214: Differential Topology

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
BUILDING MANIFOLDS

So the man gave him the bricks, and he built his house with them.

—Joseph Jacobs, "The Story of the Three Little Pigs" [Jac90]

1.1 January 16

Let's just get started.

1.1.1 Course Structure

Here are some quick notes.

- There is a bCourses page: https://bcourses.berkeley.edu/courses/1533116. For example, it has the syllabus.
- The textbook is Lee's *Introduction to Smooth Manifolds* [Lee13]. We will read most of it.
- Our instructor is Professor Eric Chen, whose email can be reached at ecc@berkeley.edu. Office hours are after class in Evans 707.
- There is a GSI, who is Tahsia Saffat, whose email is tahsin_saffat@math.berkeley.edu. He will have some office hours and grade some homeworks.
- Homework will in general be due at 11:59PM on Thursdays via Gradescope.
- There will be an in-class midterm and a final.
- Grading is 30% homework, 30% midterm, and 40% final.
- This is a math class, not so geared towards applied subjects.
- In particular, we will assume a fair amount of topology, for which we use [Elb22] as a reference.

Let's also give a couple of notes on the course content. This course is on differential topology. The topology of interest will come from manifolds, and the differential part comes from some smoothness properties.

In some sense, our goal is to "do calculus" (e.g., differentiation, integration, vector fields, etc.) on spaces which look locally like some Euclidean space, such as a sphere. We also want to understand (smooth) manifolds on their own terms, such as understanding the maps between them and understanding some classical examples and constructions such as Lie groups or quotient manifolds.

1.1.2 Topology Review

Anyway let's get started. This is a class on manifolds, so perhaps we should begin by defining a manifold. These are going to form a special kind of topological space, so let's review topologies. We will freely use topological facts which we are too lazy to prove from [Elb22].

Definition 1.1 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X satisfying the following.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- Finite intersection: given $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Union: for any subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have the union $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We say that the collection $\mathcal{T}$ is the collection of *open sets* of X. We will also suppress the collection $\mathcal{T}$ from the notation as much as possible.

Here is some helpful language.

Definition 1.2 (open, closed, neighborhood)**.** Fix a topological space $(X, \mathcal{T})$.

- An *open subset* $U \subseteq X$ is a subset in $\mathcal{T}$.
- A *closed subset* $V \subseteq X$ is one with $X \setminus V \in \mathcal{T}$.
- A *neighborhood* of a point $p \in X$ is an open subset $U \subseteq X$ containing p.

Example 1.3. Fix a metric space (X, d). Then there is a topology given by the metric. To be explicit, a set $U \subseteq X$ is open if and only if each $p \in U$ has some $\varepsilon > 0$ such that

$$\{x \in X : d(x, p) < \varepsilon\} \subseteq U.$$

See [Elb22, Example 2.13] for the details.

Sometimes it is easier to generate a topology from some subcollection.

Definition 1.4 (base)**.** Fix a topological space $(X, \mathcal{T})$. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is a *base* for $\mathcal{T}$ if and only if the following holds: for each open $U \subseteq X$ and point $p \in U$, there is some $B \in \mathcal{B}$ such that $p \in B$ and $B \subseteq U$.

Example 1.5. Fix a metric space (X, d). Then the collection $\mathcal{B}$ of open balls

$$B(p, \varepsilon) :=,$$

over all $p \in X$ and $\varepsilon > 0$, forms a base of the topology. This is immediate from the construction of the topology in Example 1.3. In fact, one can merely take $\varepsilon \in \mathbb{Q}^+$ because $\mathbb{Q}$ is dense in $\mathbb{R}$.

With our objects of topological spaces in hand, we should discuss the maps between them.

Definition 1.6 (continuous)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is *continuous* if and only if $\varphi^{-1}(U')$ is open for each open $U' \subseteq X'$.

Definition 1.7 (homeomorphism)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is a *homeomorphism* if and only if φ is a bijection and both φ and φ^{-1} are continuous. We may write $X \cong X'$.

Remark 1.8. There is a continuous bijection $[0,2\pi) \to S^1$ by $\theta \mapsto (\cos\theta, \sin\theta)$, but it is not a homeomorphism. (Here, both sets have the metric topology.) In particular, the inverse map is not continuous at 1 because the pre-image of $[0,\pi)$ is the subset $\{(x,y) \in S^1 : y > 0\} \cup \{(0,0)\}$, which is not open in S^1 (because no $\varepsilon > 0$ has $B((0,0),\varepsilon)$ lying in $\{(x,y) \in S^1 : y \geq 0\}$).

Exercise 1.9. Fix a nonnegative integer $n \geq 0$. Then $B(0,1) \cong \mathbb{R}^n$.

Proof. We proceed as in [use14]. Define the functions $f\colon B(0,1) \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to B(0,1)$ by

$$f(x) := \frac{x}{1-|x|} \qquad \text{and} \qquad g(y) := \frac{y}{1+|y|}.$$

Notably, $|g(y)| < 1$ always, so g does indeed always output to $B(0,1)$. These functions are both continuous, which can be checked on coordinates because they are rational functions in the coordinates, and the denominators never vanish on the domains. So we will be done once we show that f and g are inverse. In one direction, we note

$$f(g(y)) = \frac{g(y)}{1-|g(y)|} = \frac{\frac{y}{1+|y|}}{1-\left|\frac{y}{1+|y|}\right|} = \frac{y}{1+|y|-|y|} = y.$$

In the other direction, we note

$$g(f(x)) = \frac{f(x)}{1+|f(x)|} = \frac{\frac{x}{1-|x|}}{1+\left|\frac{x}{1-|x|}\right|} = \frac{x}{1-|x|+|x|} = x,$$

as desired. ■

We would also like to be able to build new topologies from old ones.

Definition 1.10 (subspace)**.** Fix a topological space $(X,\mathcal{T})$. Given a subset $S \subseteq X$, we form a *subspace topology* by declaring the open subsets to be

$$\{U \cap S : U \in \mathcal{T}\}.$$

Example 1.11. The metric topology on $\mathbb{R}$ and the subspace topology on $X := \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ are homeomorphic. Namely, the homeomorphism sends $x \mapsto (x,0)$, and the inverse map is $(x,0) \mapsto x$. Here are our continuity checks.

- The map $x \mapsto (x,0)$ is continuous: the pre-image V of an open subset $U \subseteq X$ is open. Namely, for any $x \in V$, we see $(x,0) \in V$, so there is $\varepsilon > 0$ such that $B((x,0),\varepsilon) \cap X \subseteq U$, so $B(x,\varepsilon) \subseteq V$.
- The map $(x,0) \mapsto x$ is continuous: the pre-image V of an open subset $U \subseteq \mathbb{R}$ is open. Namely, for each $(x,0) \in V$, we see $x \in U$, so there is $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq V$, so $B((x,0),\varepsilon) \cap X \subseteq U$.

Lastly, we will want some adjectives for our topologies.

Definition 1.12 (compact)**.** Fix a topological space X. A subset $K \subseteq X$ is *compact* if and only if any open cover can be reduced to a finite subcover. Explicitly, any collection $\mathcal{U}$ of open sets of X such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$ (this is called an *open cover*) has some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U \in \mathcal{U}'} U$.

Example 1.13. The interval $[0,1] \subseteq \mathbb{R}$ is compact. See [Elb22, Example 4.4].

Definition 1.14 (Hausdorff)**.** Fix a topological space X. Then X is *Hausdorff* if and only if any two distinct points $p_1, p_2 \in X$ have disjoint open subsets $U_1, U_2 \subseteq X$ such that $p_1 \in U_1$ and $p_2 \in U_2$.

Example 1.15. Any metric space (X,d) is Hausdorff. Namely, for distinct points $p, q \in X$, we see $d(p,q) > 0$, so set $\varepsilon := d(p,q)/2$, and we see that $p \in B(p,\varepsilon)$ and $q \in B(q,\varepsilon)$, but $B(p,\varepsilon) \cap B(q,\varepsilon) = \varnothing$. For this last claim, we note r living in the intersection would imply

$$d(p,q) \leq d(p,r) + d(r,q) < 2\varepsilon,$$

which is a contradiction to the construction of ε.

1.1.3 Topological Manifolds

For intuition, we state but not prove the following result.

Theorem 1.16 (Topological invariance of dimension)**.** Fix open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. If there is a homeomorphism $U \cong V$, then $m = n$.

Proof. The usual proofs go through (co)homology, which we may cover later in the class. For the interested, see [Elb23, Proposition 3.50]. ■

We will soon define topological manifolds. The main adjective we want is being "locally Euclidean."

Definition 1.17 (locally Euclidean)**.** Fix a topological space X. Then X is *locally Euclidean of dimension n at p* if and only if there is an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$. We say that X is *locally Euclidean of dimension n* if and only if it is locally Euclidean of dimension n at each point.

Remark 1.18. One can always take $\widetilde{U}$ to be either $B(0,1) \subseteq \mathbb{R}^n$ or even all of $\mathbb{R}^n$. Indeed, for $x \in X$, we are given an open neighborhood U of x and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi\colon U \cong \widehat{U}$. We produce open neighborhoods of x homeomorphic to $B(0,1)$ and $\mathbb{R}^n$.

- $B(0,1)$: there is $\varepsilon > 0$ such that $B(\varphi(x),\varepsilon) \subseteq \widehat{U}$. Then we let $U' := \varphi^{-1}(B(\varphi(x),\varepsilon))$ so that we have a chain of homeomorphisms

$$U' \overset{\varphi}{\cong} B(\varphi(x),\varepsilon) \cong B(0,\varepsilon) \cong B(0,1),$$

  where the second homeomorphism is a translation, and the last homeomorphism is a dilation.
- $\mathbb{R}^n$: in the light of the previous point, it suffices to note that Exercise 1.9 provides a homeomorphism $B(0,1) \cong \mathbb{R}^n$ and then post-compose with this homeomorphism.

Let's explain why we want Theorem 1.16.

Lemma 1.19. Fix a locally Euclidean space X. For each $p \in X$, there is a unique nonnegative integer n such that there exists an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$.

Proof. Suppose there are two such nonnegative integers m and n, so we get open neighborhoods $U, V \subseteq X$ and $\widetilde{U} \subseteq \mathbb{R}^m$ and $\widetilde{V} \subseteq \mathbb{R}^n$. Let $\varphi\colon U \cong \widetilde{U}$ and $\psi\colon V \cong \widetilde{V}$ be the needed homeomorphisms. Then the point is to use the intersection $U \cap V$: there is a composite isomorphism

$$\varphi(U \cap V) \cong U \cap V \cong \psi(U \cap V)$$

from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. So Theorem 1.16 completes the proof. ■

Anyway, here is our definition of a topological manifold.

Definition 1.20 (topological manifold)**.** An n-*dimensional topological manifold* is a topological space M with the following properties.

- M is Hausdorff.
- M is locally Euclidean of dimension n at each point.
- M is second countable (i.e., has a countable base).

We may abbreviate "n-dimensional topological manifold" to "topological n-manifold."

Let's give a few quick constructions.

Lemma 1.21. For each $n \geq 0$, the space $\mathbb{R}^n$ is an n-dimensional topological manifold.

Proof. Let's be quick. Being a metric space yields Hausdorff, locally Euclidean is immediate because it's $\mathbb{R}^n$, and second-countability follows by using the base

$$\left\{B(q,\varepsilon) : q \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}^+\right\}.$$

This is indeed a base because $\mathbb{Q}$ is dense in $\mathbb{R}$. Explicitly, for each $p \in \mathbb{R}^n$ living in some open subset $U \subseteq \mathbb{R}^n$, begin by replacing U with a smaller open subset of the form $B(p, \varepsilon)$ where $\varepsilon > 0$; by perhaps making ε smaller, we may assume that $\varepsilon > 0$ is rational. Now, choosing coordinates $p = (x_1, \ldots, x_n)$, choose rational numbers $q_1, \ldots, q_n$ so that $|x_i - q_i| < \varepsilon/(2\sqrt{n})$ for each i. Then $q := (q_1, \ldots, q_n)$ has $d(p, q) < \varepsilon/2$ and so

$$p \in B(q, \varepsilon/2) \subseteq B(p, \varepsilon) \subseteq U,$$

so $B(q, \varepsilon/2)$ is the needed open subset in our base. ■

The following lemma will be helpful in the sequel.

Lemma 1.22. Fix a topological space M and nonnegative integer $n \geq 0$. Suppose that there is a countable open cover $\{U_i\}_{i\in\mathbb{N}}$ of M such that each i has a homeomorphism $U_i \cong \widetilde{U}_i$ where $\widetilde{U}_i \subseteq \mathbb{R}^n$ is open. Then M is locally Euclidean of dimension n at each point, and M is second countable.

Proof. For locally Euclidean, we note that each $p \in M$ lives in some U_i, so we are done. As for second countability, we note that each $\widetilde{U}_i$ is second countable as a subspace of a second countable space (see Lemma 1.21), so each U_i is second countable by moving back through the homeomorphism, and so M is second countable by taking the union of the bases of the U_i.

To make this last step more explicitly, we note that each U_i has a countable base $\mathcal{B}_i$, so we claim that $\mathcal{B} := \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ becomes a countable base of M. Certainly $\mathcal{B}$ is countable, and every set in $\mathcal{B}$ is in one of the $\mathcal{B}_i$ and hence open in M. Lastly, to check that we have a base, we note that any open $U \subseteq M$ and $p \in M$ will have $p \in U_i$ for some i, so there is some $B \in \mathcal{B}_i \subseteq \mathcal{B}$ such that $p \in B \subseteq U \cap U_i$. ■

1.1.4 Examples and Non-Examples

Here are some non-examples to explain why we want all of these hypotheses.

Exercise 1.23. Consider the space X defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\neq 0$. (The topology on X is the quotient topology [Elb22, Definition 2.81].) This space is not Hausdorff, but it is locally Euclidean and second countable.

Proof. We run our checks.

- This space is not Hausdorff because the points $(0,0)$ and $(0,1)$ are "infinitely close together." Explicitly, any open neighborhoods U and V of $(0,0)$ and $(0,1)$, respectively, the induced topology yields some $\varepsilon>0$ such that $B((0,0),\varepsilon)\subseteq U$ and $B((0,1),\varepsilon)\subseteq V$, but then $(-\varepsilon/2,0)=(-\varepsilon/2,1)$ is in both U and V.
- This space is locally Euclidean and second countable by Lemma 1.22. Explicitly, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

Exercise 1.24. Consider the space X defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\leq 0$, again where we are using the quotient topology. Then X is Hausdorff and second countable, but it is not Euclidean of dimension 1 at $0\in X$.

Proof. We run our checks.

- This space is Hausdorff. We check this directly by casework.
  - Suppose we have distinct points $p=(x,a)$ and $q=(y,b)$ with $x\neq y$; for example, this includes the case where we may take $a=b$ and hence includes the case when $x,y\leq 0$. Then we may set $\varepsilon:=\frac{1}{2}|x-y|$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
  - We now may assume that $x=y$; then $a\neq b$. Thus, we must have $x>0$ or $y>0$. As such, we may as well take $\varepsilon:=\min\{|x|,|y|\}$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
- This space is not locally Euclidean at 0. Indeed, suppose that there is open subset $U\subseteq X$ around 0 which is homeomorphic to an open subset of $\mathbb{R}$. By shifting, we may as well assume that the homeomorphism sends 0 to 0. Additionally, the same statement will be true by any open subset of U, so we may as well as assume that U is of the form $(-\varepsilon,\varepsilon)\times\{0,1\}$ (in X). In particular, U is connected.

  But then the image $\widehat{U}$ of U in $\mathbb{R}$ is a connected open subset of $\mathbb{R}$, which must be an interval. Now, intervals have the property that deleting any point of an interval makes produces a topological space with two connected components. However, deleting 0 from U will produce three connected components: $(-\varepsilon,0)\times\{0,1\}$ and $(0,\varepsilon)\times\{0\}$ and $(0,\varepsilon)\times\{1\}$. So $\widehat{U}$ and U cannot actually be homeomorphic!
- This space is second countable by Lemma 1.22. Again, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

Remark 1.25. Essentially the same argument implies that the above space fails to be locally Euclidean of any dimension at $0\in X$. Namely, a connected open subset of $\mathbb{R}^n$ for $n\geq 2$ will remain connected after removing any point, so it cannot be homeomorphic to $(-\varepsilon,\varepsilon)\times\{0,1\}$ in X.

Morally, the second countability is being required as a smallness condition; let's see some pathological examples without second countability. The following lemma approximately explains the problem.

Lemma 1.26. Fix a topological space X. Suppose that there is an uncountable subset $Y \subseteq X$ such that each $y \in Y$ has an open neighborhood $U_y \subseteq X$ where the U_y are pairwise disjoint. Then X fails to be second countable.

Proof. Suppose we have a base $\mathcal{B}$; we show $\mathcal{B}$ is uncountable. Each $y \in U_y$ has some $B_y \in \mathcal{B}$ with $B_y \subseteq U_y$. However, $y \neq y'$ implies that $B_y \neq B_{y'}$ because $y \in B_y$ while $p_y \notin U_{y'}$ implies $p_y \notin B_{y'}$. So $\{B_y\}_{y \in Y}$ is an uncountable subcollection of $\mathcal{B}$. ■

Exercise 1.27. Consider an uncountable set S with the discrete topology (namely, every subset is open), and then we form the product $X := \mathbb{R} \times S$. Then X is Hausdorff, locally Euclidean of dimension 1, but it is not second countable.

Proof. Here are our checks.

- Note that X is a product of Hausdorff spaces and hence is Hausdorff.
- This space is locally Euclidean of dimension 1: for each $(x, s) \in X$, we note that $\mathbb{R} \times \{s\}$ is an open subset of X (because S is discrete) where $\mathbb{R} \times \{s\} \cong \mathbb{R}$ by an argument similar to Example 1.11.
- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (0, s)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \mathbb{R} \times \{s\}$. ■

Exercise 1.28. Consider the first uncountable ordinal ω_1. Then define $X := (S \times [0, 1)) \setminus \{(0, 0)\}$, and we give X the order topology where the ordering is lexicographic. (Namely, the base consists of the "intervals" $\{x : x < b\}$ or $\{x : a < x\}$ or $\{x : a < x < b\}$.) This space is Hausdorff, locally Euclidean 1, but it is not second countable.

Proof. Here are our checks.

- This space is Hausdorff because it is a dense linear order. Explicitly, for $(s, a), (t, b) \in X$, we have the following cases.
  - Suppose $s = t$. In this case, $a \neq b$; suppose $a < b$ without loss of generality. Then $\{x : x < (s, (a + b)/2)\}$ and $\{x : x > (s, (a + b)/2)\}$ are the needed open sets.
  - Suppose $s \neq t$; take $s < t$ without loss of generality. If $a > 0$, then $\{s\} \times (0, (a + 1)/2)$ and $\{s\} \times ((a+1)/2, 1) \cup \{t\} \times [0, 1)$ provide the needed open sets. Otherwise, if $a = 0$, then $\{x : x < (s, 1/2)\}$ and $\{x : x > (s, 1/2)\}$ provide the needed open sets.
- This space is locally Euclidean of dimension 1: fix any $(s, r) \in X$. Note that $s \in \omega_1$ is countable, so we claim that
$$(s + 1) \times [0, 1) \cong [0, 1),$$
sending $(0, 0)$ to 0, from which the claim follows by deleting $(0, 0)$. Because the relevant orders produce the needed topologies, we are really asking for an order-preserving bijection from $(s + 1) \times [0, 1)$ to $[0, 1)$.

  Well, for any $t \in \omega_1$, we claim that there is an increasing sequence $\{p_\alpha\}_{\alpha < t} \subseteq [0, 1)$ of order type t with $p_0 = 0$, from which the claim will follow by taking $s = t$ and sending $\alpha \times [0, 1) \subseteq (s + 1) \times [0, 1)$ to $[p_\alpha, p_{\alpha+1})$ (where we define $p_s := 1$). To see this claim, we argue by induction on s. For $s = 0$, take $p_0 := 0$. If s is a successor ordinal, divide all the existing p_α by 2 and then set $p_{s+1} := 1/2$.

  Lastly, if s is a limit ordinal, it is still only a countable limit ordinal, so we can find an increasing sequence of countable ordinals $\{s_i\}_{i \in \omega}$ approaching s. The sequence corresponding to s_0 will fit into $[0, 1/2)$ after scaling; then the sequence corresponding to s_1 but after s_0 will fit into $[1/2, 2/3)$ after scaling. We can continue this process inductively to complete the claim for s. I won't bother to write out the details.

- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (s, 1/2)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \{s\} \times (0, 1)$. ■

Remark 1.29. What makes the locally Euclidean check above annoying is that we must show $(\omega, 0) \in X$ has a neighborhood isomorphic to an open subset of $\mathbb{R}$, which is not totally obvious.

Let's return to examples.

Example 1.30. Consider the unit circle S^1. We check that S^1 is a 1-dimensional topological manifold.

- S^1 is a metric space, so it is Hausdorff.
- S^1 is second countable: it is a subspace of $\mathbb{R}^2$, and $\mathbb{R}^2$ is second countable by Lemma 1.21 again.
- S^1 is locally Euclidean: we proceed explicitly. Define $U_1^\pm := \left\{(x, y) \in S^1 : \pm x > 0\right\}$; then $U_1^\pm \cong (-1, 1)$ by $(x, y) \mapsto y$. Similarly, define $U_2^\pm := \left\{(x, y) \in S^1 : \pm y > 0\right\}$; then $U_2^\pm \cong (-1, 1)$ by $(x, y) \mapsto x$.

1.2 January 18

The first homework has been posted. It is mostly a review of point-set topology things. It is due on the 25th of January.

Remark 1.31. Please read the section on fundamental groups of manifolds on your own. We will not discuss it in class.

To review, our current goal is to define smooth manifolds. Thus far we have defined a topological space and provided enough adjectives to turn it into a topological manifold. To proceed, we need to add smoothness to our structure. We will do this later.

1.2.1 Connectivity

For now, we will content ourselves with some extra adjectives for our topological manifolds which will later be helpful. Here are two notions of connectivity.

Definition 1.32 (connected)**.** Fix a topological space X. Then X is *disconnected* if and only if there exist disjoint nonempty open subsets $U, V \subseteq X$ such that $X = U \sqcup V$. If X is not disconnected, we say that X is connected.

Example 1.33. The interval $[0, 1]$ is connected. See [Elb22, Lemma A.6].

Remark 1.34. Equivalently, we can say that X is connected if and only if X and $\varnothing$ are the only subsets of X which are both open and closed.

