essentially the second point above, given more generally.

The above inclusions finish the proof. ■

Remark 6.39. The conditions on a, b, c, d force $ad - bc \neq 0$. In particular, $ad - bc = |a|^2 - |b|^2 = 1$.

We close with a warning.

Warning 6.40. Möbius transformations are not in bijection with matrices.

The main point is that

$$f(z) = \frac{az + b}{cz + d} = \frac{waz + wb}{wcz + wd}$$

for any $w \in \mathbb{C}^\times$. As such, our Möbius transformations turn out to really be in bijection with elements of $\mathrm{PGL}_2(\mathbb{C})$, where we have modded out by the center. In particular, we can put elements of the form

$$f(z) := \frac{z - a}{1 - \overline{a}z} \in \mathrm{Aut}(B(0,1))$$

in the correct form, with some elbow grease.

6.5 April 27

Good morning, everyone.

- Homework #10 is now due on Sunday at 11:59PM. Cool.
- There is a colloquium on complex dynamics at Evans 60, by Sarah Koch.

6.5.1 The Mandelbrot Set

Today we are talking about complex dynamics. Complex dynamics is the behavior of objects under iteration. As an example, we let $c \in \mathbb{C}$ vary with the function

$$f_c(z) := z^2 + c.$$

For example, we might ask what happens to the point 0 as we iterate it through f_c.

Example 6.41. Fix $c = 1$ so that $f_1(c) = z^2 + 1$. Then we compute

$$\begin{aligned} f_1(0) &= 1, \\ f_1(1) &= 2, \\ f_1(2) &= 5, \\ f_1(5) &= 26. \end{aligned}$$

This is called "blowing up" because 0's iterations are to infinity.

It turns out that there are, roughly speaking, two options for the behavior of these iterations.

- Perhaps $\left|f^{(n)}(0)\right| \to \infty$ as $n \to \infty$.
- Perhaps $\left|f^{(n)}(0)\right|$ is bounded.

To make this easier to compute, we have the following lemma.

Lemma 6.42. Fix $c \in \mathbb{C}$ and define $\{z_n\}_{n\in\mathbb{N}}$ by $z_0 := 0$ and $z_n := f_c(z_{n-1})$ for $n > 0$. If $|z_n| > 2$ for any n, then $|z_n| \to \infty$ as $n \to \infty$.

Proof. On one hand, take $|c| \le 2$, we see that

$$|z_{n+1}| \ge |z_n|^2 - |c| > 2|z_n| - 2,$$

so

$$|z_{n+1}| - 2 > 2(|z_n| - 2)$$

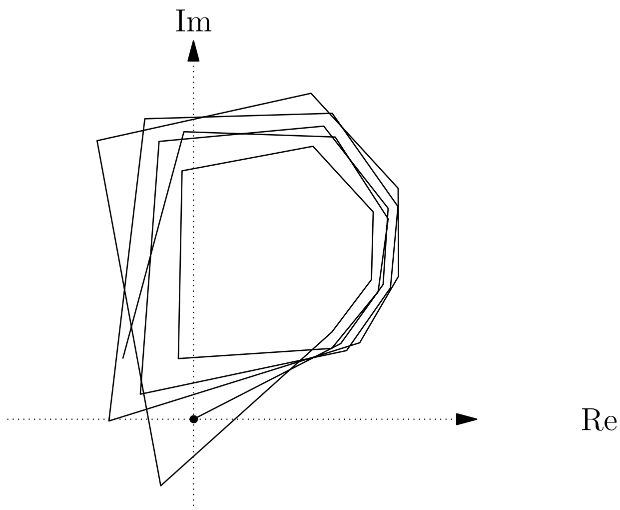
which goes to ∞ because it's constantly doubling. If $|c| > 2$, one can do something similar. ■

Example 6.43. For $c = -1$, we have

$$f_{-1}(0) = -1,$$

which will simply repeat itself, which in particular is bounded.

We want to be able to do lots of computations, so we should use a computer. Here are the iterations for $z = 0.39 + 0.2i$.



This path looks bounded.

The set of points c for which this remains bounded has a name.