Definition 1.35 (path-connected)**.** Fix a topological space X. Then X is *path-connected* if and only if any two points $p, q \in X$ has some continuous map $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Example 1.36. The space $B(0,1) \subseteq \mathbb{R}^n$ is path-connected. Indeed, we show that the path-connected component of 0 is all of $B(0,1)$; see [Elb22, Definition A.19]. In other words, we must exhibit a path from 0 to v for any $v \in B(0,1)$. Well, define $\gamma\colon [0,1] \to B(0,1)$ by $\gamma(t) := tv$. This is continuous because it is linear, and it has $\gamma(0) = 0$ and $\gamma(1) = v$ as desired.

In general, these two notions do not coincide.

Example 1.37. Consider the topological space

$$X := \{(x, \sin(1/x)) : x \in (0,1)\} \cup \{(0,y) : y \in \mathbb{R}\}.$$

Then X is connected, but it is not path-connected. See [Elb22, Exercise A.20].

But one does in general apply the other.

Lemma 1.38. Fix a topological space X. If X is path-connected, then X is connected.

Proof. See [Elb22, Lemma A.16], though we will sketch the proof. We proceed by contraposition. Suppose that X is disconnected, so we may write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. Now choose some $p \in U$ and $q \in V$, and we claim that there is no path $\gamma\colon [0,1] \to X$. Indeed, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would then be nonempty disjoint open subsets of $[0,1]$ covering $[0,1]$, which is a contradiction. ■

However, for topological manifolds, these notions do coincide.

Proposition 1.39. Fix a topological space M which is locally Euclidean of dimension n. Then M is path-connected if and only if it is connected.

Proof. The forward direction is by Lemma 1.38. Thus, we focus on showing the converse. Fix some $p \in M$, and we define the subset

$$U_p := \{q \in M : \text{there exists a path from } p \text{ to } q\}.$$

This is the path-connected component of p in M; see [Elb22, Definition A.19]. The main claim is that U_p is open.

Suppose $q \in M$, and we need to find an open neighborhood $B_q \subseteq M$ of q living inside U_p. Noting then that $U_p = \bigcup_{q \in U_p} B_q$ will complete the proof of this claim. Well, q has some open neighborhood $B \subseteq M$ equipped with a homeomorphism $\varphi\colon B \cong B(0,1)$ by Remark 1.18. Then $B(0,1)$ is path-connected by Example 1.36, so B is path-connected by going back through the homeomorphism. Thus, because U_p is an equivalence class, it is also the path-connected equivalence class of q, so U_p must contain B.

Now, let $\mathcal{U}$ denote the collection of path-connected components of M. This is a collection of disjoint open subsets covering M. Certainly it is nonempty, so select $U \in \mathcal{U}$. Then we write

$$M = U \cup \bigcup_{U' \in \mathcal{U} \setminus \{U\}} U'.$$

This is a decomposition of M into disjoint open subsets, so because M is connected, one of these must be empty. But U is empty, so instead the union of the U' must be nonempty. However, everything in $\mathcal{U}$ is nonempty, so instead we see that $\mathcal{U} \setminus \{U\}$ is empty, so $M = U$ is path-connected. ■

1.2.2 Local compactness

Here is our definition.

Definition 1.40 (local compactness)**.** Fix a topological space X. Then X is *locally compact* if and only if any $x \in X$ has some open neighborhood $U \subseteq X$ such that there exists a compact subset $K \subseteq X$ containing U.

Remark 1.41. If X is Hausdorff, then compact subsets are closed [Elb22, Corollary 4.13], and closed subsets of a compact space are still compact [Elb22, Lemma 4.10], so we may as well take $K = \overline{U}$ in the above definition.

The above remark motivates the following definition.

Definition 1.42 (precompact)**.** Fix a topological space X. An open subset $U \subseteq X$ is *precompact* if and only if $\overline{U}$ is compact.

Remark 1.43. Here is a quick check which will prove to be useful: if X is Hausdorff and $U \subseteq X$ is precompact, and $V \subseteq U$, then V is still precompact. Indeed, $\overline{U}$ is compact, and $\overline{V} \subseteq \overline{U}$ is a closed subset and hence compact [Elb22, Lemma 4.10].

Example 1.44. The topological space $\mathbb{R}$ is locally compact; see [Elb22, Example 4.71].

Non-Example 1.45. Infinite-dimensional normed vector spaces fail to be locally compact. Namely, open balls fail to be precompact, so local compactness fails.

Non-Example 1.46. The space $\mathbb{Q}$ is not locally compact. Indeed, suppose for the sake of contradiction that we have a precompact nonempty open neighborhood $U \subseteq \mathbb{Q}$ of $0 \in \mathbb{Q}$. Now, $\mathbb{Q}$ is Hausdorff (it's a metric space), so we can find some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ while $\varepsilon \notin \mathbb{Q}$, so Remark 1.43 tells us that $(\varepsilon/2, \varepsilon)$ is precompact so that $[\varepsilon/2, \varepsilon]$ is actually compact.

However, this is false. Let $\{\alpha_i\}_{i \geq 1}$ be an increasing sequence of irrationals in $[\varepsilon/2, \varepsilon]$ with $\alpha_i \to \varepsilon$. Explicitly, we can take $\alpha_i := \frac{i}{i+1} \cdot \varepsilon$. Then we define

$$U_i := [\alpha_i, \alpha_{i+1}]$$

for each $i \geq 1$. Note $[\alpha_i, \alpha_{i+1}] = (\alpha_i, \alpha_{i+1})$, so the $U_\bullet$s provide a countable sequence of disjoint open subsets covering $[\varepsilon/2, \varepsilon]$. Thus, $[\varepsilon/2, \varepsilon]$ cannot be compact.

One can check that manifolds are locally compact.

Proposition 1.47. Fix a topological n-manifold M. Then M is locally compact.

Proof. This follows from being locally Euclidean. Fix $p \in M$, and then we are promised some open subset $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. Then there is an open ball $B(\varphi(p), \varepsilon) \subseteq \widehat{U}$. Then $\overline{B(\varphi(p), \varepsilon/2)} \subseteq \widehat{U}$ is closed and bounded in $\mathbb{R}^n$ and hence compact, so $\varphi^{-1}(B(\varphi(p), \varepsilon/2))$ is a subset of the compact subset $\varphi^{-1}(\overline{B(\varphi(p), \varepsilon/2)})$. ■

Being locally compact approximately speaking allows one to understand a space by building it up from compact ones. Here is one way to do this.

Definition 1.48 (exhaustion)**.** Fix a topological space X. Then an *exhaustion* of X is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X satisfying the following.

- Ascending: $K_0 \subseteq K_1 \subseteq \cdots$.
- Covers: $X = \bigcup_{i\in\mathbb{N}} K_i$.
- Not too close: $K_i \subseteq K_{i+1}^\circ$.

Example 1.49. The space $\mathbb{R}^n$ has an exhaustion by $K_i := B(0,i)$.

Here is a way to build an exhaustion.

Proposition 1.50. Fix a topological space X. If X is second-countable, locally compact, and Hausdorff. Then X has an exhaustion. In particular, topological n-manifolds have an exhaustion.

Proof. The second claim follows from the first by Proposition 1.47 and the definition of a manifold. So we will focus on showing the first claim.

Fix a countable base $\mathcal{B}$ of X, and let $\mathcal{B}'$ be the subcollection of precompact open base elements. Quickly, we note that $\mathcal{B}'$ is still a base: certainly everything in $\mathcal{B}'$ is open, and then for any $p \in X$ and open neighborhood $U \subseteq X$, we need some $B' \in \mathcal{B}'$ such that B' is precompact.

Well, because X is locally compact, there is a precompact open neighborhood U' of p by Remark 1.41. Then $U \cap U$ is an open neighborhood of p, so we can find a base element $B \in \mathcal{B}$ containing p and inside $U' \cap U$. Then $B \subseteq U'$ is precompact by Remark 1.43.

We now construct our exhaustion. Enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$, and we proceed as follows.

1. Set $K_0 := \overline{B_0}$, which is compact by construction of B_0.
2. Now suppose we have a compact subset $K_i \subseteq X$, and we construct K_{i+1}. Note that $\mathcal{B}$ is an open cover of K_i, which can be reduced to a finite subcover, so there is some M_{i+1} such that K_i is covered by $\{B_i : i \le M_{i+1}\}$. We may as well suppose that $M_{i+1} \ge i+1$. Then we define

$$K_{i+1} := \bigcup_{i=1}^{M} \overline{B_i}.$$

   Note that the finite union of compact sets remains compact.

The above construction produces an exhaustion. Here are our checks, which will complete the proof.

- Ascending: by construction, we see that

$$K_{i+1}^\circ \supseteq \bigcup_{i=1}^{M} B_i \supseteq K_i.$$

- Covers: any $x \in X$ lives in some B_i, and by construction, we have $B_i \subseteq K_i$, so $x \in K_i$. ■

1.2.3 Paracompactness

We will want to talk about covers in some more detail.

Definition 1.51 (cover)**.** Fix a topological space X. A *cover* is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that

$$X = \bigcup_{U\in\mathcal{U}} U.$$

Definition 1.52 (locally finite)**.** Fix a topological space X. A cover $\mathcal{U}$ of X is *locally finite* if and only if any $p \in X$ has some open neighborhood $U \subseteq X$ intersecting at most finitely many elements of $\mathcal{U}$.

Definition 1.53 (refinement)**.** Fix a cover $\mathcal{U}$ of a topological space X. Then a *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that any $V \in \mathcal{U}$ is contained in some $U \in \mathcal{U}$.

And here is our definition.

Definition 1.54 (paracompact)**.** Fix a topological space X. Then X is *paracompact* if and only if every open cover has a locally finite open refinement.

Approximately speaking, the point of desiring paracompactness is that it allows "reducing to Euclidean" arguments in the future will not have to deal with intersections which are infinitely bad. Anyway, here is our result.

Proposition 1.55. Fix a topological n-manifold M. Then M is paracompact.

Proof. In fact, we are only going to use the fact that M has an exhaustion, proven in Proposition 1.50.

Fix an open cover $\mathcal{U}$, and we want to produce a locally finite open refinement. To set us up, fix an exhaustion $\{K_i\}_{i\in\mathbb{N}}$, which exists by Proposition 1.50, and define the following sets for each $i \in \mathbb{N}$.

- For $i \geq -1$, define $V_i := K_{i+1} \setminus K_i^\circ$, which is a closed subset of the compact set K_{i+1} and hence compact [Elb22, Lemma 4.10]; take $K_{-1} = \varnothing$ without concern.

- For $i \geq 0$, define $W_i := K_{i+2}^\circ \setminus K_{i-1}$, which is open; here, take $K_{-1} = \varnothing$ without concern.

For intuition, we should think about the $W_\bullet$s as being a locally finite cover from which we will build the locally finite cover refinement of $\mathcal{U}$.

For the construction, we fix some $j \geq 0$ for the time being. For each $x \in V_j$, find some $U_x \in \mathcal{U}$ containing x. Note that $\{U_x\}_{x\in V_j}$ is an open cover of V_j, and because $V_j \subseteq W_j$, in fact $\{U_x \cap W_j\}_{x\in V_j}$ is an open cover. Because V_j is compact, we can thus reduce this open cover to a finite subcover $\mathcal{A}_j$.

Now letting j vary, we define

$$\mathcal{V} := \bigcup_{j\geq 0} \mathcal{A}_j.$$

Here are our checks.

- Open cover: each $x \in X$ lives in some K_{i+1} because we have an exhaustion, so lives in some V_i, so it lives in some open subset in $\mathcal{A}_j$, so it lives in some open subset in $\mathcal{V}$.

- Refinement: by construction, each open set in $\mathcal{A}_j$ is a subset in $\mathcal{U}$.

- Locally finite: this is essentially by construction. The main point is that any $x \in X$ lives in some K_i, so by choosing the least such K_i places x in some $V_i \subseteq W_i$. We now show that only finitely many open subsets in $\mathcal{V}$ intersect W_i. Note $W_i \subseteq K_{i+2}$, so $W_i \cap W_j = \varnothing$ for $j \geq i+2$. Thus, if $V \cap W_i \neq \varnothing$, we must have $V \in \mathcal{A}_j$ for $j < i+2$. But this is only finitely many indices, and each $\mathcal{A}_j$ is finite, so this is only finitely many candidates. ■

1.2.4 Products

We now discuss an in-depth example.

Proposition 1.56. Fix finitely many topological manifolds $M_1, \ldots, M_k$. Then the product

$$M_1 \times \cdots \times M_k$$

is also a topological manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

We will do this via a sequence of lemmas.

Lemma 1.57. Fix a collection of Hausdorff topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$. Then the product

$$\prod_{\alpha\in\Lambda} X_\alpha$$

is also Hausdorff.

Proof. Fix distinct points $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$ in the product. Then there is an index $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$, so because X_β is Hausdorff, there are disjoint open neighborhoods $U_\beta, V_\beta \subseteq X_\beta$ of x_β and y_β, respectively. Then we define $U_\alpha = V_\alpha := X_\alpha$ for $\alpha \neq \beta$, and we note that the open subsets

$$\prod_{\alpha\in\Lambda} U_\alpha \quad \text{and} \quad \prod_{\alpha\in\Lambda} V_\alpha$$

are disjoint open neighborhoods of $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$, respectively, so we are done. (These are disjoint because any point in the intersection will have the β coordinate in $U_\beta \cap V_\beta = \varnothing$.) ■

Lemma 1.58. Fix finitely many second countable topological spaces $\{X_i\}_{i=1}^n$. Then the product

$$\prod_{i=1}^n X_i$$

is also second countable.

Proof. Let the product be X. For each i, let $\mathcal{B}_i$ be a countable base for X_i. Then define

$$\mathcal{B} := \left\{ \prod_{i=1}^n B_i : B_i \in \mathcal{B}_i \text{ for each } i \right\}.$$

We claim that $\mathcal{B}$ is a base for the topology on the X. Indeed, suppose $(x_1, \ldots, x_n) \in X$ lives in some open subset $U \subseteq X$. From the standard base on X, we know that there are open subsets $U_i \subseteq X_i$ for each i such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$. Now, for each U_i, we note that $x_i \in U_i$ must have some $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$ and $B_i \subseteq U_i$. But then

$$(x_1, \ldots, x_n) \in B_1 \times \cdots \times B_n \subseteq U,$$

so $B_1 \times \cdots \times B_n \in \mathcal{B}$ is the desired base element. ■

We now prove Proposition 1.56.

Proof of Proposition 1.56. We get Hausdorff from Lemma 1.57 and second countable from Lemma 1.58. So it remains to check that we are locally Euclidean. For brevity, let M be the product, and set $n_i := \dim M_i$ for each i, and let $n := n_1 + \cdots + n_k$.

Now, fix some point $(x_1, \ldots, x_k) \in M$. For each i, we get some open neighborhood $U_i \subseteq M_i$ of x_i and some open $\widehat{U}_i \subseteq \mathbb{R}^{n_i}$ with a homeomorphism $\varphi_i \colon U_i \cong \widehat{U}_i$. Now, we see that the product map

$$(\varphi_1 \times \cdots \times \varphi_k) \colon U_1 \times \cdots \times U_k \to \widehat{U}_1 \times \cdots \times \widehat{U}_k$$

is still a homeomorphism, and the target is an open subset of

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \cong \mathbb{R}^n,$$

where this last homeomorphism is obtained by simply concatenating the coordinates. So we have constructed a composite homeomorphism from an open neighborhood of $(x_1, \ldots, x_k)$ to an open subset of $\mathbb{R}^n$, as desired. ■

Example 1.59. Example 1.30 established S^1 as a topological 1-manifold, so the n-torus

$$T^n := \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is a topological n-manifold. Note that the covering space $p\colon \mathbb{R} \to S^1$ will induce the covering space $p^n\colon \mathbb{R}^n \to T^n$, so we can also view T^n as $\mathbb{R}^n/\mathbb{Z}^n$; in other words, we have the unsurprising homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to (\mathbb{R}/\mathbb{Z})^n$.

1.2.5 Open Submanifolds

We proceed with a sequence of lemmas.

Lemma 1.60. Suppose X is a Hausdorff topological space. If $X' \subseteq X$ is a subspace, then X' is still Hausdorff.

Proof. Fix distinct points $p, q \in X'$. Then X is Hausdorff, so there exist disjoint open neighborhoods $U, V \subseteq X$ of p and q, respectively, so $U \cap X'$ and $V \cap X'$ are the needed disjoint open subsets of X', respectively. ■

Lemma 1.61. Suppose that X is a second countable topological space. Then for any subset $X' \subseteq X$, the topological (sub)space X' is still second countable.

Proof. Well, let $\mathcal{B}$ be a countable base for X, and we claim that the collection

$$\mathcal{B}' := \{B \cap X' : B \in \mathcal{B}\}$$

makes a countable base for X'. Note that $\mathcal{B}'$ is certainly countable because there is a surjective map $\mathcal{B} \to \mathcal{B}'$ by $B \mapsto (B \cap X')$, and $\mathcal{B}$ is countable. (This map is surjective by construction.)

So it remains to show that $\mathcal{B}'$ is a base. Quickly, we claim that every $B' \in \mathcal{B}'$ is open in X'. Indeed, for any $B' \in \mathcal{B}'$, we can find some $B \in \mathcal{B}$ such that $B' = B \cap X'$. Now, $\mathcal{B}$ is a base, so $B \subseteq X$ is open, so $B' = B \cap X'$ is open in the subspace topology of X'.

To finish checking that we have a base, fix some $x' \in X'$ and open $U' \subseteq X'$ containing x'. Then we need some $B' \in \mathcal{B}'$ such that $x' \in B'$ and $B' \subseteq U'$. Well, by the subspace topology, we can write $U' = U \cap X'$ for some open $U \subseteq X$, but then $x' \in U$, so there is some $B \in \mathcal{B}'$ such that $x' \in B$ and $B \subseteq U$. To finish, we set

$$B' := B \cap X',$$

which is in $\mathcal{B}'$ by construction, and we have $x' \in B \cap X' = B'$ and $B' = B \cap X' \subseteq U \cap X' = U'$, so B' is indeed the required basic open set. ■

Lemma 1.62. Suppose that X is locally Euclidean of dimension n. Then for any open subset $X' \subseteq X$, the topological (sub)space X' is locally Euclidean of dimension n.

Proof. For any $x' \in X'$, we must find open subsets $U' \subseteq X'$ and $\widehat{U}' \subseteq \mathbb{R}^n$ such that $x' \in U'$ and there is a homeomorphism $U' \cong \widehat{U}'$.

Well, $x' \in X$, so there are open subsets $U \subseteq X$ and $\widehat{U} \subseteq \mathbb{R}^n$ such that $x' \in U$ and there is a homeomorphism $\varphi\colon U \cong \widehat{U}$. Now, set

$$U' := U \cap X'.$$

Then φ is a homeomorphism, so $\varphi' := \varphi|_{U'}$ continues to be a homeomorphism onto its image $\widehat{U}' := \varphi(U')$. Indeed, the inverse of the bijection $\varphi|_{U'}\colon U' \to \widehat{U}'$ is $\varphi'|_{\widehat{U}'}$. Both of these maps are continuous by, so $\varphi|_{U'}$ is in fact a homeomorphism.

Now, $U' \subseteq U$ is open, so because φ is a homeomorphism, we see that $\varphi(U') \subseteq \widehat{U}$ is open: $\varphi(U')$ is the pre-image of the open subset $U' \subseteq U$ under the continuous map $\varphi^{-1}\colon \widehat{U} \to U$, so $\varphi(U')$ being open follows. Continuing, because $\widehat{U} \subseteq \mathbb{R}^n$ is open, we conclude that $\widehat{U}' \subseteq \mathbb{R}^n$ is open.[1] So $U' \subseteq X'$ is open (by the subspace topology), contains x', and it is homeomorphic to an open subset $\widehat{U}'$ of $\mathbb{R}^n$. ■

Proposition 1.63. Fix a topological n-manifold M. For any nonempty open subset $U \subseteq M$, we have that U is a topological n-manifold.

Proof. Combine Lemmas 1.60 to 1.62. ■

1.2.6 Charts

The construction of our smooth structure will arise from more carefully understanding how a manifold is locally Euclidean. This arises from charts.

Definition 1.64 (chart)**.** Fix a topological n-manifold M. Then a *coordinate chart* or just *chart* is a pair (U, φ) where $U \subseteq M$ is open and $\varphi\colon U \cong \widehat{U}$ is a homeomorphism where $\widehat{U} \subseteq \mathbb{R}^n$ is open.

Essentially, the content of M being locally Euclidean is that it has an open cover by open subsets belonging to some chart. The reason we call it a chart is that we are (approximately speaking) providing "local coordinates" to an open subset of M.

Definition 1.65 (coordinate function)**.** Fix a chart (U, φ) if a topological n-manifold M. Then we may write

$$\varphi(p) := \big(x^1(p), \ldots, x^n(p)\big) \in \mathbb{R}^n$$

for each $p \in M$. We call these functions $x^\bullet\colon U \to \mathbb{R}$ the *coordinate functions*.

Note that these coordinate functions are continuous because they are simply the continuous function φ composed with the projection $\mathbb{R}^n \to \mathbb{R}$.

[1] Namely, an open subset of an open subset U is still an open subset. This sentence has some content because the larger open subset uses the subspace topology; the proof simply notes that being open in U is equivalent to being the intersection of an open subset and U, which is open because finite intersections of open subsets continues to be open.

Example 1.66. Fix an open subset $V \subseteq \mathbb{R}^m$, and let $F\colon V \to \mathbb{R}^n$ be a continuous function. Then the graph

$$\Gamma \coloneqq \{(x, F(x)) : x \in V\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

is a topological n-manifold. Because we are already a subspace of $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, we see that Γ is also Hausdorff and second countable. (Subspaces inherit being Hausdorff directly, and we inherit being second countable by using the intersection of the given countable base.)

The main content comes from being locally Euclidean. Namely, there is a projection map $\pi\colon \Gamma \to V$ by $(x, y) \mapsto x$ which in fact is a homeomorphism (it's continuous inverse is $(\mathrm{id} \times F)\colon x \mapsto (x, F(x))$). So we have the single chart (V, π), which establishes being a topological n-manifold.

1.3 January 23

The first homework is due on Thursday. Today we discuss smooth structures.

1.3.1 Examples of Topological Manifolds

Let's provide a few more examples of topological manifolds.

Exercise 1.67 (sphere)**.** We show that the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold.

Proof. Explicitly, for each $i \in \{1, \ldots, n+1\}$, we define

$$U_i^\pm \coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n : \pm x_i > 0\},$$

which has a projection $\pi_i^\pm\colon U_i^\pm \to B(0,1)$ (for $B(0,1) \subseteq \mathbb{R}^n$) given by erasing the x_i coordinate. One can show that the $\pi_i^\pm$ are all homeomorphisms—certainly, it is continuous, and the inverse map is given by

$$(x_1, \ldots, x_n) \coloneqq \left(x_1, \ldots, x_{i-1}, \pm\sqrt{1 - (x_1^2 + \cdots + x_n^2)}, x_i, \ldots, x_n\right),$$

which is also continuous. (We won't bother checking that the maps are mutually inverse.) Lastly, we note that the $U_i^\pm$ is an open cover of S^n because any point in S^n has some nonzero coordinate, and this nonzero coordinate will have a sign. ■

Exercise 1.68 (projective space)**.** Define the space $\mathbb{RP}^n$ as "lines in $\mathbb{R}^{n+1}$": it consists of equivalence classes of nonzero points in $\mathbb{R}^{n+1} \setminus \{0\}$, where $x \sim y$ if and only if there is some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. We show that $\mathbb{RP}^n$ is a topological n-manifold.

Proof. For notation, we let $[x_0 : \cdots : x_n]$ denote the equivalence class of $(x_1, \ldots, x_n)$ in $\mathbb{RP}^n$. Note there is a projection $p\colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$, and we give $\mathbb{RP}^n$ the induced (quotient) topology from $\mathbb{R}^{n+1} \setminus \{0\}$.

By Lemma 1.22, to achieve second countable, it suffices to provide a finite open cover by open subsets homeomorphic to open subsets of $\mathbb{R}^n$; this will also achieve locally Euclidean. Well, define

$$U_i \coloneqq \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}.$$

Note that the pre-image in $\mathbb{R}^{n+1} \setminus \{0\}$ consists of the $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $x_i \neq 0$, so $U_i \subseteq \mathbb{RP}^n$ is open. Now, by scaling, we can write elements of U_i uniquely as $[y_0 : \cdots : y_n]$ with $y_i = 1$, which provides the required element in $\mathbb{R}^n$. Explicitly, we define $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$\varphi_i\colon [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

One sees that φ_i is continuous: by the quotient topology, we are trying to show that $\varphi_i \circ \pi\colon \pi^{-1}U_i \to \mathbb{R}^n$ is just $(x_0, \ldots, x_n) \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$, which is continuous, so φ_i is continuous because $\mathbb{RP}^n$ has the quotient topology. Lastly, one notes that the inverse of φ_i is given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$, which is continuous because it is the composite of the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$ and the projection $p\colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$.

Lastly, we show that $\mathbb{RP}^n$ is Hausdorff. Doing this in a slick way is surprisingly obnoxious. We claim that there is a 2-to-1 covering space map

$$p\colon S^n \to \mathbb{RP}^n.$$

To see why this implies that $\mathbb{RP}^n$ is Hausdorff, fix two distinct points $x, y \in \mathbb{RP}^n$. Then there are lifts $x_1, x_2 \in S^n$ of x and $y_1, y_2 \in S^n$. Because S^n is already Hausdorff (it's a subspace of $\mathbb{RP}^n$), we can find disjoint open subsets $U_1, U_2, V_1, V_2 \subseteq S^n$ around $x_1, x_2, y_1, y_2 \in S^n$ respectively, and we can make them all small enough so that p is a local homeomorphism. Then $p(U_1) \cap p(U_2)$ and $p(V_1) \cap p(V_2)$ are the desired open subsets.

So we are left showing that we have a double cover p. The map is given by the composite

$$S^n \subseteq \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \twoheadrightarrow \mathbb{RP}^n,$$

which we see is continuous automatically. To see that this is a 2-to-1 local homeomorphism, we note that the pre-image of the standard open subset $U_i \subseteq \mathbb{RP}^n$ is

$$\left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0\right\},$$

whose pre-image in S^n splits into the two open subsets $U_i^\pm$. So we have our continuous map $U_i^+ \sqcup U_i^- \to U_i$; it remains to show that $U_i^\pm \to U_i$ is a homeomorphism. We may as well assume $i = 0$; then the inverse map is given by sending $[1 : x_1 : \cdots : x_n]$ to the point on the hemisphere of S^n on this line, which is

$$\pm \frac{x}{|x|},$$

where the sign depends on $U_i^\pm$. This is continuous, so we are done. ■

Remark 1.69. Note S^n is continuous, so the surjectivity of the covering space map $S^n \twoheadrightarrow \mathbb{RP}^n$ implies that $\mathbb{RP}^n$ is compact.

1.3.2 Transition Functions

Defining smooth structures will come out of transition maps between coordinate charts.

Definition 1.70 (transition map)**.** Fix charts (U, φ) and (V, ψ) on a topological n-manifold M. Then the *transition map* is the map

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V).$$

Here, we are abusing notation a little: in order to make sense of $\psi \circ \varphi^{-1}$, we really want to work with the restrictions as $\psi|_{U\cap V} \circ (\varphi|_{U\cap V})^{-1}$.

Remark 1.71. Note $\varphi(U \cap V), \psi(U \cap V) \subseteq \mathbb{R}^n$, so this is a homeomorphism from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$. Namely, $\varphi|_{U\cap V}$ and $\psi|_{U\cap V}$ are both homeomorphisms, so the above composition is still a homeomorphism.

Example 1.72 (polar coordinates)**.** Consider the topological 2-manifold $M := \mathbb{R}^2$. There is the identity chart $\operatorname{id}_M : M \to \mathbb{R}^2$, and there is also "polar coordinates" on $U := \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$ with chart $\varphi : U \to \mathbb{R}_+ \times (0, \pi)$ defined by

$$\varphi((x, y)) := \left(\sqrt{x^2 + y^2}, \arg(x, y)\right),$$

where $\arg(x, y)$ is the angle of (x, y) with the positive x-axis. Note the inverse map of φ is given by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, so φ is in fact a homeomorphism.

Now, the transition map $\psi \circ \varphi^{-1}$ sends

$$(r, \theta) \overset{\varphi^{-1}}{\mapsto} (r\cos\theta, r\sin\theta) \overset{\psi}{\mapsto} (r\cos\theta, r\sin\theta).$$

Example 1.73. Consider the topological 2-manifold $M := S^2$ from Exercise 1.67. We compute the transition maps between φ_1^+ and φ_3^+, which overlap on the open set consisting of $(x_1, x_2, x_3) \in S^2$ such that $x_1, x_3 > 0$. Well, we can directly compute that $\varphi_3^+ \circ (\varphi_1^+)^{-1}$ is given by

$$(x_2, x_3) \overset{(\varphi_1^+)^{-1}}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \overset{\varphi_3^+}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2\right).$$

In the above examples, we can note that the maps between the Euclidean smooths are smooth on their domains. This becomes our notion of smoothness.

Definition 1.74 (smoothly compatible)**.** Two charts (U, φ) and (V, ψ) of a topological manifold M are *smoothly compatible* if and only if both transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth (i.e., infinitely differentiable). Notably, this condition is vacuously satisfied if $U \cap V = \varnothing$.

1.3.3 Smooth Structures

We would like to cover M with smoothly compatible charts, so it will be helpful to have a language for such covers.

Definition 1.75 (atlas)**.** Fix a topological manifold M. An *atlas* $\mathcal{A}$ is a collection of charts "covering M" in the sense that

$$M = \bigcup_{(U,\varphi)} U.$$

An atlas is *smooth* if and only if its charts are pairwise smoothly compatible. A smooth atlas is *maximal* if and only if it is maximal in the sense of inclusion by smooth atlases.

The point of using a maximal atlas is that we would like a way to say when two atlases provide the same smooth structure for a topological manifold, but it will turn out to be easier to provide a "unique" atlas to look at, which will be the maximal smooth atlas. Quickly, we note that maximal smooth atlases exist. One could argue this by Zorn's lemma, but we don't have to.

Proposition 1.76. Fix a topological n-manifold M. Any smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas, denoted $\overline{\mathcal{A}}$.

Proof. We have to show existence and uniqueness. We will construct this directly: define $\overline{\mathcal{A}}$ to be the collection of charts (U, φ) which is smoothly compatible with each chart in $\mathcal{A}$. We show that $\overline{\mathcal{A}}$ is a maximal smooth atlas.

- Atlas: certainly $\overline{\mathcal{A}} \supseteq \mathcal{A}$, so $\overline{\mathcal{A}}$ covers M, so $\overline{\mathcal{A}}$ is an atlas.

- Smooth: fix any charts $(U_1, \varphi_1), (U_2, \varphi_2) \in \overline{\mathcal{A}}$, and we would like to show that they are smoothly compatible. If $U_1 \cap U_2 = \varnothing$, there is nothing to do, so we may assume that the intersection is nonempty. By symmetry, it will be enough to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth.

  The point is that differentiability is a local notion: explicitly, fix some $q \in \varphi_1(U_1 \cap U_2)$, and we want to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth at q. This can be checked on a small open neighborhood of q; in particular, find the $p \in U_1 \cap U_2$ such that $\varphi_1(p) = q$, and we can find some chart $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Then we note that

$$\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1} = \left(\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\psi|_{U_1 \cap U_2 \cap V})^{-1}\right) \circ \left(\psi|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1}\right)$$

  is smooth on $\varphi_1(U_1 \cap U_2 \cap V)$ as it is the composition of smooth maps. So our left-hand side is smooth on $U_1 \cap U_2 \cap V$ and in particular at $q \in \varphi_1(U_1 \cap U_2 \cap V)$.

- Maximal: suppose $\mathcal{A}'$ is a smooth atlas containing $\mathcal{A}$. We must show that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$; by supposing further that $\mathcal{A}'$ contains $\overline{\mathcal{A}}$, we achieve the maximality of $\overline{\mathcal{A}}$. Well, for each $(U, \varphi) \in \mathcal{A}'$, we see that (U, φ) is smoothly compatible with each chart in $\mathcal{A}$, so $(U, \varphi) \in \overline{\mathcal{A}}$. Thus, $(U, \varphi) \in \overline{\mathcal{A}}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$.

- Unique: suppose $\mathcal{A}'$ is a maximal smooth atlas containing $\mathcal{A}$. Then the previous point establishes that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$, but then we must have equality because $\mathcal{A}'$ is a maximal smooth atlas. ■

So we may make the following definition.

Definition 1.77 (maximal smooth atlas)**.** Fix a topological n-manifold M. Given a smooth atlas $\mathcal{A}$ on M, we let $\overline{\mathcal{A}}$ denote the unique maximal smooth atlas containing $\mathcal{A}$, which we know exists and is unique by Proposition 1.76.

Corollary 1.78. Fix a topological n-manifold M. Given smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a smooth atlas, then

$$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}.$$

Proof. Define $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$ and hence both $\mathcal{A}_1$ and $\mathcal{A}_2$, so we see that $\overline{\mathcal{A}_1} = \overline{\mathcal{A}}$ and $\overline{\mathcal{A}_2} = \overline{\mathcal{A}}$. Notably, we are using the uniqueness of Proposition 1.76. ■

At long last, here is our definition.

Definition 1.79 (smooth manifold)**.** Fix a topological n-manifold M. A *smooth structure* on M is a maximal smooth atlas on M. A *smooth n-manifold* is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a smooth structure on M.

Remark 1.80. Adjusting the "smoothness" on the manifold M produces different notions of manifold. For example, we can have twice differentiable manifolds, real analytic manifolds, complex manifolds, etc.

1.4 January 25

The first homework is due later today.

1.4.1 A Couple Lemmas on Atlases

Here are some basic properties of smooth manifolds which one can check.

Lemma 1.81. Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$, then for any open subset $U' \subseteq U$, we have $(U', \varphi|_{U'}) \in \mathcal{A}$.

Proof. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U', \varphi|_{U'})\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas of charts. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that $(U', \varphi|_{U'})$ and (V, ψ) are smoothly compatible. (The charts in $\mathcal{A}$ are already smoothly compatible with each other.) In other words, we must show that the transition functions are diffeomorphism: the transition maps are

$$\varphi|_{U'\cap V} \circ \psi|_{U'\cap V}^{-1} = \left(\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}\right)|_{\psi(U'\cap V)}$$

and

$$\psi|_{U'\cap V} \circ \varphi|_{U'\cap V}^{-1} = \left(\psi|_{U\cap V} \circ \varphi|_{U\cap V}^{-1}\right)|_{\varphi(U'\cap V)},$$

and these are both smooth as the restrictions of smooth maps. (Namely, we are using the fact that (U, φ) and (V, ψ) are smoothly compatible already.) ■

Lemma 1.82. Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$ and diffeomorphism $\chi \colon \varphi(U) \to V$ for some open subset $V \subseteq \mathbb{R}^n$, we have $(U, \chi \circ \varphi) \in \mathcal{A}$.

Proof. The argument is similar to that of the above lemma. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U, \chi \circ \varphi)\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that (V, ψ) and $(U, \chi \circ \varphi)$ are smoothly compatible. (Indeed, the charts in $\mathcal{A}$ are already smoothly compatible with each other.) Well, the transition maps are

$$(\chi \circ \varphi)|_{U\cap V} \circ \psi|_{U\cap V}^{-1} = \chi|_{\varphi(U\cap V)} \circ \left(\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}\right)$$

and

$$\psi|_{U\cap V} \circ (\chi \circ \varphi)|_{U\cap V}^{-1} = \psi|_{U\cap V} \circ \varphi|_{U\cap V}^{-1} \circ \chi|_{\varphi(U\cap V)}^{-1},$$

which are smooth maps because (U, φ) and (V, ψ) are already smoothly compatible, and χ is a diffeomorphism. ■

Lemma 1.83. Fix a smooth n-manifold $(M, \mathcal{A})$. If $\varphi \colon U \to \mathbb{R}^n$ is an injective map with $U \subseteq M$ is such that each $p \in U$ has some open neighborhood $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, then actually $(U, \varphi) \in \mathcal{A}$.

Proof. By the definition of being a maximal smooth atlas, it suffices to show that (U, φ) is smoothly compatible with all charts in $\mathcal{A}$. Well, pick up some chart (V, ψ), and we would like to show that the transition map

$$\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}$$

is a diffeomorphism. Well, we can being a diffeomorphism locally by checking it at all point $\psi(p) \in \psi(U \cap V)$ where $p \in U \cap V$. But for some fixed p, we are promised some open subset $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, so the map

$$\varphi|_{U_p\cap V} \circ \psi|_{U_p\cap V}^{-1} = \left(\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}\right)|_{\psi(U_p\cap V)}$$

is a diffeomorphism. So we produce smoothness at the images of p of the function and its inverse. ■

1.4.2 Examples of Smooth Manifolds

We go through some examples of smooth manifolds.

Example 1.84. Recall from Lemma 1.21 that $\mathbb{R}^n$ is a topological n-manifold. Then $\operatorname{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ provides an atlas on $\mathbb{R}^n$ consisting of a single chart, which is vacuously smooth; note Proposition 1.76 then gives us a smooth structure.

More generally, we have the following.

Proposition 1.85. Fix a smooth n-manifold $(M, \mathcal{A})$. For any nonempty open subset $M' \subseteq M$, we have that M' is a topological n-manifold, and

$$\mathcal{A}' := \{(U, \varphi) \in \mathcal{A} : U \subseteq M'\}$$

is a smooth structure on M.

Proof. By Proposition 1.63, we see that M' is a topological n-manifold. It remains to show that $\mathcal{A}'$ is a smooth structure. Here are our checks.

- Chart: for any $x \in M'$, we know $\mathcal{A}$ is a chart on M, so there is a chart $(U, \varphi) \in \mathcal{A}$ with $x \in U$. Now, $U \subseteq M$ is open, so Lemma 1.81 tells us that $(U \cap M', \varphi|_{U \cap M'})$ is a chart in $\mathcal{A}$. But now $U \cap M' \subseteq M'$, so $(U \cap M', \varphi|_{U \cap M'}) \in \mathcal{A}'$ by construction, so we conclude because $x \in U \cap M'$.

- Smooth: for any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}'$, we note that these charts belong to the smooth atlas $\mathcal{A}$ already, so they are already smoothly compatible.

- Maximal: by definition of being a maximal smooth atlas, it suffices to show that if (U, φ) is a chart of M' smoothly compatible with $\mathcal{A}'$, then it must be in $\mathcal{A}'$. Well, $U \subseteq M'$ already, so it suffices to show that $(U, \varphi) \in \mathcal{A}$. Because $\mathcal{A}$ is already a maximal smooth atlas, it suffices to show that (U, φ) is compatible with all the charts in $\mathcal{A}$. Well, for any chart $(V, \psi) \in \mathcal{A}$, we need the composite

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}$$

to be a diffeomorphism. But we simply note that $(U \cap V, \psi|_{U \cap V}) \in \mathcal{A}$ by Lemma 1.81 will live in $\mathcal{A}'$, so the above is a diffeomorphism because the hypothesis on (U, φ) implies that it would be smoothly compatible with $(U \cap V, \psi|_{U \cap V}) \in \mathcal{A}'$. ■

Example 1.86. Any nonempty open subset of $\mathbb{R}^n$ is a smooth n-manifold by combining Example 1.84 and Proposition 1.85. For example,

$$\operatorname{GL}_n(\mathbb{R}) := \left\{M \in \mathbb{R}^{n \times n} : \det M \neq 0\right\}$$

is an open subset of $\mathbb{R}^{n\times n}$, so $\operatorname{GL}_n(\mathbb{R})$ is a smooth manifold. (Notably, $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a polynomial and hence continuous, so the pre-image of $\mathbb{R} \setminus \{0\}$ is open.)

Example 1.87. From Example 1.66, we know that the graph Γ of a smooth function $f\colon V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^m$ is open, is a topological n-manifold, where we have a chart given by the projection $\pi\colon \Gamma \to V$. Using this chart alone produces a smooth atlas and makes Γ into a smooth n-manifold as well.

Example 1.88. We claim that the charts on S^n provided in Exercise 1.67 provide a smooth atlas on S^n and hence a smooth structure by Proposition 1.76. Indeed, we must show that the transition maps

$$\varphi_i^\pm|_{U_i^\pm\cap U_j^\pm}\circ\varphi_j^\pm|_{U_i^\pm\cap U_j^\pm}^{-1}(x_1,\ldots,x_n)=\left(x_1,\ldots,\widehat{x}_j,\ldots,x_{i-1},\pm\sqrt{1-(x_1^2+\cdots+x_n^2)},x_i,\ldots,x_n\right)$$

is a diffeomorphism (for any choice of signs). The above equation shows that our map is smooth for $i>j$, and the computation for $i<j$ simply switches the ith and jth coordinates. On the homework, we will see how to use stereographic projection to provide a smooth structure (in fact, the same smooth structure) on S^n.

Example 1.89. Fix an n-dimensional $\mathbb{R}$-vector space V. Then we claim

$$\mathcal{A}\coloneqq\{(V,\varphi):\varphi\text{ is an isomorphism to }\mathbb{R}^n\}$$

is a smooth atlas on V and hence provides a smooth structure. Indeed, certainly this is an atlas: there is some isomorphism $\varphi\colon V\to\mathbb{R}^n$, and this chart will cover V. Further, these are smoothly compatible because the transition map between the two arbitrary charts (V,φ) and (V,ψ) is the linear isomorphism $\left(\varphi\circ\psi^{-1}\right):\mathbb{R}^n\to\mathbb{R}^n$, which is linear and hence smooth.

Example 1.90. Fix the topological 1-manifold $\mathbb{R}$ of Lemma 1.21. Example 1.84 tells us $\mathcal{A}\coloneqq\{(\mathbb{R},\mathrm{id}_\mathbb{R})\}$ provides a smooth atlas, and $\mathcal{A}'\coloneqq\{(\mathbb{R},\varphi)\}$ given by $\varphi\colon x\mapsto x^3$ is also a smooth atlas (again, smoothness is vacuous). However, $\mathcal{A}$ and $\mathcal{A}'$ provide smooth structures: otherwise, they would be contained in the same maximal smooth atlas, so $(\mathbb{R},\mathrm{id}_\mathbb{R})$ and $(\mathbb{R},\varphi)$ would be smoothly compatible, but then the composite $\left(\mathrm{id}_\mathbb{R}\circ\varphi^{-1}\right):x\mapsto\sqrt[3]{x}$ is not a smooth function $\mathbb{R}\to\mathbb{R}$.

Example 1.91. Recall that $\mathbb{RP}^n$ is a topological n-manifold by Exercise 1.68. We claim that the charts (U_i,φ_i) actually form a smooth atlas on $\mathbb{RP}^n$, thus making $\mathbb{RP}^n$ into a smooth atlas. We already checked that these charts cover $\mathbb{RP}^n$, and they are smoothly compatible because we can compute the transition between (U_i,φ_i) and (U_j,φ_j) is

$$\varphi_i|_{U_i\cap U_j}\circ\varphi_j|_{U_i\cap U_j}^{-1}(x_0,\ldots,\widehat{x}_j,\ldots,x_n)=\left(\frac{x_0}{x_i},\ldots,\frac{x_{j-1}}{x_i},\frac{1}{x_i},\frac{x_{j+1}}{x_i},\ldots,\frac{x_n}{x_i}\right),$$

which we can see is a rational and hence smooth function.

Example 1.92. Fix smooth manifolds $(M_1,\mathcal{A}_1),\ldots,(M_k,\mathcal{A}_k)$, where M_i is a smooth n_i-manifold. The product $M\coloneqq M_1\times\cdots\times M_k$ is a smooth manifold by Proposition 1.56, and the proof implies that

$$\mathcal{A}\coloneqq\{(U_1\times\cdots\times U_k,\varphi_1\times\cdots\times\varphi_k):(U_i,\varphi_i)\in\mathcal{A}_i\text{ for each }i\}$$

is an atlas on M. In fact, this is a smooth atlas, thus providing M with a smooth structure by Proposition 1.76. Well, the transition map between the charts $(U,\varphi)\coloneqq(U_1\times\cdots\times U_k,\varphi_1\times\cdots\times\varphi_k)$ and $(V,\psi)\coloneqq(V_1\times\cdots\times V_k,\psi_1\times\cdots\times\psi_k)$ is

$$\varphi|_{U\cap V}\circ\psi|_{U\cap V}^{-1}=\left(\varphi_1|_{U_1\cap V_1}\circ\psi_1|_{U_1\cap V_1}^{-1}\right)^{-1}\times\cdots\times\left(\varphi_k|_{U_k\cap V_k}\circ\psi_k|_{U_k\cap V_k}^{-1}\right),$$

which we can see is smooth as it is the product of smooth functions.

Remark 1.93. In fact, if a topological n-manifold has some smooth structure, there are uncountably many distinct smooth structures on M. On the other hand, for n-manifolds of small dimensions (e.g., $n \leq 3$), it turns out that these are diffeomorphic.

Remark 1.94. However, there do exist topological n-manifolds with no smooth structure, in dimensions $n \geq 4$. Even worse, there are topological n-manifolds with distinct smooth structures up to diffeomorphism, again in dimensions $n \geq 4$. Even for S^n, the story is complicated: there is only one smooth structure for $n \leq 3$, we don't understand $n = 4$, and the story is complicated but somewhat understood for $n \geq 5$.