Definition 6.44 (Mandelbrot set)**.** The *Mandelbrot set* is the set of all $c \in \mathbb{C}$ such that

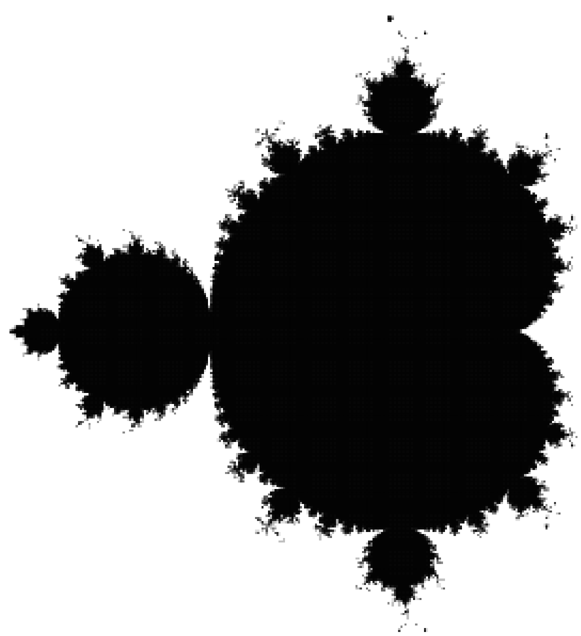
$$\left\{ f_c^{(n)}(0) : n \in \mathbb{N} \right\}$$

is bounded.

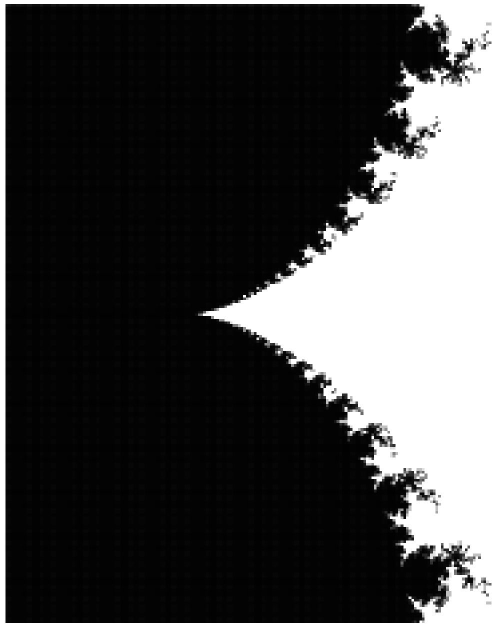
Remark 6.45. The Mandelbrot set is named by Benoit B. Mandelbrot.

Remark 6.46. The "B" in Benoit B. Mandelbrot stands for "Benoit B. Mandelbrot."

If we color all the points c for which we remain bounded, we get the following figure. (The graphics were created with Asymptote.)



This is more fun to Zoom it; here we have zoomed in to $z = 0.25$.



There are fun things that we can say about the Mandelbrot set, even though it looks very strange.

Theorem 6.47. The Mandelbrot set is connected.

The proof is about 200 pages, in French, like all good mathematics.

Remark 6.48. It is conjectured that the Mandelbrot set is "locally connected"—every point has a connected neighborhood.

6.5.2 Julia Sets

One might ask what happens if we fix c and then let the starting point z vary instead. This gives the following definition.

Definition 6.49 (Julia set)**.** Fix c. Then the set

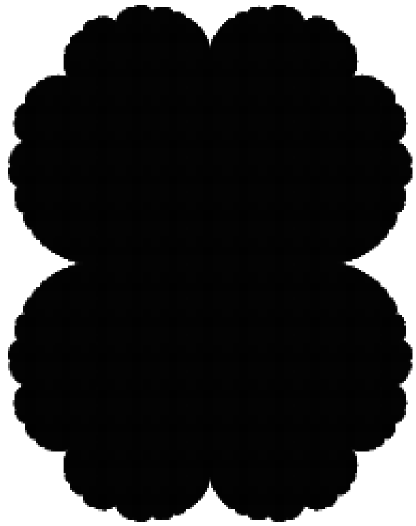
$$\left\{x \in \mathbb{C} : \left|f_c^{(n)}(x)\right| \text{ is bounded as } n \to \infty\right\}$$

is the *filled Julia set of* c.

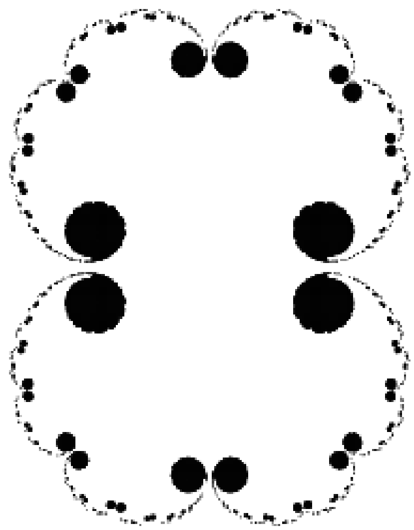
Remark 6.50. Gaston Julia, for whom Julia sets are named after, is often pictured wearing a mask because he lost his nose in World War I.