1.4.3 Grassmannians

The construction of smooth manifolds is rather long: we build a topological space, define some charts, and then check that the charts are smoothly compatible. Here's a lemma to do all of this at once.

Lemma 1.95. Fix a set M with a nonnegative integer $n \geq 0$ and a collection of functions $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ where $U_\alpha \subseteq M$ and $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ is open. Further, suppose the following.

(i) $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ is open for all $\alpha, \beta \in \kappa$.

(ii) The composite $\varphi_\alpha|_{U_\alpha\cap U_\beta} \circ \varphi_\beta|_{U_\alpha\cap U_\beta}^{-1}$ is smooth for all $\alpha, \beta \in \kappa$.

(iii) M is covered by a countable subcollection of $\{U_\alpha\}_{\alpha\in\kappa}$.

(iv) For distinct $p, q \in M$, either there is $\alpha \in \kappa$ such that $p, q \in U_\alpha$, or there are disjoint U_α and U_β containing p and q, respectively.

Then M is a smooth n-manifold with smooth atlas given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$.

Proof. We sketch the steps.

1. We provide M with a topology. We would like for Well, we say that $A \subseteq M$ is open if and only if $\varphi_\alpha(A \cap U_\alpha)$ is open for all $\alpha \in \kappa$.
2. Then condition (i) makes the φ_α into homeomorphisms onto their images. Thus, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ is an atlas.
3. Condition (ii) implies that $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas.
4. Condition (iii) implies that M becomes second countable.
5. Lastly, condition (iv) implies that M is Hausdorff.

We leave the checks to the reader. ■

Let's see an example of this.

Exercise 1.96. Fix nonnegative integers $k \leq n$. Then let $M := \mathrm{Gr}_k(\mathbb{R}^n)$ denote the set of k-dimensional linear subspaces V of $\mathbb{R}^n$. We show that M is a smooth $k(n-k)$-manifold.

Sketch. We use Lemma 1.95. For concreteness, let we choose our index set I to consist of pairs (P, Q) of subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = P \oplus Q$ and $\dim P = k$ and $\dim Q = n - k$. The point is that we are choosing a complement for our k-dimensional subspaces in order to help count them. In particular, we may define the subset

$$U_\alpha := \{V \in \mathrm{Gr}_k(\mathbb{R}^n) : V \cap Q = \{0\}\}.$$

Notably, for any $V \in U_\alpha$, there is a unique linear map $M_{P,Q,V} \colon P \to Q$ such that

$$V = \{x + M_{P,Q,V}x \in P \oplus Q : x \in P\}.$$

Approximately speaking, we are viewing V as a graph. Anyway, this construction provides a map $\varphi_\alpha \colon U_\alpha \to \operatorname{Hom}_{\mathbb{R}}(P,Q)$ given by $V \mapsto M_{P,Q,V}$, where we identify $\operatorname{Hom}_{\mathbb{R}}(P,Q) \cong \mathbb{R}^{k(n-k)}$. We now conclude by noting that we can check the properties from Lemma 1.95. For example, to see that the transition maps are smooth, suppose we have two pairs $(P,Q),(P',Q') \in I$, and the vector space V decomposes into the two separate ways, and these matrices have rational functions in their coordinates, so smoothness follows. As another example, one can actually cover M by finitely many charts, and the last check follows because any k-dimensional subspaces $V,V' \subseteq \mathbb{R}^n$ has some $(n-k)$-dimensional subspace $Q \subseteq \mathbb{R}^n$ such that $V \cap Q = V' \cap Q = \{0\}$. ■

1.4.4 Manifolds with Boundary

Before moving on from our discussion of a single manifold, we discuss manifolds with boundary.

Definition 1.97 (topological manifold with boundary)**.** Fix a nonnegative integer n. A *topological n-manifold with boundary* is a Hausdorff, second countable topological space M with the following variant of being locally Euclidean: for any $p \in M$, there are open subsets $U \subseteq M$ and

$$\widehat{U} \subseteq \mathbb{H} \coloneqq \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

such that $p \in U$ and $U \cong \widehat{U}$. We continue to call (U,φ) a chart.

Example 1.98. Any topological n-manifold is a topological n-manifold with boundary: one can simply make the charts output to $\mathbb{H}^\circ$.

Example 1.99. The space $\mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a topological n-manifold with boundary.

The point is that we can pick up some "boundary" like the one in $\mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Anyway, let's discuss smoothness. This requires understanding smoothness on $\partial\mathbb{H}^n$.

Definition 1.100. Fix a subset $A \subseteq \mathbb{R}^n$. A function $f \colon A \to \mathbb{R}^m$ is *smooth* if and only if there is an open subset $V \subseteq \mathbb{R}^n$ containing A and a smooth extension $\widetilde{f} \colon V \to \mathbb{R}^n$ of f.

Remark 1.101. It turns out that (by Seeley's theorem) if $V \subseteq \mathbb{H}^n$ is open, it is enough to check that the partial derivatives of some function $f \colon V \to \mathbb{R}^m$ extend continuously to the boundary.

Definition 1.102 (smooth manifold with boundary)**.** Fix a nonnegative integer n. A *smooth n-manifold with boundary* is a pair $(M,\mathcal{A})$ where M is a topological n-manifold with boundary, and $\mathcal{A}$ is a maximal smooth atlas, where we are taking atlas in the sense

We will not bother to redo the proof of Proposition 1.76 to explain that the notion of a maximal smooth atlas makes sense with subsets of $\mathbb{H}^n$ in addition to subsets of $\mathbb{R}^n$; all the proofs are the same.

Note that boundary is in fact an intrinsic notion.

Definition 1.103 (boundary, interior)**.** Fix a smooth n-manifold with boundary M and a point $p \in M$.

- Then p is a *boundary point* if and only if there is a smooth chart (U,φ) such that $\varphi(p) \in \partial\mathbb{H}^n$.
- Then p is an *interior point* if and only if there is a smooth chart (U,φ) such that $\varphi(p)$ is in the interior of $\mathbb{H}^n$.

We will show in Theorem 1.104 that any point in M is exactly one of a boundary point or an interior point.

1.5 January 30

Here we go.

1.5.1 Smooth Manifolds with Boundary

We would like for the boundary of a smooth manifold with boundary to make sense.

Theorem 1.104. Fix a smooth n-manifold with boundary M, and fix some $p \in M$. Given two charts (U, φ) and (V, ψ) with $p \in U \cap V$), then $\varphi(p) \in \partial\mathbb{H}^n$ if and only if $\psi(p) \in \partial\mathbb{H}^n$.

Proof. Suppose this is not the case. Then, up to rearranging, we get $\varphi(p) \in (\mathbb{H}^n)^\circ$ and $\psi(p) \in \partial\mathbb{H}^n$. Our transition maps are smooth, so we have produced a diffeomorphism from the open subsets $U' \subseteq \mathbb{H}^n$ and $V' \subseteq \mathbb{H}^n$ such that $U' \cap \partial\mathbb{H}^n = \varnothing$ but $V' \cap \partial\mathbb{H}^n \neq \varnothing$. Now, for smoothness, the transition map $\tau \colon V' \to U'$ must have an extension $\widetilde{\tau} \colon \widetilde{V}' \to \widetilde{U}'$. But then $\widetilde{\tau}$ is an invertible map, so the Inverse function theorem implies that τ is locally invertible and in particular must be an open map. But V' goes to U', which is not open in $\mathbb{R}^n$, so we have our contradiction. ■

Remark 1.105. In fact,

$$\psi \circ \varphi^{-1}|_{\partial\mathbb{H}^n \cap \varphi(U \cap V)} \colon (\partial\mathbb{H}^n \cap \varphi(U \cap V)) \to (\partial\mathbb{H}^n \cap \psi(U \cap V))$$

is a smooth transition map, though we will not check this here.

Remark 1.106. People in the modern day might allow ∂M to be a manifold with boundary itself, which is a "manifold with corners."

Remark 1.107. One can remove the smoothness assumption here as well, but it will require some cohomology or similar.

The boundary/interior for a smooth manifold may not actually be its boundary/interior when embedded into a space.

Example 1.108. Consider $M := \{x \in \mathbb{R}^n : x_n > 0\}$. Then M is a smooth manifold with boundary, but $\partial M = \{x \in \mathbb{R}^n : x_n = 0\}$ when viewed as a subset of $\mathbb{R}^n$.

Example 1.109. Consider $M = S^n \subseteq \mathbb{R}^{n+1}$. Then M is a smooth manifold (without boundary), but as a subspace of $\mathbb{R}^{n+1}$, we have $\partial M = M$.

Example 1.110. Consider $M := \mathbb{H}^n \cap B(0,1)$. Then M is a smooth manifold whose boundary (as a manifold) is $\partial\mathbb{H}^n \cap B(0,1)$, but the topological boundary is $\partial\mathbb{H}^n \cup (\partial B(0,1) \cap \mathbb{H}^n)$.

Theme 2
Maps Between Manifolds

I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial.

—Andre Weil, [Wei59]

2.1 January 30-map

We continue.

2.1.1 Smooth Maps to $\mathbb{R}^n$

We will define smooth maps in steps. To begin, we say what it means to have a smooth map $M \to \mathbb{R}^n$. Basically, we look locally at the points on our manifold and check smoothness on charts.

Definition 2.1 (smooth)**.** Fix a smooth manifold M, possibly with boundary. Then a function $f\colon M \to \mathbb{R}^m$ is *smooth* if and only if each $p \in M$ has some smooth chart (U, φ) with $p \in U$ and

$$f|_U \circ \varphi|_U^{-1}$$

is a smooth map $\varphi(U) \to \mathbb{R}^m$.

Example 2.2. Any smooth map $f\colon U \to \mathbb{H}^m$, where $U \subseteq \mathbb{H}^n$ is open, is smooth in the above sense. Indeed, U as an n-manifold has a smooth atlas given by $\{(U, \mathrm{id}_U)\}$, and this witnesses the smoothness of f for any $p \in U$.

Here is a quick sanity check: the charts don't matter.

Lemma 2.3. Fix a smooth map $f\colon M \to \mathbb{H}^m$, where M is a smooth manifold, possibly with boundary. For any smooth chart (V, ψ), the composition $f|_U \circ \varphi|_U^{-1}$ is smooth.

Proof. This is a matter of tracking through all the definitions. Fix some $p \in V$, and we would to test smoothness around p. Well, p has some smooth chart (U, φ) such that $p \in U$ and $f|_U \circ \varphi|_U^{-1}$ is smooth. But now we write

$$f|_{U\cap V}\circ\psi|_{U\cap V}^{-1}=\left(f|_{U\cap V}\circ\varphi|_{U\cap V}^{-1}\right)\circ\left(\varphi|_{U\cap V}\circ\psi|_{U\cap V}^{-1}\right),$$

which is the composition of smooth maps: the left map is smooth by construction of (U,φ), and the right map is smooth by compatibility of smooth charts. ■

We are now ready to define smooth maps between manifolds. Approximately speaking, we simply add in a check locally on the target.

Definition 2.4 (smooth)**.** Fix smooth manifolds M and N, possibly with boundary. A map $F\colon M\to N$ is *smooth* if and only if each $p\in M$ has smooth charts (U,φ) and (V,ψ) such that $p\in U$ and $F(U)\subseteq V$ and the composite

$$\psi\circ F|_U\circ\varphi|_U^{-1}$$

is a smooth map $\mathbb{H}^m\to\mathbb{H}^n$. We may call the above composite a *coordinate representation*.

Example 2.5. Any smooth map $F\colon U\to V$, where $U\subseteq\mathbb{R}^m$ and $V\subseteq\mathbb{R}^n$ are open, is smooth in the above sense. Indeed, U and V have smooth atlases given by $\{(U,\mathrm{id}_U)\}$ and $\{(V,\mathrm{id}_V)\}$ (respectively), and these charts witness that F is smooth at each $p\in U$ because the composite

$$\mathrm{id}_V\circ F\circ{\mathrm{id}_U}^{-1}=F$$

is smooth by hypothesis.

Here's the same sanity check: the charts don't matter.

Lemma 2.6. Fix a smooth map $F\colon M\to N$ of manifolds, possibly with boundary. If (U,φ) and (V,ψ) are smooth charts on M and N, respectively, and $F(U)\subseteq V$, then the composite $\psi\circ F|_U\circ\varphi|_U^{-1}$ is smooth.

Proof. Again, we track through locally, tracking through all the definitions. To check that $\psi\circ F|_U\circ\varphi|_U^{-1}$ is smooth, it suffices to check it an open cover of $\varphi(U)$. Pick $\varphi(p)\in\varphi(U)$ where $p\in U$, and we know that we have smooth charts (U_p,φ_p) and (V_p,ψ_p) in M and N, respectively, such that $F(U_p)\subseteq V_p$ and the composite $\psi_p|_{F(U_p)}\circ F|_{U_p}\circ\varphi_p|_{U_p}^{-1}$ is smooth. Then we see that

$$\psi\circ F|_U\circ\varphi|_U^{-1}=\left(\psi|_{V\cap V_p}\circ\psi_p|_{V\cap V_p}^{-1}\right)\circ\left(\psi_p\circ F|_U\circ\varphi_p|_{U_p}^{-1}\right)|_{\varphi_p(U\cap U_p)}\circ\left(\varphi_p|_{U\cap U_p}\circ\varphi|_{U\cap U_p}^{-1}\right)$$

is smooth, where the left and right maps are smooth by smooth compatibility, and the middle map is smooth by construction. ■

Remark 2.7. One can write out the above proof diagrammatically by noting that having smooth charts (U,φ) and (U',φ') of M and smooth charts (V,ψ) and (V',ψ') of N such that $F(U)\subseteq V$ and $F(U')\subseteq V'$ will have the following diagram.

$$\begin{array}{ccc}\varphi(U) & \xrightarrow{\psi\circ F\circ\varphi^{-1}} & \psi(V)\\ \updownarrow & & \updownarrow\\ \varphi'(U') & \xrightarrow[\psi'\circ F\circ(\varphi')^{-1}]{} & \psi'(V')\end{array}$$

Here, the vertical maps are only defined on the corresponding intersections, but it is smooth when defined by the smooth compatibility.

Remark 2.8. Please read more of chapter 2 to get helpful properties of smooth maps.

2.1.2 Partition of Unity

By way of motivation, suppose we have two smooth functions $f, g\colon \mathbb{R} \to \mathbb{R}$, and we want to build a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $f|_{(-\infty,-1)} = h|_{(-\infty,-1)}$ and $g|_{(1,\infty)} = h|_{(1,\infty)}$. One way to do this is to find smooth functions $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \varphi|_{(-\infty,-1)} = 1, \\ \varphi|_{(1,\infty)} = 0. \end{cases}$$

Then $h \coloneqq \varphi f + (1-\varphi)g$ is smooth by construction, and it satisfies the restriction conditions also by construction. This idea of "splitting up the 1 function" is known as partition of unity.

Definition 2.9 (partition of unity)**.** Fix a topological space X, and let $\{U_\alpha\}_{\alpha\in\kappa}$ be an open cover on M. Then a *partition of unity subordinate to* $\{U_\alpha\}_{\alpha\in\kappa}$ is a collection of continuous functions $\{\varphi_\alpha\}_{\alpha\in\kappa}$ on X satisfying the following.

- $\operatorname{im}\varphi_\alpha \subseteq [0,1]$ always.
- $\operatorname{supp}\varphi_\alpha \subseteq U_\alpha$ for each α.
- The collection $\{\operatorname{supp}\varphi_\alpha\}_{\alpha\in\kappa}$ is locally finite.
- For each $x \in X$, we have
$$\sum_{\alpha\in\kappa}\varphi_\alpha(x) = 1.$$

Of course, we must show that these exist.

2.2 February 1

The second homework is due later today. We began class by completing a proof, so I edited directly into those notes.

2.2.1 Partition of Unity for Manifolds

We will show that partitions of unity exist for manifolds.

Theorem 2.10. Fix a smooth manifold M. For any open cover $\{U_\alpha\}_{\alpha\in\kappa}$, there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in\kappa}$ (of smooth functions) subordinate to $\{U_\alpha\}_{\alpha\in\kappa}$.

Proof. We begin by constructing smooth functions $\{\widetilde{\varphi}\}_{\alpha\in\kappa}$ satisfying the following constraints.

- $\operatorname{im}\widetilde{\varphi}_\alpha \subseteq [0,\infty)$.
- $\operatorname{supp}\widetilde{\varphi}_\alpha \subseteq U_\alpha$.
- The collection $\{\operatorname{supp}\widetilde{\varphi}_\alpha\}_{\alpha\in\kappa}$ is a locally finite open cover of M.

Dividing out by the summation of the $\widetilde{\varphi}_\bullet$s completes the proof. Notably, for each $x \in M$, the sum

$$\widetilde{\varphi}(x) \coloneqq \sum_{\alpha\in\kappa}\widetilde{\varphi}_\alpha(x)$$

is finite (x can only belong to finitely many of the supports); in fact, there is an open neighborhood U of x such that U only intersects finitely many of the supports, so

$$\widetilde{\varphi}|_U = \sum_{\substack{\alpha\in\kappa \\ \operatorname{supp}\widetilde{\varphi}_\alpha\cap U\neq\varnothing}}\widetilde{\varphi}_\alpha|_U$$

is just a finite sum of smooth functions, so $\widetilde{\varphi}$ is smooth on U. Thus, by gluing, $\widetilde{\varphi}$ is smooth on M, and we note that it is nonzero because each $x \in M$ is in some support, so we can define $\varphi_\alpha := \widetilde{\varphi}_\alpha/\varphi$ to satisfy all the needed conditions, most notable being that these functions are smooth, have support contained in U_α, and $\sum_{\alpha\in\kappa}\varphi_\alpha = 1$.

It remains to construct the $\widetilde{\varphi}_\alpha$s. We proceed in steps.

1. We construct a nice open cover. For each $x \in M$, we can find some open neighborhood U such that we have a homeomorphism $\varphi\colon U \to B(\varphi(x), 2)$. Then $\{\varphi^{-1}(B(\varphi(x), 1))\}_{x\in M}$ is an open cover of M, so we can refine this to a locally finite open cover $\mathcal{U}$ of precompact open sets. By looking down on compact, we may as well assume that $\mathcal{U}$ is made of coordinate balls $B(\varphi(x), r)$ contained in larger coordinate balls $B(\varphi(x), r')$ for $r' > r$.

2. Now, for each coordinate ball $\varphi\colon U \cong B(0, r)$ for $U \in \mathcal{U}$ extending to $\varphi'\colon U' \cong B(0, r')$. Then we construct f_U which is nonzero on $B(0, r)$ but vanishes on $B(0, r')$.

   Now, for each $U \in \mathcal{U}$, select $\alpha_U \in \kappa$ such that $\overline{U} \subseteq U_{\alpha_U}$. From here, we may set

$$\widetilde{\varphi}_\alpha := \sum_{\alpha_U=\alpha} f_U,$$

   which satisfies all the needed conditions. For example, one finds that the support of $\widetilde{\varphi}_\alpha$ is

$$\overline{\bigcup_{U\subseteq U_\alpha} U} \subseteq \bigcup_{U\subseteq U_\alpha} \overline{U} \subseteq U.$$

   One needs local finiteness in order to verify the first inclusion; the point is that one can reduce this large union to a finite one around any given point, so the closures must agree. ■

Let's give some applications.

Corollary 2.11. Fix a smooth manifold M. For any closed set $A \subseteq M$ contained in an open set $U \subseteq M$, there exists a smooth function $\psi\colon M \to \mathbb{R}$ such that $\psi|_A = 1$ and $\psi|_{M\setminus U} = 0$.

Proof. Consider the open cover $\{U, M \setminus A\}$; this is an open cover because $U \cup (M \setminus A) = M$ is equivalent to $A \subseteq U$. Then Theorem 2.10 produces two nonnegative smooth functions ψ_0 and ψ_1 such that $\operatorname{supp}\psi_0 \subseteq U$ and $\operatorname{supp}\psi_1 \subseteq M \setminus A$ and $\psi_0 + \psi_1 = 1$ everywhere. But now ψ_0 is the desired function: $\operatorname{supp}\psi_0 \subseteq U$ implies $\psi_0|_{M\setminus U}$, and $\psi_0|_A + \psi_1|_A = 1$, but $\psi_1|_A = 0$ because $\operatorname{supp}\psi_1 \subseteq M \setminus A$. ■

Corollary 2.12 (Extension lemma)**.** Fix a smooth manifold M. Further, fix a closed subset $A \subseteq M$ contained in an open set $U \subseteq M$. Given a smooth function $f\colon A \to \mathbb{R}^k$, there is a smooth function $\widetilde{f}\colon M \to \mathbb{R}^k$ extending f and with $\operatorname{supp}\widetilde{f} \subseteq U$.

Proof. Omitted. ■

Corollary 2.13. Fix a smooth manifold M. There is a nonnegative function $f\colon M \to \mathbb{R}$ such that all the sets

$$f^{-1}([0, c])$$

are compact for any $c \geq 0$.

Proof. Fix a countable cover $\{U_n\}_{n\in\mathbb{N}}$ of M by precompact open subsets, and let $\{\psi_n\}_{n\in\mathbb{N}}$ be the corresponding partition of unity. Then we set

$$f := \sum_{n=0}^{\infty} n\psi_n.$$

Notably, for each $c \in \mathbb{R}$, we see

$$f^{-1}([0,c]) \subseteq \bigcup_{n\le c} \operatorname{supp}\psi_n,$$

so $f^{-1}([0,c])$ is a closed subset of a finite union of compact sets (which is compact), so we are done. ■

Corollary 2.14. Fix a closed subset K of a smooth manifold M. Then there is a nonnegative smooth function $f\colon M\to\mathbb{R}$ such that $f^{-1}(\{0\}) = K$.

Proof. One begins with $M = \mathbb{R}^n$ and then does the general case from there. ■

2.2.2 Diffeomorphisms

Here is our definition.

Definition 2.15 (diffeomorphism)**.** Fix a map $F\colon M\to N$ of smooth manifolds, possibly with boundary. Then F is a *diffeomorphism* if and only if F is bijective, smooth, and has smooth inverse.

Remark 2.16. Invariance of the boundary under smooth charts implies F must send boundary points to boundary points.

Remark 2.17. If F is a diffeomorphism, then $\dim M = \dim N$. Simply put, we can work locally on a chart, and then we are providing a diffeomorphism $\mathbb{R}^m \to \mathbb{R}^n$, but this can only happen when $m = n$. For example, it means that DF and DF^{-1} are invertible linear maps $\mathbb{R}^m\to\mathbb{R}^n$ and $\mathbb{R}^n\to\mathbb{R}^m$, respectively, which manifestly requires $m = n$.

Remark 2.18. It turns out that topological n-manifolds with a smooth structure admit a unique smooth structure up to diffeomorphism, for $n \ge 3$. For $n \ge 4$, even $\mathbb{R}^4$ fails to have a unique smooth structure.

Remark 2.19. The collection $\operatorname{Diff}(M)$ of diffeomorphisms $M \to M$ is a group, and one can give it a topology. For example, one can compute that $\operatorname{Diff}\left(S^2\right)$ is homotopy equivalent to $O(3)$, given approximately by rotations.

2.2.3 Tangent Spaces

Fix a smooth n-manifold M. One would like to provide each point $p\in M$ with an n-dimensional tangent vector space T_pM. If M is embedded into Euclidean space reasonably, we can imagine using the embedding to realize the tangent space; for example, if M is a (smooth) curve in $\mathbb{R}^2$, we can imagine that the tangent vectors tell us what direction we are moving in. We would also like to actually be able to compute these things in charts.

Anyway, here is our definition of tangent vectors. This definition is a bit awkward to handle because we want to be invariant.

Definition 2.20 (tangent space)**.** Fix a smooth n-manifold M and some point $p \in M$. A *derivation at* p is an $\mathbb{R}$-linear map $v\colon C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = f(p)v(g) + g(p)v(f)$$

for any $f, g \in C^\infty(M)$. Then the *tangent space* $T_p(M)$ *at* p is the collection of derivations.

Remark 2.21. Note that $T_p(M)$ is an $\mathbb{R}$-subspace of the collection of linear maps $C^\infty(M) \to \mathbb{R}$.

Example 2.22. Fix $M := \mathbb{R}^n$ and some $p \in M$. Then any $v \in \mathbb{R}^n$ has a "directional derivative" given by

$$f \mapsto \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_p.$$

This is simply by the product rule in multivariable calculus.

2.3 February 6

Today we continue talking about tangent vectors.

2.3.1 Derivations

Let's provide some basic properties of derivations.

Lemma 2.23. Fix a smooth n-manifold M and a derivation $v\colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. If $f\colon M \to \mathbb{R}$ is constant, then $v(p) = 0$.

Proof. By scaling, it suffices to do the case where $f \equiv 1$. Then we see that $f^2 = f$, so

$$v(f) = v\left(f^2\right) = 2f(p)v(f) = 2v(f),$$

so $v(f) = 1$ is forced. ■

Lemma 2.24. Fix a smooth n-manifold M and a derivation $v\colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. Given $f, g \in C^\infty(M)$ such that $f|_U = g|_U$ for some open $U \subseteq M$ containing p, we have $v(f) = v(g)$.

Proof. Set $h := f - g$ so that we want to show $v(h) = 0$ by linearity. The moral of the story is to extend being zero on U to all of M; in other words, we will want some bump functions. Because M is locally Euclidean, we can find a precompact open neighborhood V of p such that $\overline{V} \subseteq U$. Thus, Corollary 2.11 provides a smooth bump function $\psi\colon M \to \mathbb{R}$ such that $\psi|_{\overline{V}} \equiv 1$, and $\operatorname{supp}\psi \subseteq U$. Notably, $\psi \cdot h$ has support contained in U, but h vanishes on U, so $\psi \cdot h = 0$, so

$$0 = v(\psi \cdot h) = \psi(p)v(h) + h(p)v(\psi) = v(h),$$

as desired. ■

Manifolds are understood by passing to local charts, and the above lemma somewhat allows us to do this. As such, we are now motivated to understand local charts.

Lemma 2.25. Fix a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$. For each $v \in \mathbb{R}^n$, define $D_v|_a \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ by

$$D_v|_a(f) := \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a .$$

Then $D_v|_a$ is a derivation at a. In fact, the map $D\colon \mathbb{R}^n \to T_a\mathbb{R}^n$ given by $v \mapsto D_v|_a$ is an isomorphism of vector spaces.

Proof. To check that $D_v|_a$ is a derivation, one proceeds via the product rule in multivariable calculus. We omit this check. It remains to check that we have an isomorphism.

- Linear: given $c, d \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, we compute

$$D_{cv+dw}|_a f = \sum_{i=1}^n (cv_i + dw_i) \frac{\partial f}{\partial x_i}\bigg|_a = c\sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a + d\sum_{i=1}^n w_i \frac{\partial f}{\partial x_i}\bigg|_a = (cD_v|_a + dD_v|_a)f,$$

  as desired.

- Injective: by linearity, it is enough to show that having $D_v|_a = 0$ implies $v = 0$. Well, it is enough to check that $v_j = 0$ for each j. For this, we let $p_j \colon \mathbb{R}^n \to \mathbb{R}$ denote the jth projection so that

$$\frac{\partial p_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else}, \end{cases}$$

  so we see that

$$D_v|_a(p_j) = \sum_{i=1}^n v_i \frac{\partial p_j}{\partial x_i}\bigg|_a = v_j$$

  must vanish for each j, as desired.

- Surjective: this is the heart of the matter. Fix a derivation $v \in T_a\mathbb{R}^n$. We need a candidate vector, so we define $u_i := v(p_i)$, where $p_i \colon \mathbb{R}^n \to \mathbb{R}$ is the ith projection. We claim that

$$v = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\bigg|_a ,$$

  which will complete the proof. This requires a quick digression into a Taylor expansion. Given a smooth function $f \colon M \to \mathbb{R}$ and points $x, a \in \mathbb{R}^n$, we see

$$\begin{aligned} f(x) &= f(a) + \int_0^1 \frac{d}{dt} f(x + t(x-a))\, dt, \\ &= f(a) + \sum_{i=1}^n \left((x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\, dt}_{h_i(x) :=} \right), \end{aligned}$$

  where in the last equality we have used the multivariable chain rule. Applying the derivation, we see

$$v(f) = \underbrace{v(f(a))}_{0} + \sum_{i=1}^n v(x_i - a_i) h_i(a) + \sum_{i=1}^n \underbrace{(a_i - a_i)}_{0} v(h_i),$$

  where $v(f(a)) = 0$ by Lemma 2.23. Additionally, $v(x_i - a_i) = v(x_i) = u_i$ using Lemma 2.23 again. Notably, $h_i(a) = \frac{\partial f}{\partial x_i}|_a$, so

$$v(f) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}\bigg|_a ,$$

  as desired. ■

2.3.2 Differentials of Smooth Maps

Derivations explain how to take derivatives of functions in $M \to \mathbb{R}$. We now upgrade to taking derivatives of functions between manifolds.

Definition 2.26 (differential)**.** Fix smooth manifolds M and N. Given a smooth map $F\colon M \to N$, the *differential of F at $p \in M$* is the map $dF_p\colon T_pM \to T_{F(p)}N$ defined by

$$dF_p(v)(f) := v(f \circ F)$$

for any $f \in C^\infty(N)$.

Remark 2.27. The composition of smooth functions is smooth, so $f \circ F$ is smooth, so the definition of dF_p at least makes sense. Notably, $f \mapsto (f \circ F)$ is a map $C^\infty(N) \to C^\infty(M)$ of $\mathbb{R}$-algebras, so $f \mapsto v(f \circ F)$ remains a derivation. Explicitly, it is surely $\mathbb{R}$-linear (as the composition of $\mathbb{R}$-linear maps), and we satisfy the Leibniz rule because

$$\begin{aligned} dF_pv(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))dF_pv(g) + g(F(p))dF_pv(f). \end{aligned}$$

Remark 2.28. The map $dF_p\colon T_pM \to T_{F(p)}N$ is linear, essentially by definition. Namely, for $a, b \in \mathbb{R}$ and $v, w \in T_pM$ and $f \in C^\infty(N)$, we compute

$$dF_p(av + bw)(f) = (av + bw)(f \circ F) = av(f \circ F) + bw(f \circ G) = (adF_p(v) + bdF_p(w))(f).$$

Example 2.29. Take $M := \mathbb{R}^m$ and $N := \mathbb{R}^n$, and let $F\colon M \to N$ be a smooth map, which we may as well write as $F = (F_1, \ldots, F_n)$. Now, fix some $p \in M$, and identify $\mathbb{R}^m \cong T_pM$ and $\mathbb{R}^n \cong T_{F(p)}N$ as in Lemma 2.25. Well, given some smooth $f\colon N \to \mathbb{R}$, we see

$$dF_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(f) = \frac{\partial}{\partial x_i}(f \circ F) \stackrel{*}{=} \sum_{j=1}^n \left.\frac{\partial f}{\partial y_j}\right|_{F(p)} \left.\frac{\partial F_j}{\partial x_i}\right|_p = \left(\sum_{j=1}^n \left.\frac{\partial F_j}{\partial x_i}\right|_p \cdot \left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right)(f),$$

where the main point is the application of the Chain rule in $\stackrel{*}{=}$.

Remark 2.30. Differentials behave under composition. Explicitly, let $F_1\colon M_1 \to M_2$ and $F_2\colon M_2 \to M_3$ be smooth maps. Given $p \in M_1$, we claim that

$$d(F_2 \circ F_1)_p \stackrel{?}{=} (dF_2)_{F_1(p)} \circ (dF_1)_p.$$

This can be checked directly.

Example 2.31. Fix a point p on a smooth n-manifold M. Then we claim $d(\operatorname{id}_M)_p = \operatorname{id}_{T_pM}$. Indeed, we simply compute

$$d(\operatorname{id}_M)_p(v)(f) = v(f \circ \operatorname{id}_M) = v(f).$$

2.3.3 Back to Tangent Spaces

Now that we understand how to take differentials of maps, we may realize the remark that derivations ought to be understood locally, as alluded to in Lemma 2.24.

Proposition 2.32. Fix a smooth n-manifold M. Given an open neighborhood U of a point $p \in M$, the inclusion $i\colon U \hookrightarrow M$ is smooth, and $di_p\colon T_pU \to T_pM$ is an isomorphism of vector spaces.

Proof. Remark 2.28 tells us that this map is linear. It remains to check injectivity and surjectivity, which we do by hand.

- Injective: if $di_p(v) = 0$, then $v(f \circ i) = 0$ for all $f \in C^\infty(M)$, or equivalently, $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We would now like to show that v is actually zero. Well, pick up some $g \in C^\infty(U)$, and we want to show that $v(g) = 0$.

  Well, choose some open precompact open neighborhood B around p such that $\overline{B} \subseteq U$. Then Corollary 2.11 provides us with a smooth bump function $\psi\colon M \to \mathbb{R}$ which is 1 on $\overline{B}$ and vanishes outside U. Then $g\psi$ is actually smooth (it is smooth on U because g and ψ are both smooth there, and it is smooth outside U because the function is zero there), so $v(g\psi|_U) = 0$. But $g\psi$ and g agree on B, so $v(g) = v(g\psi|_U)$ by Lemma 2.24, as needed.

- Surjective: fix some derivation $\widetilde{v} \in T_pM$, and we want some $v \in T_pU$ such that $\widetilde{v}(f) = v(f|_U)$ for all $f \in C^\infty(M)$. The main point is the construction of U.

  Given a smooth function $f \in C^\infty(U)$, we want to define $\widetilde{v}(f)$. Well, as in the previous step, we may define $\widetilde{f}\colon M \to \mathbb{R}$ such that there is an open neighborhood $B \subseteq U$ of p with $f|_B = \widetilde{f}|_B$. Then we define $v(f) \coloneqq \widetilde{v}(\widetilde{f})$. Note that $v(\widetilde{f})$ does not depend on the choice of $\widetilde{f}$ and B: well, given another pair of $\widetilde{f}'$ and B', we see that $\widetilde{f}|_{B\cap B'} = \widetilde{f}|_{B\cap B'}$, so they have the same value of $\widetilde{v}$ under Lemma 2.24.

  Additionally, we note that v is in fact a derivation: given $f, g \in C^\infty(U)$ and smooth extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ agreeing on $B_f, B_g \subseteq M$, respectively, we see

  $$\widetilde{v}(\widetilde{f}\widetilde{g}) = \widetilde{f}(p)\widetilde{v}(\widetilde{g}) + \widetilde{g}(p)\widetilde{v}(\widetilde{f})$$

  because $\widetilde{v}$ is a derivation, but then this immediately produces $v(fg) = f(p)v(g) + g(p)v(f)$ by checking the definitions. Similarly, we have

  $$\widetilde{v}(a\widetilde{f} + b\widetilde{g}) = a\widetilde{v}(\widetilde{g}) + b\widetilde{v}(\widetilde{g}),$$

  so $v(af + bg) = av(f) + bv(g)$, so v is linear.

  Lastly, we note that $\widetilde{v}(f) = v(f|_U)$ for any $f \in C^\infty(M)$ by construction. Namely, f is a perfectly fine extension of $f|_U$ agreeing on some open neighborhood of p contained in U (for example, taking U to be the needed open neighborhood itself will work), so we conclude. ■

Corollary 2.33. Fix a smooth n-manifold M. For any $p \in M$, we have $\dim_{\mathbb{R}} T_pM = n$.

Proof. Fix a smooth chart (U, φ) around $p \in M$. Then we have the sequence of isomorphisms

$$T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U) \cong T_p\mathbb{R}^n \cong \mathbb{R}^n.$$

The first and third isomorphisms are by Proposition 2.32. The second isomorphism is by functoriality of the tangent space from Remark 2.30 and Example 2.31; namely, the differential of a diffeomorphism must be an isomorphism by functoriality. And the last isomorphism is by Lemma 2.25. ■

While we're here, we take a moment to understand how these derivations behave under coordinates.

Remark 2.34. Please read about how to provide the differential of a smooth map on coordinates.

So here are some coordinate computations.

- Fix a smooth n-manifold M and a point $p \in M$. Given a smooth chart (U, φ) around p, we give φ its coordinates $\varphi := (x_1, \ldots, x_n)$. For example, given $f \in C^\infty(U)$, we are able to define

$$\left.\frac{\partial}{\partial x_i}\right|_p f := \left.\frac{\partial f}{\partial \widetilde{x}_i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right),$$

  where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are the coordinates of M. By tracking the isomorphisms of Corollary 2.33 through, we can see that the above derivations form a basis for T_pM. Indeed, it suffices to show that they are a basis for the derivations on T_pU, and by passing through φ, it is enough to see that $\partial f/\partial \widetilde{x}_i|_{\varphi(p)}$ form a basis of derivations on $T_{\varphi(p)}U$. But it's now enough to see that we have a basis on $T_p\mathbb{R}^n$, which is simply Lemma 2.25.

- We examine change of coordinates. Fix a smooth n-manifold M and a point $p \in M$ covered by the charts (U, φ) and (V, ψ). As above, we give coordinates as $\varphi := (x_1, \ldots, x_n)$ and $\psi := (y_1, \ldots, y_n)$, and we give the target spaces the coordinates $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$, respectively.

  Well, on the restrictions, we will choose coordinate representations by

$$(\psi \circ \varphi^{-1})(\widetilde{x}) := (\overline{y}_1(\widetilde{x}), \ldots, \overline{y}_n(\widetilde{x})),$$

  and we in particular see that

$$\begin{aligned}
\left.\frac{\partial}{\partial y_j}\right|_p y_k &= \left(\left(d\psi^{-1}\right)_{\psi(p)} \left.\frac{\partial}{\partial \widetilde{y}_j}\right|_{\psi(p)}\right) y_k \\
&= \left.\frac{\partial}{\partial \widetilde{y}_j}\right|_{\psi(p)} \left(y_k \circ \psi^{-1}\right) \\
&= \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{\psi(p)} \widetilde{y}_k \\
&= 1_{j=k}.
\end{aligned}$$

  The moral of the story is that some $v = \sum_{k=1}^m v_k \partial/\partial y_k|_p$ will have

$$\left.\frac{\partial}{\partial x_i}\right|_p = \sum_{k=1}^n \left.\frac{\partial \overline{y}_k}{\partial \widetilde{x}_i}\right|_{\varphi(p)} \left.\frac{\partial}{\partial y_j}\right|_p.$$

2.4 February 8

Here we go.

2.4.1 Velocity Vectors

Let's discuss a more geometric variant of tangent vectors.

Definition 2.35 (velocity vector)**.** Fix a smooth n-manifold M and a point $p \in M$. Define the space $\mathcal{J}_pM$ to be the set of smooth curves $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ (and $\varepsilon > 0$). We say that $\gamma_1, \gamma_2 \in \mathcal{J}_p$ are equivalent, written $\gamma_1 \sim \gamma_2$, if and only if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any $f \in C^\infty(M)$.

Remark 2.36. We won't bother checking that $\sim$ is an equivalence relation; it holds because we are basically checking equalities after passing to $\mathbb{R}^{C^\infty(M)}$ by sending $\gamma \mapsto ((f \circ \gamma)'(0))_f$.

And here is how this relates to tangent vectors.

Lemma 2.37. Fix a smooth n-manifold M and a point $p \in M$. Then T_pM is in natural bijection with $\mathcal{J}_pM/\sim$.

Proof. In one direction, one can send some $[\gamma] \in (\mathcal{J}_pM/\sim)$ to the derivation $v_{[\gamma]} \colon f \mapsto (f \circ \gamma)'(0)$. Note that this only depends on the class $[\gamma]$ rather than the representative γ by definition of the equivalence relation $\sim$. This map is injective essentially by construction, and one can show by hand that it is surjective, for example by working locally on charts and then using lines as the needed curve to realize a differential in T_pM. ■

2.4.2 The Tangent Bundle

Let's glue our tangent spaces together.

Remark 2.38. Given $p, q \in \mathbb{R}^n$, there is a natural identification $T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. One can see this on velocity vectors by moving the curves over by hand. Alternatively, let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation sending $T \colon p \mapsto q$, which is a diffeomorphism, and then we know we have an isomorphism $dT_p \colon T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. (Recall functoriality of T_p implies that diffeomorphisms produce isomorphisms.)

In general, it is somewhat difficult to identify these tangent spaces naturally.

Definition 2.39 (tangent bundle)**.** Fix a smooth n-manifold M. Then *tangent bundle* TM is

$$TM := \bigsqcup_{p \in M} T_pM.$$

Morally, TM consists of all the tangent spaces glued together.

Proposition 2.40. Fix a smooth n-manifold M. Then TM is a smooth $2n$-manifold.

Proof. We will use Lemma 1.95. Quickly, note that we have a projection $\pi \colon TM \to M$ given by $\pi(p, v) := p$.

Now, for each smooth chart (U, φ) on M, we define the chart $\left(\pi^{-1}U, \widetilde{\varphi}\right)$ on TM, where $\widetilde{\varphi} \colon \pi^{-1}U \to (\operatorname{im}\varphi) \times \mathbb{R}^n$ is defined by

$$\widetilde{\varphi} \colon \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\bigg|_p \mapsto (\varphi(p), (v_1, \ldots, v_n)).$$

Recall $(\partial/\partial x_i)|_p = d\varphi^{-1}_{\varphi(p)}(\partial/\partial \widetilde{x}_i)$, where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are coordinates chosen on is We now have to check our various conditions. For example, $\widetilde{\varphi}$ is a bijection to an open subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{n+1}$ by construction.

(i) Given two (U, φ) and (V, ψ), we need $\widetilde{\varphi}\left(\pi^{-1}U \cap \pi^{-1}V\right)$ to be open in $\mathbb{R}^{2n}$. But this is $\widetilde{\varphi}\left(\pi^{-1}(U \cap V)\right)$, which is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ because $(U \cap V, \varphi|_{U \cap V})$ is a smooth chart on M, so the argument above applies.

(ii) Given two (U, φ) and (V, ψ), we need the composite $\widetilde{\varphi} \circ \widetilde{\psi}^{-1}$ to be smooth, when suitably restricted. Well, one simply commutes the change-of-coordinates for the part on the tangent spaces, and on points, we simply use that $\varphi \circ \psi^{-1}$ is smooth already. Explicitly, one finds that this is

$$(\widetilde{x}, v) \mapsto \left(\left(\varphi \circ \psi^{-1}\right)(\widetilde{x}), \sum_{i=1}^n v_i \frac{\partial \widetilde{y}_\bullet}{\partial \widetilde{x}_i} \frac{\partial}{\partial \widetilde{y}_\bullet}\right).$$

(iii) A countable cover of M by charts produces a countable cover of TM by charts upon pulling back by π.

(iv) Fix distinct $(p, v), (q, w) \in TM$. If $p \neq q$, then we can find disjoint smooth charts (U, φ) and (V, ψ) on M, so $(\pi^{-1}U, \widetilde{\varphi})$ and $\left(\pi^{-1}V, \widetilde{\psi}\right)$ provided the needed disjoint charts. Otherwise, $p = q$, and then p and q are of course contained in the same chart (U, φ), so (p, v) and (q, w) are contained in the same chart $(\pi^{-1}U, \widetilde{\varphi})$. ■

Example 2.41. One has $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Example 2.42. One has $TS^1 = S^1 \times \mathbb{R}$ and $TS^3 = S^3 \times \mathbb{R}^3$ and even $TS^7 = S^7 \times \mathbb{R}^7$.

Example 2.43. For even n, one has $TS^n \neq S^n \times \mathbb{R}^n$, which is essentially a consequence of the Hairy ball theorem: one would be able to produce n linearly independent elements of $S^n \times \mathbb{R}^n$ and then pull them back to n linearly independent vector fields TS^n, which do not exist for even n. The same inequality holds for odd $n \notin \{1, 3, 7\}$.

2.4.3 Maps of Constant Rank

We are going to want some inverse function theorems. Here is the most basic case. Morally, the statement is that invertible derivative should mean locally invertible.

Theorem 2.44 (Inverse function). Fix a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Given $x_0 \in \mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}\mathbb{R}^n \to T_{f(x_0}\mathbb{R}^n$ is invertible, then there is an open neighborhood $U \subseteq \mathbb{R}^n$ around x_0 such that $f|_U$ is a diffeomorphism.

By working on charts, the following result is basically immediate.

Theorem 2.45 (Inverse function). Fix a smooth function $f\colon M \to N$ of n-manifolds. Given $x_0 \in \mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}M \to T_{f(x_0}N$ is invertible, then there is an open neighborhood $U \subseteq M$ around x_0 such that $f|_U$ is a diffeomorphism.

This condition is good enough to make into a definition.

Definition 2.46. Fix a smooth function $F\colon M \to N$ of n-manifolds. Then F is a *local diffeomorphism* at p if and only if dF_p is invertible. Equivalently, by Theorem 2.45, there is an open neighborhood U of p such that $F|_U$ is a diffeomorphism onto its image.

Remark 2.47. Of course, the converse direction (local diffeomorphism implies invertible derivative) is just by functoriality of the tangent space construction.

Remark 2.48. By gluing, if F has invertible derivative at all points, and F is a bijection, then one can see that F^{-1} must be locally a diffeomorphism at all points, so in particular F^{-1} is smooth, so F is fully a diffeomorphism.

Example 2.49. The map $F\colon \mathbb{R} \to S^1$ given by $x \mapsto (\cos x, \sin x)$ is not injective, but it is a local diffeomorphism.