Example 6.51. Fix $c = 0$ so that we are looking at $f^{(n)}(z) = z^{2^n}$. Then we can see that the Julia set is just $\overline{B(0,1)}$—everything outside here will have exploding norm, and certainly $|z| \le 1$ implies $\left|z^{2^n}\right| \le 1$.

Connectivity of Julia sets is a somewhat strange phenomenon. Here is the filled Julia set for $c = 0.25$.



And here is the filled Julia set for $c = 0.26$.



So indeed, connectivity looks sporadic, in some sense. Here is an amazing result.

Theorem 6.52. The Mandelbrot set is precisely the values of $c \in \mathbb{C}$ so that the filled Julia set of c is connected.

In general, dynamics questions are somewhat easy to state but very hard to answer. Here is an example.

Definition 6.53. A complex number $z \in \mathbb{C}$ is *preperiodic* for a polynomial $f(z) \in \mathbb{C}[z]$ if and only if there are distinct m and n so that

$$f^{(m)}(z) = f^{(n)}(z).$$

The image here is that the points should "loop" in on themselves, in some sense. And here is our result.

Theorem 6.54. Fix an integer $d \geq 2$ and complex numbers $a, b \in \mathbb{C}$. Then the set of parameters $c \in \mathbb{C}$ such that both a and b are preperiodic for the polynomial $f(z) := z^d + c$ is infinite if and only if $a^d = b^d$.

This is a really hard result, proven in roughly the last decade.

6.6 April 29

Good morning, everyone. Welcome to the last day of class.

6.6.1 Complex Numbers and Their Topology

Today we summarize the course. We began our story with the complex numbers.

Definition 2.1 (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here i is some symbol such that $i^2 = -1$ formally.

However, we wanted to turn this into a space, more specifically a metric space.

Definition 2.11 (Distance on $\mathbb{C}$)**.** Given complex numbers $z = a + bi$ and $w = c + di$, we define the *distance* between z and w to be

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}.$$

From here, we could define open balls and open subsets.

Definition 2.18 (Open ball)**.** Given some $z_0 \in \mathbb{C}$, then *open ball* centered at z_0 with radius $r > 0$ is

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Observe $z_0 \in B(z_0, r)$.

Then the open balls formed a basis of our topological space, giving our open sets.

Definition 2.25 (Open)**.** A subset $X \subseteq \mathbb{C}$ is *open* if and only if, for each $z \in X$, there exists $r > 0$ such that $B(z, r) \subseteq X$.

6.6.2 Complex Functions

With a topology in hand, we were able to talk about continuity; here are a few equivalent conditions.

Lemma 2.92. Suppose that $f\colon X \to \mathbb{C}$.

(a) Then f is continuous at w if and only if every sequence $\{z_n\} \subseteq X$ such that $z_n \to z$ implies $f(z_n) \to f(z)$.

(b) We have that f is continuous on X if and only if every open set $U \subseteq \mathbb{C}$ has $f^{-1}(U)$ open in X.

(c) We have that f is continuous on X if and only if each closed set $V \subseteq X$ has $f^{-1}(V)$ closed in X.

(d) Lastly, we have that f is continuous at if and only if, for each $\varepsilon > 0$ and $z \in \mathbb{C}$, we have that $f^{-1}(B(z, \varepsilon))$ is open in X.

However, to really be able to talk about complex analysis, we need to introduce our notion of differentiation.

Definition 3.2 (Differentiable)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ a function. Then f is *complex differentiable at* $z_0 \in \Omega$ *with derivative* $\alpha \in \mathbb{C}$ if and only if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \alpha.$$

We write this as $f'(z_0) = \alpha$.

Complex differentiability (as above) turns out to be very strong because the limit is taking place in the two-dimensional plane $\mathbb{C}$.

Functions differentiable everywhere had a special name.

Definition 3.4 (Holomorphic, entire)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ a function. Then f is *holomorphic on* Ω if and only if f is complex differentiable at each $z_0 \in \mathbb{C}$. If $\Omega = \mathbb{C}$, then we say f is *entire*.

We were able to show that a variety of functions were holomorphic, from polynomials to power series. Not all "smoothish" functions were holomorphic, such as $z \mapsto |z|$ and $z \mapsto \operatorname{Re} z$.

As our first taste of the power of complex differentiability, we saw that it was a strictly stronger condition than merely being differentiable as a function $\mathbb{R}^2 \to \mathbb{R}^2$: we had to satisfy some partial differential equations.