More generally, one could require something weaker than full invertibility.

Definition 2.50 (immersion, submersion, full rank, constant rank)**.** Fix a map $F\colon M \to N$ of smooth manifolds, where $m := \dim M$ and $n := \dim N$.

- F is an *immersion* if and only if dF_p is injective for all $p \in M$.
- F is a *submersion* if and only if dF_p is surjective for all $p \in M$.
- F has *full rank* if and only if $\operatorname{rank} dF_p = \min\{m, n\}$ for all $p \in M$ (notably, this is as large as possible).
- F has *constant rank* if and only if dF_p has the same rank for all $p \in M$ (notably, this is as large as possible).

We now state the following theorem.

Theorem 2.51. Fix a map $F\colon M \to N$ of smooth manifolds. If dF_p has full rank for some $p \in M$, then there is an open neighborhood U of p such that $F|_U$ has full rank.

Proof. The condition that dF_p having full rank is equivalent to the determinant of some largest submatrix being nonzero. So one has a map $M \to \mathbb{R}^N$ for some large N taking $p \in M$ to the list of determinants of these submatrices of dF_p, and this map is continuous, so the set of points not going to zero is open and contains p. ■

Example 2.52. Fix two manifolds M and N, and fix some $y_0 \in N$.

- The map $x \mapsto (x, y_0)$ is an immersion.
- The projection map $M \times N \to M$ is a submersion.

Example 2.53. Fix a smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ with non-vanishing derivative everywhere. Then γ is an immersion.

2.5 February 13

Here we go.

2.5.1 The Rank Theorem

It will turn out that maps of constant rank basically look like projections.

Example 2.54. The projection $F\colon \mathbb{R}^2 \to \mathbb{R}$ given by $F\colon (x, y) \mapsto x$ is a submersion. Namely, $dF = (1, 0)$ for each p, so $\operatorname{rank} dF_p = 1$ for all p.

Our result will arise from some change of basis.

Proposition 2.55. Fix a linear map $L\colon V \to W$ of finite-dimensional $\mathbb{R}$-vector spaces of rank r. Then there is a basis of V and a basis of W such that L has matrix representation given by

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I is an $r \times r$ identity matrix.

Proof. Put any given matrix L in row-reduced Echelon form and then move the columns around as needed. Row and column operations correspond to adjusting bases of V and W. ■

So here is our result.

Theorem 2.56 (Constant rank)**.** Fix a smooth m-manifold M and a smooth n-manifold N, and fix a smooth map $F\colon M \to N$ of constant rank r. For each p, there are smooth coordinate charts (U, φ) on M and (V, ψ) such that $p \in U$, $F(U) \subseteq V$, and F has a coordinate representation given by

$$F(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

Proof. Smoothness allows us to choose some coordinate representation, so we may assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. In our choice of coordinate representation, we may also assume that $p = 0 \in \mathbb{R}^m$ and $F(p) = 0 \in \mathbb{R}^n$. We are basically trying to "straighten out" F around p.

The name of the game is to find a diffeomorphism φ on an open neighborhood $U \subseteq \mathbb{R}^m$ of 0 and a diffeomorphism ψ on an open neighborhood $V \subseteq \mathbb{R}^n$ of 0 such that

$$\psi \circ F \circ \varphi^{-1}$$

is going to look as in the statement. We proceed in steps.

1. Using change-of-basis isomorphisms $A\colon \mathbb{R}^m \to \mathbb{R}^m$ and $B\colon \mathbb{R}^n \to \mathbb{R}^n$ so that $d(B \circ F \circ A)_0 = dB_0 \circ dF_0 \circ dF_0$ now looks like
$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$
(We are using Proposition 2.55 to find A and B.) The point is that F looks how we want locally at 0.

2. We apply the Inverse function theorem to straighten out the first r coordinates. While we're here, we establish our coordinate as follows: given the domain of F the coordinates $(x_1, \ldots, x_r, y_1, \ldots, y_{m-r})$, and give the codomain of F the coordinates $(x_1', \ldots, x_r', y_1', \ldots, y_{n-r}')$. Under these coordinates, say F is $F(x, y) = (Q(x, y), R(x, y))$.
To straighten out Q, we set $\varphi(x, y) \coloneqq (Q(x, y), y)$. We would like for φ to be a diffeomorphism local at 0, which we can compute as $\mathrm{id}_{\mathbb{R}^m}$: on the first r coordinates, we are $Q(x, y)$, which is I_m locally, and on the last $n - r$ coordinates, we are y, which continues to be the identity. Thus, φ is in fact locally a diffeomorphism on some open neighborhood U of 0. So we may compute
$$\left(F \circ \varphi^{-1}\right)(x, y) = \left(x, (R \circ \varphi^{-1})(x, y)\right).$$

3. We remove the dependence of $F \circ \varphi^{-1}$ on y. Computing our current differential, we get
$$d(F \circ \varphi^{-1})_{(x,y)} = \begin{bmatrix} I_r & 0 \\ \frac{\partial(R \circ \varphi^{-1})}{\partial x} & \frac{\partial(R \circ \varphi^{-1})}{\partial y} \end{bmatrix}.$$
However, for F to have constant rank r, we see that we must have $\frac{\partial(R \circ \varphi^{-1})}{\partial y} = 0$; in other words, this composite does not depend on y. (In other words, it is constant with respect to y.) So we set $S(x) \coloneqq (R \circ \varphi^{-1})(x, y)$. So we now have
$$\left(F \circ \varphi^{-1}\right)(x, y) = (x, S(x)).$$

4. We straighten out the remaining $n - r$ coefficients using the Inverse function theorem. Namely, define $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ by
$$\psi(x', y') \coloneqq (x', S(x') - y').$$
Computing the differential at 0 shows that ψ is locally a diffeomorphism, so we may use it as a chart. We now conclude by computing $\left(\psi \circ F \circ \varphi^{-1}\right)(x, y) = (x, 0)$, as required. ■

Remark 2.57. Please read the Global rank theorem.

2.5.2 Embeddings

Here is our definition.

Definition 2.58 (embedding)**.** Fix smooth manifolds M and N. A smooth map $F: M \to N$ is an *embedding* if and only if F is an injective immersion and a homeomorphism onto its image.

Remark 2.59. The image of a smooth map does not necessarily make sense as a smooth manifold, which is why we are only requiring a homeomorphism onto the image instead of a diffeomorphism onto its image.

Here is how one might check this.

Lemma 2.60. Fix a smooth map $F: M \to N$. Then F is an embedding if and only if F is an injective immersion, and given any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ and $x \in M$ such that $Fx_n \to Fx$ as $n \to \infty$, we have $x_n \to x$ as $n \to \infty$.

Proof. The forward direction is clear because the inverse homeomorphism must take convergent sequences to convergent sequences. The reverse direction amounts to checking the continuity of F^{-1}, which is basically what the condition says on sequences. ■

Example 2.61. Fix smooth manifolds M and N. For $p \in N$, the inclusion map $M \times \{p\} \to M \times N$ is an embedding.

Non-Example 2.62. Any curve $\gamma: [0,1] \to \mathbb{R}^n$ with self-intersection fails to be injective, so γ fails to be an embedding.

Non-Example 2.63. Consider the map $\gamma: [0, 2\pi) \to \mathbb{R}^2$ by $\gamma(x) := (\cos x, \sin x)$. Then as $x \to 2\pi$, we have $\gamma(x) \to \gamma(0)$, which contradicts Lemma 2.60.

Non-Example 2.64. Consider the map $F: \mathbb{R}^+ \to \mathbb{R}^2$ by $F(t) := (t, \sin 1/t)$. One can see that F is in fact an embedding, but if we add in some $(-1,1) \to \mathbb{R}^2$ by $s \mapsto (0, s)$, then $F: (\mathbb{R}^+ \sqcup (-1,1)) \to \mathbb{R}^2$ is no longer an embedding. The point is that there are points in $\operatorname{im} F$ converging to $\{0\} \times (-1,1)$, but this is bad news because points in $\mathbb{R}^+$ are not going to converge to $(-1,1)$.

Non-Example 2.65. Fix $T^2 := S^1 \times S^1$, and realize S^1 as $\mathbb{R}/\mathbb{Z}$. Then $F: \mathbb{R} \to T^2$ defined by $t \mapsto (\alpha t, \beta t)$ for $\alpha, \beta \in \mathbb{R}^\times$ is never an embedding.

- If $\alpha/\beta \in \mathbb{Q}$, then one can see that F is periodic, so it fails to be injective. Namely, if $\beta = (r/s)\alpha$, then $F(st)$.
- When $\alpha/\beta \notin \mathbb{Q}$, some Diophantine approximation implies that $\operatorname{im} F$ is dense in T^2, so it cannot be an embedding.

Non-Example 2.66. Consider $F: \mathbb{R} \to \mathbb{R}$ by $F(t) := t^3$. Then F does not have constant rank, so F is not an embedding.

Compactness makes many of these pathologies disappear.

Proposition 2.67. Fix an injective immersion $F\colon M \to N$ of smooth manifolds. Then F is an embedding.

Proof. We need to show that F is a homeomorphism onto its image. Because F is a continuous injection, it suffices to show that the map $F\colon M \to \operatorname{im} F$ is an open map, for which it suffices to show that it is actually a closed map. Well, any closed subset $V \subseteq M$ is compact because M is compact, so $F(V)$ is compact, so $F(V) \subseteq \operatorname{im} F$ is closed because $\operatorname{im} F \subseteq N$ is Hausdorff. ■

Similarly, looking locally makes many of these pathologies disappear.

Proposition 2.68. Fix an immersion $F\colon M \to N$. Given $p \in M$, there is an open neighborhood U of p such that $F|_U$ is an embedding.

Proof. This follows somewhat quickly from Theorem 2.56. ■

Remark 2.69. If $\dim M = \dim N$, then the above result follows rather quickly from the Inverse function theorem.

Remark 2.70. Please read about submersions and smooth covering maps.

2.5.3 Submanifolds

Our naïve definition is simply that we are a subset with inherited smooth structure.

Definition 2.71 (embedded smooth submanifolds)**.** Fix a smooth manifold M. Then a subspace $S \subseteq M$ is an *embedded smooth submanifold* if and only if S is a manifold with smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth embedding. In other words, we are asking that S is the image of a smooth embedding $F\colon N \to M$.

Example 2.72. Fix an open subset $S \subseteq M$. Then the inclusion $S \hookrightarrow M$ is of course an embedding, so S is a submanifold.

Example 2.73. Fix a countable discrete set of points $S \subseteq M$. Then the inclusion $S \hookrightarrow M$ is smooth of rank 0.

2.6 February 15

The midterm is in two weeks.

2.6.1 Proper Embeddings

The following notion will be useful.

Definition 2.74. An embedded smooth submanifold $S \subseteq M$ is *properly embedded* if and only if the inclusion $S \hookrightarrow M$ is proper; i.e., the inverse image of a compact subset of M is still compact in S.

Non-Example 2.75. There is an embedding $\mathbb{R}^2 \to S^2$ by inverting the stereographic projection map $\left(S^2 \setminus \{(0,0,1)\}\right) \to \mathbb{R}^2$. However, this is not proper: all of S^2 is compact, but its pre-image in $\mathbb{R}^2$ is all of $\mathbb{R}^2$, which is not compact.

Here is a nice way to check properness.

Proposition 2.76. Fix an embedded smooth submanifold $S \subseteq M$. Then S is properly embedded if and only if $S \subseteq M$ is closed.

Proof. We have two directions to show.

- Suppose $S \subseteq M$ is closed. Well, for any compact subset $K \subseteq M$, we see that $S \cap K$ is closed in M (it is the intersection of two closed subsets of M), so $S \cap K$ is a closed subset of the compact set K, so $S \cap K$ continues to be compact.
- Suppose $S \subseteq M$ is properly embedded. Then we want to show that $S \subseteq M$ is closed. Well, it suffices to check that S contains all of its limit points, so suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of points in S which converges to some point $x \in M$; then we want to show that $x \in S$.

  Well, we note that the subset $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact (any open cover has an open neighborhood of x, and this open neighborhood has all but finitely many of the x_ns), so $(\{x_n : n \in \mathbb{N}\} \cup \{x\}) \cap S$ continues to be compact by the proper embedding. But if $x \notin S$, then $\{x_n : n \in \mathbb{N}\}$ fails to be compact, so instead we must have $x \in S$. ■

2.6.2 Slice Charts

Here is our definition.

Definition 2.77 (slice). Fix a smooth n-manifold and a smooth chart (U, φ), where we give φ the coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then a *$k$-slice of* (U, φ) is the slice

$$S(c_{k+1}, \ldots, c_n) \coloneqq \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k\}.$$

Conversely, a chart (U, φ) is a *k-slice chart* for a given subset $S \subseteq U$ if and only if $S = S(c_{k+1}, \ldots, c_n)$ for some real numbers $(c_{k+1}, \ldots, c_n)$. Then a subset $S \subseteq M$ satisfies the *local k-slice condition* if and only if any $p \in S$ has a smooth chart (U, φ) around p such that (U, φ) is a k-slice chart for $S \cap U$.

Example 2.78. Fix a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$, and define the graph

$$\Gamma(f) \coloneqq \left\{(x, f(x) \in \mathbb{R}^{m+n} : x \in \mathbb{R}^m\right\}.$$

Then $\Gamma(f) \subseteq \mathbb{R}^{m+n}$ is a (global) m-slice chart for the chart $(\mathbb{R}^{m+n}, \varphi)$, where φ is the map $\varphi(x, y) \coloneqq (x, y - f(x))$. (Note that φ is of course smooth, and it has smooth inverse given by $(x, y) \mapsto (x, y + f(x))$.) Namely,

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : \varphi(x, y) = (x, 0)\},$$

so we are indeed a slice chart.

Here is our theorem. Approximately, we are saying embedded submanifolds locally look like slices.

Theorem 2.79 (Slice). Fix a smooth n-manifold M. A subset $S \subseteq M$ is an embedded k-dimensional submanifold if and only if S satisfies the local k-slice condition.

Proof. We have two implications to show, which we do separately.

- Suppose that S is an embedded k-dimensional submanifold of M, and let $F\colon S \to M$ to be the embedding. We need to show that S satisfies the local k-slice condition. Well, fix some $p \in S$, and we need a k-slice chart (U, φ) around $p \in U$. For this, we use Theorem 2.56, which provides us with smooth charts (U, φ) and (V, ψ) around $p \in S$ and $F(p) \in M$, respectively, such F has a coordinate representation given by
$$\widehat{F}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0),$$
where $\widehat{F} := \psi \circ F \circ \varphi^{-1}$.

  We are almost done, except for a technicality that V might contain other parts of S. For brevity, let $\widehat{U} := \varphi(U)$ and $\widehat{V} := \psi(V)$ to be subsets of Euclidean space; notably, $\widehat{F}(\widehat{U}) = \widehat{U} \times \{0\}$. To begin our restriction, set $\widehat{V}' := \widehat{V} \cap \left(\widehat{U} \times \mathbb{R}^{n-k}\right)$ and $V' := \psi^{-1}(\widehat{V}')$, so we are excluding points of S not in U which are near p. To exclude points not near p, note we can write $U = U' \cap S$ where $U' \subseteq M$ is open, so we define
$$V'' := V' \cap U'.$$
We set $\psi'' := \psi|_{V''}$.

  We now claim that (V'', ψ'') is the needed local k-slice chart of S around p. Indeed, we claim that
$$S \cap V'' \stackrel{?}{=} \{q \in V'' : \psi''_\ell(q) = 0 \text{ for } \ell > k\}.$$
In one direction, $q \in V'' \cap S$ implies $q \in U$ by construction, but then $\psi''(q) = \psi(q) \in \mathbb{R}^{n-k} \times \{0\}$ by definition of ψ. In the other direction, if $q \in V''$ has $\psi''_\ell(q) = 0$ for $\ell > k$, then (for example) $\psi(q) \in \widehat{U} \times \mathbb{R}^{n-k}$ because that is where V' goes to, so actually $\psi(p) \in \widehat{U} \times \{0\} = \widehat{F}(\widehat{U})$, so $p \in \varphi^{-1}(\widehat{U})$ by undoing $\widehat{F}$, so $p \in S$ by definition.

- Suppose that S satisfies the local k-slice condition. Then we want to give a smooth structure to S so that the inclusion makes S into a smooth embedded submanifold. Well, give $S \subseteq M$ the subspace topology; then this makes S a homeomorphism onto its image automatically, so notably S is Hausdorff and second countable.

  It remains to give S some smooth charts. Well, fix some $p \in S$, and satisfying the k-slice chart condition promises us a chart (U, φ) around p so that
$$S \cap U = \{q \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k\}$$
for some given real numbers $c_{k+1}, \ldots, c_n$. These last $(n-k)$ coordinates shouldn't matter, so we let $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ denote the projection onto the first k coordinates. As such, we set $V := U \cap S$ and $\widehat{V} := (\pi \circ \varphi)(V)$, which is an open subset of
$$\varphi(U) \cap \{x \in \mathbb{R}^n : x_\ell = c_\ell \text{ for } \ell > k\}.$$
The above is open in the subspace defined by the plane at the right, so it is open when projected down to π, which can be checked because π is a quotient map.

  So we will let $(V, \pi \circ \varphi)$ become the relevant chart. For example, we can check that $\pi \circ \varphi$ is a homeomorphism: indeed, its inverse map is given by $\varphi^{-1} \circ j$, where $j(x_1, \ldots, x_k) := (x_1, \ldots, x_k, c_{k+1}, \ldots, c_n)$, and φ^{-1} and j are both smooth. This concludes the proof that S is a topological k-manifold.

  We now check smooth compatibility of the given charts to show that we have actually given S a smooth structure. Well, choose two charts (V, ψ) and (V', ψ') of S which are constructed as above from charts (U, φ) and (U', φ') of M. Well, the transition map $\psi' \circ \psi^{-1}$ is given by
$$\pi' \circ \varphi' \circ \varphi^{-1} \circ j,$$
where j and π and j' and π' are given as above. This transition map is smooth because it is the composition of smooth maps.

  Lastly, we must check that the embedding $S \to M$ is smooth. Well, for any $p \in S$, choose a smooth chart (V, ψ) arising from the smooth chart (U, φ) on M, as constructed above. Then the inclusion $F\colon S \subseteq M$ sends $V \subseteq U$, and the composite $\varphi \circ F \circ \psi^{-1}$ is just the identity, so it is smooth. ■

Here is a consequence of the above proof.

Corollary 2.80. Fix a smooth n-manifold M, and let $S \subseteq M$ be a smooth embedded submanifold. Then for any k-slice chart (U, φ) of S, one finds that $(U \cap S, (\varphi_1, \ldots, \varphi_k))$ where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a coordinate expansion.

Proof. The second part of the proof of Theorem 2.79 establishes this. ■

2.6.3 Level Sets

A common way to build embedded submanifolds is via level sets. Let's begin with a couple examples.

Example 2.81. Consider the smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 + y^2$. For example, $f^{-1}(\{1\}) = S^1$ and $f^{-1}(\{0\}) = \{(0,0)\}$ and $f^{-1}(\{-1\}) = \varnothing$.

Example 2.82. Consider the smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 - y^2$. Then $f^{-1}(\{1\})$ is a hyperbola with two connected components, but $f^{-1}(\{0\})$ looks like two crossing lines.

Remark 2.83. Given any closed set $A \subseteq M$, we remarked earlier that there is a smooth function $f\colon M \to \mathbb{R}$ such that $f(\{0\}) = A$. So it cannot be the case that level sets always give nice submanifolds.

We do expect that we should get a submanifold "generically." Here is one instance of this.

Theorem 2.84. Fix a smooth map $F\colon M \to N$ of constant rank r between the m-manifold M and n-manifold N. Then any $q \in \operatorname{im} F$ makes the level set $F^{-1}(\{q\})$ is a proper embedded submanifold of M of dimension $(m - r)$.

Morally, the dimensions of M must go somewhere, and there are r dimensions going out into N.

Example 2.85. Consider the smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 - y^2$. Then

$$df_{(x,y)} = \begin{bmatrix} 2x & -2y \end{bmatrix},$$

so the function $f|_{\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})}$ is a smooth map of constant rank 1, so Theorem 2.84 tells us that all of its fibers will be proper submanifolds of M of dimension $2 - 1 = 1$.

2.7 February 20

The fun never ends.

2.7.1 More on Level Sets

Last class we were stated the following result.

Theorem 2.84. Fix a smooth map $F\colon M \to N$ of constant rank r between the m-manifold M and n-manifold N. Then any $q \in \operatorname{im} F$ makes the level set $F^{-1}(\{q\})$ is a proper embedded submanifold of M of dimension $(m - r)$.

Proof. We apply Theorem 2.79 to $S := F^{-1}(\{q\})$, for which we will use Theorem 2.56. For each $p_0 \in S$, we receive smooth charts (U, φ) on M (around p_0) and (V, ψ) on N (with $F(U) \subseteq V$) such that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

In particular, write $\psi(q) = (c_1, \ldots, c_r, 0, \ldots, 0)$, and we see that

$$S \cap U = \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell \leq r\},$$

which is in fact an $(m-r)$-slice. Thus, Theorem 2.79 applies, and to finish up, we note that $S \subseteq M$ is certainly closed and hence proper by Proposition 2.76. ■

Example 2.86. If $F\colon M \to N$ is a submersion, then $F^{-1}(\{q\})$ is a proper embedded submanifold of dimension $(\dim M) - (\dim N)$ for any $q \in N$.

2.7.2 Regularity

We will want to understand the differential of a smooth map pointwise, for which we provide some language.

Definition 2.87 (regular, critical)**.** Fix a smooth map $F\colon M \to N$.

- A point $p \in M$ is *regular* if and only if dF_p is surjective; otherwise, $p \in M$ is *critical*.
- A value $q \in N$ is *regular* if and only if all points in $F^{-1}(\{q\})$ are regular; otherwise, $q \in N$ is *critical*.

Example 2.88. Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we see that the point $x_0 \in \mathbb{R}$ is regular if and only if $f'(x_0) = 0$, based on some Jacobian computation reducing $T_{x_0} f$ to $\frac{d}{dx} f\big|_{x_0}$. Accordingly, the critical values are exactly when some point in the fiber is critical.

Example 2.89. Continuing from Example 2.85, we see that the regular points of $\mathbb{R}^2$ are just $\mathbb{R}^2 \setminus \{0\}$, so the collection of regular values is $\mathbb{R} \setminus \{0\}$, which has pre-image $\mathbb{R}^2 \setminus \{(x, y) : xy = 0\}$.

It will turn out that the set of critical values will always be small (namely, measure zero).