Theorem 3.19 (Cauchy–Riemann)**.** Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f\colon \Omega \to \mathbb{C}$ a function differentiable at some $z_0 = x_0 + y_0 i \in \mathbb{C}$. If we write $f(x + yi) = u(x, y) + i(x, y)$, then

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

In fact, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

The above result had a pretty natural proof, essentially by writing down what we need for complex differentiability on the real and imaginary axis.

However, it turns out that this real and imaginary information was also sufficient.

Theorem 3.26. Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f\colon \Omega \to \mathbb{C}$ a function. Writing $f(x + yi) = u(x, y) + iv(x, y)$ and fixing some $z_0 := x_0 + y_0 i$, then suppose we have the following.

- We have u_x, u_y, v_x, v_y all exist and are continuous (!).
- We have
$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

Then f is differentiable at z_0.

6.6.3 Integration

Having talked about derivatives, we were able to integrate.

Definition 4.25 (Integration)**.** Fix $\Omega \subseteq \mathbb{C}$ an open and connected subset with a C^1 path $\gamma\colon [a,b] \to \Omega$. Now, given a continuous function $f\colon \Omega \to \mathbb{C}$, we define the *integral*

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt,$$

if the integral exists.

This definition was extended to piecewise C^1 paths in the natural way.

The point of studying integration was for the Cauchy integral formula. More concretely, the story of integration tied nicely into the story of analytic functions.

Definition 3.47 (Analytic)**.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f\colon X \to \mathbb{C}$ a function. Then f is *analytic* at $z_0 \in \mathbb{C}$ if and only if f has a power series expansion at z_0. Explicitly, there is a power series $S(z) = \sum_{k=0}^\infty a_k z^k$ and positive real number $r > 0$ (less than the radius of convergence) such that

$$f(z) = S(z - z_0) = \sum_{k=0}^\infty a_k (z - z_0)^k$$

for any $z \in B(z_0, r)$. Then f is *analytic* if and only if it is analytic at each $z_0 \in \mathbb{C}$.

Because power series were differentiable, we were able to get the following result.

Lemma 3.51. Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f\colon X \to \mathbb{C}$ an analytic function. Then f' is also analytic.

It turns out that the converse is also true: holomorphic functions were analytic.

To codify our connection, we needed to talk about winding numbers. Roughly speaking, $\operatorname{Ind}(\gamma, z_0)$ refers to the number of times γ goes around z_0 (with counterclockwise orientation). So our first hint that integration would be helpful for us is that it actually let us compute winding numbers.

Lemma 4.54. Fix $\gamma\colon [0,1] \to \mathbb{C}$ a closed, piecewise C^1 path. Further, fix $z_0 \in \mathbb{C} \setminus \operatorname{im}\gamma$. Then

$$\operatorname{Ind}(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_0}\,dz.$$

From here, we could define more generalized winding numbers.

Definition 4.59 (Index)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ and a closed piecewise C^1 path $\gamma\colon [a,b] \to \Omega$. Given a function $f\colon \Omega \to \mathbb{C}$ which is continuous on $\operatorname{im}\gamma$, we define

$$\operatorname{Ind}_f(\gamma, w) := \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - w}\,dz$$

The point of this? The Cauchy integral formula.

Theorem 4.63 (Cauchy integral formula)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and some $z_0 \in \Omega$ with $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$. Further, fix the path $\gamma\colon [0,1] \to \Omega$ given by

$$\gamma(t) := z_0 + r\exp(2\pi i t).$$

Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic, then any $w \in B(z_0,r)$ has

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \operatorname{Ind}_f(\gamma,w).$$

From here, we could in fact, prove our goal.

Corollary 4.71. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ some holomorphic function. Then f is analytic at any $z_0 \in \Omega$. In fact, for any $r > 0$ such that $\overline{B(z_0,r)} \subseteq \Omega$, the path

$$\gamma(t) := z_0 + r\exp(2\pi i t)$$

gives

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{n+1}}\,dz.$$

The main ingredient in the proof of Theorem 4.63 was the Cauchy–Goursat theorem.

Theorem 4.70. Fix an open, connected, star-like subset $\Omega \subseteq \mathbb{C}$ with respect to z_0. Further, fix a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$. Then, if $f\colon \Omega \to \mathbb{C}$ is holomorphic,

$$\oint_\gamma f(z)\,dz = 0.$$

The Cauchy–Goursat theorem was first proven for triangles by some geometric argument and then generalized to star-like domains.

The Cauchy integral formula gave us all sorts of lovely corollaries. Let's start with Liouville's theorem.