Remark 2.90. Note that the set of regular points M' of M is open: the map sending $p \in M$ to the ordered list of determinants of the largest square minors of M is continuous by checking on charts (where this function is a polynomial), and being regular means that we are interested in the pre-image where at least one coordinate is nonzero. Thus, so $F|_{M'}\colon M' \to N$ will be a submersion provided that there is some regular input to F.

Anyway, we get the following result.

Proposition 2.91. Fix a smooth map $F\colon M \to N$ from the m-manifold M to the n-manifold N, and let $q \in N$ be a regular value. Then $F^{-1}(\{q\})$ is a proper embedded submanifold of dimension $m - n$.

Proof. Let $U \subseteq M$ be the set of regular points in M, which is nonempty because N has a regular value; in particular, $F^{-1}(\{q\}) \subseteq U$. Now, $F|_U\colon U \to N$ is a submersion by the regularity of each $p \in U$, so Example 2.86 tells us that $F^{-1}(\{q\}) \subseteq M$ is an embedded submanifold of dimension $m - n$. Lastly, $F^{-1}(\{q\})$ is still proper by Proposition 2.76 because it is closed. ■

Example 2.92. Define $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ by $F(p) := |x|^2$. Then $S^n = F^{-1}(\{1\})$ will be a proper embedded submanifold of dimension n by Proposition 2.91. Indeed, it is enough to check that $1 \in \mathbb{R}$ is a regular value of F. Well, for $p = (x_0, \ldots, x_n) \in F$, we can compute dF_p as the Jacobian matrix

$$\begin{bmatrix} 2x_0 & \cdots & 2x_n \end{bmatrix}.$$

Notably, this has full rank 1 unless $p = 0$, and $F^{-1}(\{1\}) \cap \{0\} = \varnothing$, so we are safe.

Example 2.93. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) := \left(x^2 + y^2 - 1\right)^2$. Then for $p = (x_0, y_0)$, we can compute dF_p as the Jacobian matrix

$$\begin{bmatrix} 4x_0\left(x_0^2 + x_1^2 - 1\right) & 4x_0\left(x_0^2 + x_1^2 - 1\right) \end{bmatrix},$$

so $S^1 \subseteq \mathbb{R}^2$ now contains entirely critical points even though $S^1 = F^{-1}(\{1\})$ is a perfectly fine smooth embedded submanifold of dimension 1.

Example 2.94. Consider the torus $T^2 := S^1 \times S^1$, and define $F\colon T^2 \to \mathbb{R}$ by some kind of height function, achieved by embedding $T^2 \subseteq \mathbb{R}^3$. Then the pre-image of the critical values of this height make figure-8s, which are not smooth embedded submanifolds.

These regular values also allow us to sensibly discuss defining functions.

Definition 2.95 (defining function)**.** Fix a smooth embedded submanifold $S \subseteq M$, where $k := \dim S$ and $m := \dim M$. Then a smooth function $F\colon M \to N$ is a *defining function for S* if and only if $S = F^{-1}(\{q\})$ for some regular value $q \in N$. Locally over some open subset $U \subseteq M$, we say that a smooth map $F\colon M \to N$ is a *local defining function for F at some $p \in S$* if and only if $S \cap U = F^{-1}(\{q\})$ for some regular value $q \in N$.

The local notion is useful because it is universal.

Proposition 2.96. Fix a subset S of a smooth m-manifold M. Then S is a k-dimensional embedded submanifold of M if and only if any $p \in S$ has some open neighborhood $U \subseteq M$ of p such that there is a local defining function $F\colon U \to \mathbb{R}^{m-k}$ for any $p \in S$.

Sketch. Use Theorem 2.79 to realize F as a projection onto the relevant coordinates. ■

2.7.3 Tangent Vectors

Embedded submanifolds will produce a natural embedding on tangent spaces, which we now use.

Definition 2.97 (tangent vector)**.** Fix an embedded k-submanifold S of the smooth m-manifold M. For any $p \in S$, we define $T_p^{\text{extrinsic}} S := \operatorname{im} d\iota_p$, where $\iota\colon S \to M$ is the inclusion. Namely, we are viewing T_pS as a k-dimensional subspace of T_pM.

Example 2.98. Let (U, φ) be a local k-slice chart for S so that

$$S \cap U = \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k\}.$$

Then we see $T_p^{\text{extrinsic}}$ is just the span of $\left\{ \frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_k}\big|_p \right\}$.

Example 2.99. Let $F\colon U \to N$ be a local defining function for S so that $U \cap S = F^{-1}(\{q\})$ for some regular value $q \in N$. Then $T_p^{\text{extrinsic}} S = \ker dF_p$ by tracking through what being a defining function means.

Example 2.100. Consider the subset

$$\mathrm{O}(n) := \left\{ A \in \mathbb{R}^{n\times n} : A^\intercal A = I_m \right\}.$$

We have a natural defining map $F\colon \mathbb{R}^{n\times n} \to \mathrm{Sym}(n)$ by $A \mapsto A^\intercal A$, and F is certainly smooth because it is a polynomial in the coordinates. We claim that $I_m \in \mathrm{Sym}(n)$ is a regular value for F, which implies $\mathrm{O}(n) \subseteq \mathbb{R}^{n\times n}$ is a smooth embedded submanifold of codimension 1 by Proposition 2.91.

Well, we compute dF_A for $A \in \mathbb{R}^{n\times n}$ via curves. A curve producing the differential $B \in T_A\mathbb{R}^{n\times n}$ is simply given by $t \mapsto A + tB$, so

$$dF_A(B) = \frac{d}{dt}F(A+tB)\bigg|_{t=0} = \frac{d}{dt}(A^\intercal + tB^\intercal)(A+tB)\bigg|_{t=0} = \frac{d}{dt}\left(A^\intercal A + t(B^\intercal A + A^\intercal B) + t^2 B^\intercal B\right)\bigg|_{t=0},$$

which is $B^\intercal A + A^\intercal B$. So we need the map $B \mapsto B^\intercal A + A^\intercal B$ to be surjective, so we will just check that it has kernel of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. Well, B lives in the kernel if and only if $B^\intercal A = -A^\intercal B$, or equivalently $A^\intercal B$ is alternating. Taking A to be invertible, we are looking at A times the space of alternating matrices, which is in fact of dimension $\frac{1}{2}n(n-1)$.

Remark 2.101. While we're here, we note that we have already computed $T_{I_m}\mathrm{O}(n)$ extrinsically as

$$\ker dF_A = \left\{ B \in \mathbb{R}^{n\times n} : B^\intercal + B = 0 \right\},$$

which we will later understand as the Lie alegbra.

2.8 February 22

The midterm is next week. It will be about four questions. More information will be sent out soon.

2.8.1 Null Sets

Sard's theorem will tell us that most values are regular values. In particular, we will show that critical values have measure zero. The notion of measure zero will be glued together from charts.

Definition 2.102 (null set)**.** A subset $A \subseteq \mathbb{R}^n$ has *measure zero* or is a *null set* if and only if any $\varepsilon > 0$ has some countable list of balls $\{B(x_i, r_i)\}_{i\geq 1}$ such that

$$A \subseteq \bigcup_{i\geq 1} B(x_i, r_i) \qquad \text{and} \qquad \sum_{i=1}^{\infty} r_i^n < \varepsilon.$$

Example 2.103. According to the above definition, any countable subset is a null set, even if we are in $\mathbb{R}^0$.

The point of the r_i^n is that it is the volume of $B(x_i, r_i)$, up to a constant only depending on the dimension, so we are saying that A can be covered by sets of arbitrarily small measure.

Remark 2.104. We can replace the balls in this definition with cubes.

Here are some quick checks.

Lemma 2.105. Fix a positive integer n.

(a) If $A \subseteq \mathbb{R}^n$ is a null set and $B \subseteq A$, then B is a null set.

(b) If $\{A_j\}_{j\geq 1}$ is a countable collection of null sets, then $\bigcup_{i=1}^\infty A_i$ is a null set.

(c) If $A \subseteq \mathbb{R}^n$ makes $A \cap \left(\{c\} \times \mathbb{R}^{n-1}\right) \subseteq \{c\} \times \mathbb{R}^{n-1}$ into a null set for each $c \in \mathbb{R}$, then A is a null set.

(d) If $f: U \to \mathbb{R}$ is a continuous function with $U \subseteq \mathbb{R}^{n-1}$ measurable, then the graph

$$\Gamma(f) \coloneqq \{(x, f(x)) : x \in U\}$$

is a null set.

(e) Every nontrivial affine subspace of $\mathbb{R}^n$ (not equal to $\mathbb{R}^n$) is a null set.

(f) If $A \subseteq \mathbb{R}^n$ is a null set, then $\mathbb{R}^n \setminus A$ is dense.

(g) A subset $A \subseteq \mathbb{R}^n$ is a null set if and only if each $p \in A$ has some open neighborhood $U_p \subseteq \mathbb{R}^n$ such that $A \cap U_p$ is a null set.

(h) If a subset $A \subseteq \mathbb{R}^n$ is a null set, and a function $F\colon A \to \mathbb{R}^n$ is Lipschitz, then $F(A)$ is a null set.

(i) Let $S \subseteq \mathbb{R}^n$ be a submanifold of positive codimension. Then S has measure zero.

Proof. Here we go.

(a) For any $\varepsilon > 0$, we get a countable list $\{B(x_i, r_i)\}_{i\geq 1}$ such that

$$A \subseteq \bigcup_{i\geq 1} B(x_i, r_i) \qquad \text{and} \qquad \sum_{i=1}^\infty r_i^n < \varepsilon.$$

Thus, we see that $B \subseteq A \subseteq \bigcup_{i\geq 1} B(x_I, r_i)$ too, so we are done.

(b) Fix $\varepsilon > 0$. For each j, build a countable list $\{B(x_{ij}, r_{ij})\}_{i\geq 1}$ such that

$$A_j \subseteq \bigcup_{i\geq 1} B(x_{ij}, r_{ij}) \qquad \text{and} \qquad \sum_{i=1}^\infty r_{ij}^n < \frac{\varepsilon}{2^j}.$$

Now, set $\mathcal{B} \coloneqq \{B(x_{ij}, r_{ij})\}_{i,j\geq 1}$ to be a countable union of balls, and we see that $\bigcup_{j\geq 1} A_j$ is contained in $\bigcup_{j\geq 1}\bigcup_{i\geq 1} B(x_{ij}, r_{ij})$, and

$$\sum_{j\geq 1}\sum_{i\geq 1} r_{ij}^n < \sum_{j\geq 1} \frac{\varepsilon}{2^j} = \varepsilon,$$

as required.

(c) Use Fubini's theorem, integrating over $c \in \mathbb{R}$. Explicitly, now using some heavier measure theory,

$$\mu(A) = \int_{\mathbb{R}^n} 1_A(x)\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} 1_A(c, x)\, dx\, dc = \int_{\mathbb{R}} 0\, dc = 0.$$

(d) We induct on n. If $n = 1$, there is nothing to do because $\Gamma(f)$ is a single point. For the induction, we use the previous part: it is enough to check that $\Gamma(f) \cap \left(\{c\} \times \mathbb{R}^{n-1}\right) \subseteq \{c\} \times \mathbb{R}^{n-1}$ has measure zero. But this amounts to restricting f to $\{c\} \times \mathbb{R}^{n-1}$, so this intersection is now the graph of a continuous function in $n-2$ variables whose graph lives in $\mathbb{R}^{n-1}$. So our dimension is one smaller, so we complete the induction.

(e) Apply the previous part because an affine subspace is the image of a linear map composed with a translation.

(f) If $\mathbb{R}^n \setminus A$ fails to be dense, then there is an open subset in the complement of $\mathbb{R}^n \setminus A$, so A contains an open ball, so A cannot be a null set.

(g) The forward direction is immediate by taking subsets. In the reverse direction, we loop over all $p \in \mathbb{R}^n$ produces an open $\{U_p\}_{p\in\mathbb{R}^n}$ cover of $\mathbb{R}^n$. However, $\mathbb{R}^n$ is countably compact, so we can refine this to a countable cover $\{U_p\}_{p\in S}$ of $\mathbb{R}^n$, where $S \subseteq \mathbb{R}^n$ is some finite subset. Thus, A is the union of the countably many null sets $\{U_p \cap A\}_{p\in S}$, so we are done by a previous part.

(h) For $\varepsilon > 0$, cover A with open balls of measure smaller than ε; by shrinking the balls if necessary, we may assume that F has a smooth extension to the (compact!) closure of each ball. Thus, F becomes Lipschitz on each ball with a Lipschitz constant of (say) K,[1] so passing the open balls through F will have image contained in an open ball with K times the radius. So we have bounded the measure of A by $K^n\varepsilon$, up to some constants, which vanishes as $\varepsilon \to 0^+$.

(i) Use k-slice charts to realize S^k locally as a slice chart, which have measure zero. Notably, if $A \subseteq U$ is a null set where $U \subseteq \mathbb{R}^n$ is open, and $\varphi\colon U \to \widehat{U}$ is a diffeomorphism to some other $\widehat{U} \subseteq \mathbb{R}^n$, then $\varphi(A)$ continues to be a null set by using (g) to allow us to check locally and then note that diffeomorphisms are locally Lipschitz by taking the Lipschitz constant to be the norm of the Jacobian matrix. ■

And now let's glue.

Definition 2.106 (null set)**.** Let M be a smooth n-manifold. Then a subset $A \subseteq M$ is a *null set* if and only if any smooth chart (U, φ) of M makes $\varphi(A \cap U) \subseteq \mathbb{R}^n$ into a null set.

Remark 2.107. We remark that one can check that A is a null set on a particular choice of smooth charts: suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ is a collection of smooth charts covering A for which $\varphi_\alpha(A\cap U_\alpha)$ is a null set. Then we must check that A is a null set. Well, pick up any new chart (U, φ), and we want to check that $\varphi(U \cap A)$ is a null set. Any open cover of A can be refined with a countable subcover, so we may replace our cover with a countable one $\{(U_i, \varphi_i)\}_{i\geq 1}$. Then $\varphi(U\cap A)$ is the countable union of the $\varphi(U\cap U_i\cap A)$s, so it is enough to check that these are null sets. But then

$$\varphi(U \cap U_i \cap A) = \left(\varphi \circ \varphi_i^{-1}\right)^{-1} (\varphi_i(U \cap U_i \cap A))$$

is the image of a null set along a smooth map (of Euclidean spaces), which is a null set by Lemma 2.105.

Remark 2.108. If $A \subseteq \mathbb{R}^n$ is a null set, then actually A has measure zero where we view $\mathbb{R}^n$ as an n-manifold. The backward direction is clear because $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a smooth chart; the forward direction follows because having measure zero is a diffeomorphism invariant as argued in Lemma 2.105.

Remark 2.109. The image of a null set $A \subseteq M$ along a smooth map $F\colon M \to N$ continues to be a null set. Indeed, for each $p \in M$, choose charts (U, φ) of p and (V, ψ) of $F(p)$ so that $F(U) \subseteq V$. Then we want to check that $\psi(F(A) \cap V)$ is a null set, where we know that $\varphi(A \cap U)$ is a null set. Well,

$$\psi(F(A) \cap V) = \psi(F(A \cap U)) = \left(\psi \circ F \circ \varphi^{-1}\right) (\varphi(A \cap U))$$

is the image of a null set along a smooth map (of Euclidean spaces) and hence a null set by Lemma 2.105.

[1] For example, one can use some sort of multivariable mean value theorem on passing through a norm.

Remark 2.110. We still want to know that any null set $A \subseteq M$ is small. Concretely, we check that $M \setminus A$ is dense in M. Well, choose any open subset $U \subseteq M$, and we want to show that $U \setminus A$ has a point. By shrinking U if necessary, we may suppose that (U, φ) is a smooth chart, so we are told that $\varphi(U \cap A)$ is a null set of $\varphi(U) \subseteq \mathbb{R}^n$. Thus, $\varphi(U \setminus A) \subseteq \varphi(U)$ has some point by Lemma 2.105, so we are done.

2.8.2 Sard's Theorem

Recall from our examples that there simply were not many critical values; for example, see Examples 2.85 and 2.92. This is in general true.

Theorem 2.111 (Sard)**.** Fix a smooth map $F\colon M \to N$. Then the set of critical values of F has measure zero.

Remark 2.112. Here's a heuristic argument when $\dim M = \dim N$. Let $C \subseteq M$ consist of the critical points. Then one has

$$\mu(F(c)) = \int_{F(c)} 1\, dy \le \int_C |\det dF_p(x)|\, dx = 0,$$

where the content is in justifying the inequality above via some change-of-variables argument.

Anyway, let's start the proof.

Proof of Theorem 2.111. Let $D \subseteq N$ be the set of critical points. By Lemma 2.105, we know that it suffices to show that each $q \in N$ has some open neighborhood U_q such that $D \cap U_q$ is a null set. As such, it suffices to replace N with $\mathbb{R}^n$ (using diffeomorphism invariance of null sets) where $n := \dim N$, and then restrictions of F by pullback mean that we may as well replace M also with an open subset $U \subseteq \mathbb{R}^m$ where $m := \dim M$.

We are going to induct on m. Starting with $m = 0$, it means that M is a 0-manifold, so M is countable, so $F(M)$ is countable, so its image has measure zero. We also note that if $n = 0$, then the image is always countable and hence a null set. So we are left with the case $m, n \ge 1$.

To set up, let $C \subseteq U$ denote the critical points of F, and we set

$$C_k := \left\{ p \in U : \left. \frac{\partial F_\bullet}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \right|_p = 0 \text{ for all } \ell \le k \text{ and } i_1, \ldots, i_\ell \in \{1, \ldots, m\} \right\}.$$

Notably, we have a chain $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$; note all these sets are closed because taking these derivatives is continuous. The game for the proof is to show that the differences are small, and that these sets are small for large k. Explicitly, we find

$$F(C) = \big(F(C \setminus C_1)\big) \cup \bigcup_{i=2}^{k} \big(F(C_i \setminus C_{i+1})\big) \cup F(C_{k+1})$$

where k is some large integer to be determined later. So we see that our sets divide up into three classes (as above), and we will show that each class is a null set.

1. We show that $F(C \setminus C_1)$ is a null set. Well, choose some $p \in C \setminus C_1$; we would like an open subset $U_p \subseteq U$ such that $F(C \cap U_p)$ is a null set, which will complete the argument by looping over all p and then reducing to a countable cover of C. Because C_1 is closed, we may as well replace U by $U \setminus C_1$, meaning that some partial derivative of F fails to vanish at each point in U. We can cover U by the open subsets where each partial derivative fails to vanish, of which there are finitely many, so we may as well assume that there's a fixed partial derivative that fails to vanish by passing to this open set. By rearranging, we may then assume that $\frac{\partial F_1}{\partial x_1} \ne 0$, and by scaling, we'll just go ahead and take $\frac{\partial F_1}{\partial x_1} = 1$.

   Set $y_1 := F_1$ and $y_i := x_i$ for each $2 \le i \le m$ so that the matrix of partial derivatives $\left[\frac{\partial y_j}{\partial x_i}\right]_{1 \le i,j \le m}$ is invertible at p. In particular, $\Phi := (y_1, \ldots, y_m)$ is a local diffeomorphism around p, so passing to an open neighborhood of $p \in U$ allows us to make Φ into a genuine diffeomorphism $\Phi\colon U \to U'$. Because Φ is a diffeomorphism, we see that showing the critical values of F is a null set is then equivalent to show that the critical values of $\widetilde{F} := F \circ \Phi^{-1}$ is a null set, so we will focus on $\widetilde{F}$.

Now, the point of passing to $\widetilde{F}$ is that

$$(F_1(x_1, \ldots, x_m), \ldots) = F(x_1, \ldots, x_m) = \left(\widetilde{F} \circ \Phi\right)(x_1, \ldots, x_m) = (y_1(x_1, \ldots, x_m), \ldots),$$

so the moral of the story is that

$$\widetilde{F}(x_1, \ldots, x_m) = (x_1, \ldots),$$

where the "…" simply means that we have some other functions that we haven't bothered to write out. The point is that we can compute the Jacobian of $\widetilde{F}$ as a block matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & \partial\widetilde{F}_2/\partial x_2 & \cdots & \partial\widetilde{F}_2/\partial x_m \\ \vdots & \vdots & \ddots & \vdots \\ * & \partial\widetilde{F}_n/\partial x_2 & \cdots & \partial\widetilde{F}_n/\partial x_m \end{bmatrix}.$$

The moral of the story is that surjectivity of F is equivalent to surjectivity of $\widetilde{F}$. Now set

$$\widetilde{C}_s := C \cap \left(\{s\} \times \mathbb{R}^{n-1}\right)$$

to be the critical points of F whose first coordinate is s. So we can integrate over s to get the desired null sets, using the inductive hypothesis because we moved down in coordinates.

2. We show that $F(C_k \setminus C_{k+1})$ is a null set. Note that $p \in C_k \setminus C_{k+1}$ must have some $(k+1)$-derivative which is nonzero, say

$$\left.\frac{\partial^{k+1} F^j}{\partial x_{i_1} \cdots \partial x_{i_{k+1}}}\right|_p \neq 0,$$

so we set $h := \partial^k F_j / \left(\partial x_{i_1} \cdots \partial x_{i_k}\right)$ to be a function $M \to \mathbb{R}$. Then $h(p) = 0$ but $\frac{\partial}{\partial x_{k+1}} h\big|_p \neq 0$. Thus, p is a regular point (having nonzero derivative is enough for a map to $\mathbb{R}$), so we may as well take $U_p \subseteq M$ to be the regular locus of h.

In particular, we see that $h^{-1}(\{0\}) \cap U_p$ is a lower-dimensional embedded submanifold $S \subseteq M$, and $C_k \cap U_p \subseteq h^{-1}(\{0\}) \cap U_p$, so $F(C_k \cap U_p)$ is contained in the critical values of $F|_S\colon S \to N$, which we see has measure zero by the induction. Looping over all $p \in M$ (and then reducing $\{U_p\}_{p \in M}$ to a countable subcover), we conclude.

3. We show that $F(C_k)$ is a null set for $k > \frac{m}{n} - 1$. This is rather technical. Recall we realized M as an open subset $U \subseteq \mathbb{R}^m$, so we may as well show that each $p \in M$ is contained in some cube $Q \subseteq \mathbb{R}^m$ such that $F(C_k \cap Q)$ is a null set. By shifting and scaling, we may as well assume that $Q = [0,1]^m$.

Take some large N to be determined later. The point is that F has very slow polynomial growth on the scale of $1/N$ when living in C_k, made rigorous by Taylor's theorem, so we are able to bound the size of the image of F. Indeed, we go ahead and subdivide the cube Q into the N^m cubes $\{Q_v\}_{v \in (\mathbb{Z} \cap [0,N))^m}$ given by

$$Q_v = \prod_{i=1}^{m} \left[\frac{v_i}{N}, \frac{v_i + 1}{N}\right].$$

Now, for each $v \in (\mathbb{Z} \cap [0,N))^m$, we bound the size of $F(Q_v)$ under the assumption that $C_k \cap Q_v$ is nonempty. Say $a \in C_k \cap Q_v$.