Theorem 4.80 (Liouville's)**.** Fix an entire function $f\colon \mathbb{C} \to \mathbb{C}$. If f is bounded, then f is constant.

From here followed the Fundamental theorem of algebra.

Theorem 4.81 (Fundamental theorem of algebra)**.** Fix a polynomial $p(z) \in \mathbb{C}[z]$ of degree $n > 0$. Then p has a root in $\mathbb{C}$.

My personal favorite corollary was the Identity theorem.

Theorem 5.1 (Identity)**.** Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ with two holomorphic functions $f_1, f_2\colon \Omega \to \mathbb{C}$. Further, set

$$Z := \{z \in \Omega : f_1(z) = f_2(z)\}.$$

If Z contains an accumulation point, then $f_2 = f_2$ on Ω.

6.6.4 Singularities

Another consequence of the Cauchy integral formula was that it let us study singularities. The most basic form was removable singularities.

Theorem 4.84 (Riemann removable singularity)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$, and pick up some $z_0 \in \Omega$. If $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then f extends to a holomorphic function on Ω.

More generally, we had the following classification of singularities.

Definition 5.10 (Regular, singular)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$.

- A point $z_0 \in \overline{\Omega}$ is *regular* if and only if f is holomorphic at z_0.
- A point $z_0 \in \overline{\Omega}$ is a *singularity* otherwise.

Definition 5.11 (Isolated singularity)**.** Fix an open and connected subset $\Omega \subseteq \mathbb{C}$ with a function $f\colon \Omega \to \mathbb{C}$. A point $z_0 \in \overline{\Omega}$ is an *isolated singularity* if and only if we can find $r > 0$ with $B(z,r) \subseteq \mathbb{C}$ such that f is holomorphic on $B(z_0,r) \setminus \{z\}$.

- z_0 is *removable* if and only if f is bounded near z_0.
- z_0 is a *pole* if and only if f is not bounded near z_0, but z_0 is a removable singularity of $1/f(z)$.
- z_0 is an *essential singularity* if and only if z_0 is neither removable nor a pole.

We could understand removable singularities from the Riemann removable singularity theorem above, but more work was required to understand poles and essential singularities.

To begin, we started with poles. The key to understanding them was the Laurent series.

Definition 5.21 (Laurent expansion)**.** In the context of Lemma 5.20, the "power series" expansion

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k$$

is the *Laurent expansion* of f at z_0; here m is the order of the pole at z_0.

Having access to Laurent expansions gave us a Residue theorem.

Theorem 5.24 (Residue)**.** Fix a primitive domain $\Omega \subseteq \mathbb{C}$ and some finite subset $S \subseteq \Omega$ such that we have a holomorphic function $f\colon \Omega \setminus S \to \mathbb{C}$, where S consists of the poles of f. Now, if $\gamma\colon [0,1] \to \Omega$ is a closed, piecewise C^1 path such that $\operatorname{im}\gamma \cap S = \varnothing$, then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \in S} \operatorname{Res}_{z_0}(f)\operatorname{Ind}(\gamma, z_0).$$

In particular, if f were holomorphic within the interior of γ, then Theorem 4.70 could tell us that the integral should be 0.

It is possible to generalize Theorem 5.24 by simply removing the condition on poles.

Theorem 6.26 (Residue)**.** Fix a domain Ω and a finite set $S \subseteq \Omega$ so that $f\colon (\Omega \setminus S) \to \mathbb{C}$ can be a holomorphic function with isolated singularities S. Given a closed, piecewise C^1 path $\gamma\colon [0,1] \to \Omega$ such that $\operatorname{im}\gamma \cap S = \varnothing$ with inside contained in Ω (i.e., homologous to 0 in Ω), we have

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \in S} \operatorname{Res}_{z_0}(f)\operatorname{Ind}(\gamma, z_0).$$

The main ingredient in the proof of Theorem 6.26 was a more general version of the Cauchy integral formula.

Theorem 6.22 (Cauchy integral formula)**.** Fix a domain Ω with a cycle Γ. Then, given a holomorphic function $f\colon\Omega\to\mathbb{C}$, we have

$$\oint_\Gamma f(z)\,dz=0,$$

and for any $w\in\Omega\setminus\operatorname{im}\gamma$, we have

$$f(w)\operatorname{Ind}(\Gamma,w)=\frac{1}{2\pi i}\oint_\Gamma\frac{f(z)}{z-w}\,dz.$$

We then closed the course by discussing Möbius transformations and complex dynamics, for fun.

Remark 6.55. In a future course, one might see Weierstrass factorization, the Riemann mapping theorem, and much more. We'll see you there.

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List of Definitions