So we claim that

$$|F(x) - F(a)| \stackrel{?}{\leq} C\,|x - a|^{k+1}$$

for some constant $C > 0$ depending only on F. Let's quickly see why this is enough. Indeed, it follows that the value of F on Q_v is contained in a cube of radius $C(1/N)^{k+1}$. But there are only N^m total cubes, so the volume of our images is bounded above by

$$N^m(1/N)^{n(k+1)},$$

up to some unnamed constant depending only on F. Because $k > \frac{m}{n} - 1$, sending $N \to \infty$ will complete our bound.

It remains to show the bound of the previous paragraph. This follows from an analogue of Taylor's theorem. It suffices to get this bound when F is valued in $\mathbb{R}$ by working on each coordinate function $f := F_\ell$ and then summing the bounds for each coordinate. (Note now that the derivatives for f all vanish to the order k.) So now we claim more generally that

$$f(x) \stackrel{?}{=} f(a) + \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I = i}} \partial_I f(a)(x-a)^I + R_k(x), \tag{2.1}$$

where our remainder is

$$R_k(x) := \frac{1}{k!} \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I = k+1}} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a + (t-a)x)\, dt.$$

This is enough for our inequality because all the terms vanish except for $f(a) + R_k(x)$, and we can upper-bound our remainder by hand because these derivatives are taking place over the compact set Q, the integral can be bounded. One now shows (2.1) by an induction on k: if $k = 0$, there is nothing to say (this is just the Fundamental theorem of calculus), and for the induction, one uses integration by parts to expand out $\partial_I f$ again. ■

2.9 February 27

Today we completed the proof of Sard's theorem. I have edited there for completeness.

2.9.1 Applications of Sard's Theorem

Here are some applications.

Corollary 2.113. Fix a smooth map $F\colon M \to N$ where $\dim M < \dim N$. Then $F(M)$ has measure zero.

Proof. Because $\dim M < \dim N$, it is required that every value of F is critical: $dF_p\colon T_pM \to T_{F(p)}N$ can never be surjective! So we conclude by Theorem 2.111. ■

For the next application, we need the following notion.

Definition 2.114 (regular domain)**.** A regular domain D of a smooth manifold M is a properly embedded codimension-0 smooth submanifold (possibly with boundary).

Corollary 2.115. Fix a closed subset K of a smooth manifold M. Then there are descending regular domains $\{Q_i\}_{i\in\mathbb{N}}$ such that

$$M \supseteq Q_0 \supseteq Q_1 \supseteq \cdots$$

and $K = \bigcap_{i\in\mathbb{N}} Q_i$.

Proof. To begin, we recall that we can find a nonnegative smooth function $f \in C^\infty(M)$ such that $f^{-1}(\{0\}) = K$. Now, Theorem 2.111 allows us to find a regular sequence of values $\{s_i\}_{i\in\mathbb{N}}$ such that $s_i \to 0^+$ monotonically. Then $Q_i := f^{-1}([0, s_i])$ will work. (We will not show that $f^{-1}([0, s_i])$ is a regular domain; this is essentially on the homework.) ■

2.9.2 The Whitney Embedding Theorem

As another application, we will show that any smooth manifold can be embedded into some Euclidean space. To begin, we discuss how to decrease the dimensionality of the target space.

Lemma 2.116. Fix a smooth m-manifold M embedded in some $\mathbb{R}^N$. For each $v \in \mathbb{R}^N \setminus \mathbb{R}^{N-1}$, let $\pi_v \colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ denote the projection map with kernel $\mathbb{R}v$. If $N > 2m + 1$, then there exists some v for which $\pi_v|_M$ is an injective immersion $M \to \mathbb{R}^{N-1}$.

Proof. Injectivity of $\pi_v|_M$ is equivalent to asking for $p - q$ to never be parallel to v for $p, q \in M$. Being a smooth immersion is equivalent to asking for $T_pM \cap \ker d(\pi_v)_p = 0$; note $(\pi_v)_p = \pi_v$ up to the identification $T_p\mathbb{R}^N = \mathbb{R}^N$, so we are asking for T_pM to not have any nonzero vectors parallel to v.

We now build a smooth map to check these two facts. Set $\Delta_M \subseteq M \times M$ to be the diagonal subset $\{(p,p) : p \in M\}$; this allows us to define $\kappa \colon (M \times M) \setminus \Delta_M \to \mathbb{RP}^{N-1}$ by $\kappa(x,y) := [x - y]$. Analogously, we define $M_0 := \{(p, 0) \in TM : p \in M\}$ by $\tau \colon TM \setminus M_0 \to \mathbb{RP}^{N-1}$ by $\tau(p, w) := [w]$. We are now choosing $v \in \mathbb{RP}^{N-1}$ to avoid the images of κ and τ, which are both null sets by Corollary 2.113, so we conclude. ■

Next up, we show that we can embed compact manifolds.

Lemma 2.117. Fix a smooth compact m-manifold M. Then M can be embedded in $\mathbb{R}^N$ for some $N > 0$.

Proof. Choose a finite smooth atlas $\{(U_i, \varphi_i)\}_{i=1}^d$. By adding in some more charts (and then using compactness to reduce), we may assume that $\operatorname{im}\varphi_i = B(0,1) \subseteq \mathbb{R}^m$ by some shifting and that the open subsets $\varphi_i^{-1}(B(0,1/2))$ actually fully cover M. By smoothly extending, we are able to find some $\eta \colon B(0,1) \to [0,1]$ which is 0 on $\partial B(0,1)$ but 1 on $B(0,1/2)$. We now define

$$F := ((\eta \circ \varphi_1)\varphi_1, \ldots, (\eta \circ \varphi_m)\varphi_m).$$

A quick counting argument tells us that the target is $\mathbb{R}^{m(n+1)}$. Now one checks that F is injective and an immersion and hence a smooth embedding (by compactness of M). ■

Remark 2.118. Please read the rest of the proof of the Whitney embedding theorem, which extends the above result to the general case.

Here is the total result, whose proof we will not complete.

Theorem 2.119 (Whitney embedding)**.** Fix a smooth n-manifold M. Then there is an embedding $M \to \mathbb{R}^{2n+1}$.

2.10 March 5

The midterms will be graded by next week.

2.10.1 The Whitney Approximation Theorem

Let's give another application of Theorem 2.111.

Proposition 2.120 (Whitney approximation)**.** Fix a continuous map $F\colon M \to \mathbb{R}^k$ such that $F|_A$ is smooth on a closed subset $A \subseteq M$. Given a positive continuous "error" function $\delta\colon M \to \mathbb{R}_{>0}$, there exists a smooth function $\widetilde{F}\colon M \to \mathbb{R}^k$ such that $\widetilde{F}|_A = F|_A$ and

$$\left|\widetilde{F}(x) - F(x)\right| < \delta(x)$$

for all $x \in M$.

Remark 2.121. Do note that we may take $A = \varnothing$, which tells us that arbitrary continuous functions can be approximated by smooth ones.

Proof. By Corollary 2.12, we certainly get some smooth function $F_0\colon M \to \mathbb{R}^k$ such that $F_0|_A = F|_A$. It remains to adjust F_0 to be close to F. Well, define

$$U_0 \coloneqq \{x \in M : |F_0(x) - F(x)| < \delta(x)\}.$$

Intuitively, U_0 is the set of points where F_0 is already close to F; for example, $A \subseteq U_0$. Additionally, for each $x \notin A$, we choose an open neighborhood $U_x \subseteq M \setminus A$ of x such that

$$|F(x) - F(y)| < \delta(y)$$

for all $y \in U_x$; continuity of F and δ means that U_x is actually open. Intuitively, U_x asserts that F does not move much around x.

The point is that M is covered by the open collection $\{U_0\} \cup \{U_x\}_{x \in M\setminus A}$, so we get a partition of unity subordinate to this open cover, which we denote $\{\psi_0\} \cup \{\psi_x\}_{x\in M\setminus A}$. As such, we set

$$\widetilde{F}(y) \coloneqq \psi_0(y)F_0(y) + \sum_{x\in M\setminus A} \psi_x(y)F(x).$$

Note $\widetilde{F}$ in any neighborhood of some $y \in M$ is a finite sum of smooth functions and hence smooth, so $\widetilde{F}$ is itself smooth. Now, for our bounding, we see that

$$F(y) = \psi_0(y)F(y) + \sum_{x\in M\setminus A} \psi_x(y)F(y)$$

by the partition of unity, so the difference is bounded as

$$\left|\widetilde{F}(y) - F(y)\right| \le \psi_0(y)\left|F(y) - F_0(y)\right| + \sum_{x\in M\setminus A} \psi_x(y)\left|F(y) - F(x)\right|.$$

Each difference on the right-hand side is at most $\delta(y)$ by construction, so the entire sum continues to be at most $\delta(y)$. ■

Example 2.122. Fix a smooth manifold M and a continuous function $\delta\colon M \to \mathbb{R}_{>0}$. Then there is smooth $\widetilde{\delta}\colon M \to \mathbb{R}_{>0}$ such that $0 < \widetilde{\delta} < \delta$ pointwise. Indeed, use Proposition 2.120 to approximate $\delta/2$ with error given by $\delta/2$.

We haven't used Theorem 2.111 yet, but we will do so soon, in the guise of Theorem 2.119. In particular, we would like to upgrade Proposition 2.120 to smoothly approximate arbitrary continuous functions $F\colon N \to M$ (for suitable definition of approximation). The obstruction is that we took linear combinations in the proof of Proposition 2.120, which is not possible in general. To fix this, we fix an embedding $M \subseteq \mathbb{R}^N$, and we know that we can approximate in $\mathbb{R}^N$, but we now need a way to retract the target to stay inside M.

2.10.2 Tubular Neighborhoods

Our current goal will be to understand retractions to embedded submanifolds $M \subseteq \mathbb{R}^k$. This requires a notion of being perpendicular to M (so that we can retract to M).

Definition 2.123 (normal bundle). Fix an embedded submanifold $M \subseteq \mathbb{R}^k$. Then the *normal space* at $x \in M$ is

$$N_xM := \{v \in \mathbb{R}^n : v \perp T_xM\},$$

where T_xM is identified with its image in $T_x\mathbb{R}^k \cong \mathbb{R}^k$. Then the *normal bundle* is defined as

$$NM := \bigsqcup_{x \in M} N_xM.$$

Remark 2.124. It turns out that NM is a smooth manifold of dimension $\dim M + (k - \dim M) = k$. In fact, NM is an embedded submanifold of $T\mathbb{R}^k \cong \mathbb{R}^{2k}$, which is checked on slice charts. Roughly speaking, one may assume that M itself is a slice chart by checking locally, and the normal bundle of a hyperplane (given by a slice chart) is essentially another hyperplane.

Remark 2.125. Note that there is a subset $M_0 \subseteq NM$ given by pairs of the form $(x, 0) \in NM$. Then $M_0 \subseteq NM$ is also an embedded submanifold.

Remark 2.126. One can check that the map $E\colon NM \to \mathbb{R}^k$ given by $(x, v) \mapsto (x + v)$ is smooth. Indeed, it is the restriction of a smooth map on $T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$. We remark that $E(M_0) = M$.

This definition allows us the notion of a tubular neighborhood.

Definition 2.127 (tubular neighborhood). Fix an embedded submanifold $M \subseteq \mathbb{R}^k$, and let U be an open neighborhood of M. Then U is a *tubular neighborhood* if and only if there is an open neighborhood $V \subseteq NM$ of M_0 such that $E|_V : V \to U$ is a diffeomorphism.

Morally, E as addition with a normal tangent vector means that $E(V)$ should be thought of as a small tube sitting around M.

Remark 2.128. Let U be a tubular neighborhood of M. Then the projection $E(x, v) \mapsto x$ will provide a smooth submersion and a retraction to M. Note r is smooth by construction, and the composite

$$M \subseteq U \xrightarrow{r} M$$

is the identity by construction, which implies that r is a submersion by examining tangent spaces.

Anyway, we should probably show that tubular neighborhoods exist.

Proposition 2.129. Every embedded submanifold $M \subseteq \mathbb{R}^k$ has a tubular neighborhood.

Proof. We proceed in steps.

1. We claim that the map $E\colon NM \to \mathbb{R}^k$ is a local diffeomorphism at any $x \in M_0$. It suffices to check that $d_{(x,0)}E$ is an isomorphism for each $(x, 0) \in M_0$, which is done by showing that its image contains $T_xM + N_xM = T_x\mathbb{R}^k$.

2. Now, each $x \in M$ has some $V_x \subseteq NM$ such that $E|_{V_x}$ is a local diffeomorphism. Then one can shrink the V_x so that E is injective on $V := \bigcup_{x\in M} V_x$, which makes E a diffeomorphism. (Namely, as soon as E is injective, it becomes a diffeomorphism onto its image: the inverse map exists by injectivity and is smooth by checking locally.) ■

2.10.3 Back to Whitney Approximation

Now here is our upgraded result.

Theorem 2.130 (Whitney approximation)**.** Let M be a smooth manifold without boundary. Fix a continuous map $F\colon N \to M$ of smooth manifolds such that $F|_A$ is smooth for some closed subset $A \subseteq N$. Then F is homotopic (relative to A) to a smooth map $\widetilde{F}\colon N \to M$.

Here, being homotopic relative to A means that one has a continuous homotopy $H_\bullet\colon N \times [0,1] \to M$ such that $H_0 = F$ and $H_1 = \widetilde{F}$ and $H_\bullet|_A = F$.

Proof. By Theorem 2.119, we may fix a smooth embedding $M \subseteq \mathbb{R}^k$. Additionally, Proposition 2.129 grants us a tubular neighborhood $U \subseteq \mathbb{R}^k$ of M, and we note Remark 2.128 provides a smooth retraction $r\colon U \to M$.

We now use Proposition 2.120 to perturb $F\colon N \to \mathbb{R}^k$ inside U. For our error, define

$$\delta(x) := \sup\{\varepsilon \le 1 : B_\varepsilon(x) \subseteq U\}.$$

One can see that $\delta(x) > 0$ for each $x \in M$ because $M \subseteq U$ and U is open. Further, we note that δ is continuous: by chaining balls together, we see

$$\delta(x') \ge \delta(x) - |x - x'|$$

for any $x, x' \in M$, so δ is in fact Lipschitz continuous by some rearranging. So Proposition 2.120 grants us $\widetilde{F}\colon N \to \mathbb{R}^k$ such that $\widetilde{F}$ and F do not differ by any more than δ everywhere, so we see that $\widetilde{F}$ outputs to U by construction.

The smooth composite $r \circ \widetilde{F}$ will be the desired smooth approximation. Morally, because U is locally convex, we can build a homotopy between F and $\widetilde{F}$ directly, and then composition with r completes the construction. Explicitly, we define

$$H_t(p) := r\left((1-t)F(p) + t\widetilde{F}(p)\right).$$

Note $(1-t)F(p) + t\widetilde{F}(p)$ will live inside $B(F(p), \delta(F(p))) \subseteq U$ always, so we are in fact allowed to input that point into r. Now, H is continuous as the composite of continuous functions, and it satisfies the needed restriction properties by construction. ■

2.10.4 Transverse Intersections

Here is our definition.

Definition 2.131 (transverse)**.** Fix a smooth map $F\colon N \to M$ of smooth manifolds. Then F intersects *transversally* with an embedded submanifold $S \subseteq M$ if and only if

$$\operatorname{im} dF_x + T_{F(x)}S = T_{F(x)}M$$

whenever $F(x) \in S$. In particular, taking F to be an embedding, we say two embedded submanifolds $S_1, S_2 \subseteq M$ intersect *transversally* if and only if $T_pS_1 + T_pS_2 = T_pM$ for all $p \in S_1 \cap S_2$.

Transverse intersections should provide smooth intersections. For a counterexample without transverse intersections, one can view level sets as intersections of a hyperplane with a graph and then take any example where a level set fails to be a submanifold. Anyway, here is our result.

Theorem 2.132. Fix embedded submanifolds $S_1, S_2 \subseteq M$. If S_1 and S_2 intersect transversally (with nonempty intersection), then $S_1 \cap S_2$ is an embedded submanifold with codimension $\operatorname{codim}_M S_1 + \operatorname{codom}_M S_2$.

We can restate this in terms of the more general notion of transverse intersection.

Theorem 2.133. Fix a smooth map $F\colon N \to M$ of smooth manifolds. If F is transverse to an embedded submanifold $S \subseteq M$, then $F^{-1}(S) \subseteq N$ is an embedded submanifold of codimension $\operatorname{codim}_M S$.

Example 2.134. Suppose $p \in M$ is a regular value of M. Then we know the level set $F^{-1}(\{p\})$ (if nonempty) is an embedded submanifold of codimension $\dim M$ by Proposition 2.91.

Notably, Theorem 2.132 follows from Theorem 2.133 by letting F be an embedding. So it remains to prove Theorem 2.133.

Proof of Theorem 2.133. Set $n := \dim N$ and $m := \dim M$. One can check the result locally on M, so we may use k-slice charts in order to assume that $M \subseteq \mathbb{R}^m$ is open, and $S \subseteq M$ is a hyperplane in M of codimension k. Then let $\varphi\colon S \to \mathbb{R}^k$ be a local defining function for S by taking an orthogonal projection to the hyperplane S, and we check that $\varphi \circ F$ continues to have $0 \in \mathbb{R}^k$ as a regular value, which completes by appealing to Proposition 2.91.

Let's discuss the check that $\varphi \circ F$ has $0 \in \mathbb{R}^k$ as a regular value. Note that $dF_\bullet\colon M \to \mathbb{R}^{m-k}$ will be surjective by the transverse intersection, so adding in parts from $T_\bullet S$ (which are granted by examining what φ does to the differential) completes the check. ■

2.11 March 7

The homework has been pushed back.

Remark 2.135. Note that continuity is a requirement for smooth approximation via Theorem 2.130. For example, a surjection $S^2 \to S^1$ has no continuous approximation, so of course it has no smooth approximation.

2.11.1 More on Transverse Intersections

It should generically be true that submanifolds intersect transversally. However, we need a way to discuss what "generically" means in this context. This is the content of our next result.

Definition 2.136 (smooth family)**.** Fix smooth manifolds S, N, and M. Then a *smooth family of maps* is a smooth map $F_\bullet\colon N \times S \to M$. Here, S is viewed as a parameter so that $F_s\colon N \to M$ is a smooth map for each $s \in S$, and somehow the map F_s itself varies smoothly in s.

Proposition 2.137 (Parametric transversality)**.** Fix a smooth family of maps $F_\bullet\colon N \times S \to M$. Fix a smooth submanifold $X \subseteq M$. If the family F is transverse to X, then F_s is transverse to X for almost all every $s \in S$. (Namely, the conclusion holds outside a null set.)

The use of a null set tells us that we are going to use Theorem 2.111. Morally, the intuition is that we should expect two generic manifolds to intersect transversally. For example, one can fix a hypersurface $X \subseteq M$ and then use $N \times SS$ so that $F_\bullet$ parameterizes hyperplanes on M, and we are being told that almost all hyperplanes intersect X transversally.

Proof of Proposition 2.137. Set $W := F^{-1}(X) \subseteq (N \times S)$, which is an embedded submanifold of $N \times S$ by Theorem 2.133. We want a result for almost every $s \in S$, so we will need to consider regular values of some function outputting to S. As such, we will look at the restriction of the projection $\pi\colon (N \times S) \to S$ to W.[2]

So by Theorem 2.111, it remains to show that $s_0 \in S$ is a regular value for $\pi|_W$ implies that F_{s_0} intersects transversally to X. Well, choose $p \in F_{s_0}^{-1}(X)$ so that $(p, s_0) \in S$. Set $q := F_{s_0}(p)$. By the regularity of s_0, we know (p, s_0) is regular for $\pi|_W$, so

$$d\pi_{(p,s_0)}(T_{(p,s_0)}W) = T_{s_0}S.$$

As such, up to some identifications, we may write

$$T_{(p,s)}(N \times S) = T_pN \oplus T_{s_0}S = T_pN \oplus \operatorname{im} d\pi_{(p,s_0)},$$

which we now carry over to M as

$$(dF_{s_0})_p(T_pN)+T_qX = (dF)_{(p,s_0)}(T_{(p,s_0)}(N\times\{s_0\}))+T_qX \stackrel{*}{=} (dF)_{(p,s_0)}(T_{(p,s_0)}(N\times\{s_0\})+T_{(p,s_0)}W)+T_qX,$$

where $\stackrel{*}{=}$ holds because dF maps TW to TX already, so we haven't gained anything. But now this is T_qM because F itself is transverse to X. ■

As an application, we show that any embedding can be perturbed to smooth transverse one.

Proposition 2.138 (Transversality homotopy)**.** Fix a smooth map $f\colon N \to M$ and an embedded submanifold $X \subseteq M$. Then there is a smooth embedding $g\colon N \to M$ which is transverse to X and homotopic to f.

Proof. The idea is that we should be able to work in a tubular neighborhood to perturb f a small amount to achieve the transverse intersection. To discuss tubular neighborhoods, we go ahead and use Theorem 2.119 to place M inside some $\mathbb{R}^k$, from which we are able to extract a tubular neighborhood $U \subseteq \mathbb{R}^k$ of M; let $r\colon U \to M$ be the corresponding smooth retraction. In order to make sure we only ever make small perturbations, define $\delta_0\colon M \to \mathbb{R}_{>0}$ by

$$\delta_0(x) := \max\{r \geq 1 : B(x,r) \subseteq U\},$$

and use Example 2.122 to get some smooth $\delta\colon M \to \mathbb{R}_{>0}$ with $\delta < \delta_0$.

We now build our family to make perturbations. Set $S := B(0,1) \subseteq \mathbb{R}^k$ and $F\colon N \times S \to M$ by

$$F_s(p) := r(f(p) + \delta(f(p))s).$$

Note F is smooth as some smooth composite, and F is actually a submersion: r is a submersion, so it is enough to check that $(p,s) \mapsto (f(p)+\delta(f(p))s)$ is a submersion, but actually $s \mapsto (f(p)+\delta(f(p))s)$ is already a smooth submersion. So Proposition 2.137 grants s_0 such that F_{s_0} is transverse to X, so a smooth map connecting s and s_0 provides a homotopy from $F_0 = f$ to the transverse embedding F_{s_0}. ■

2.11.2 Remarks on Cohomology

We conclude with some remarks about using transversal intersections for (co)homology.

Remark 2.139. Fix a smooth compact n-manifold M without boundary, and let $S \subseteq M$ be a closed submanifold of codimension 1. We claim that the existence of a smooth retraction $r\colon M \to S$ implies that $M \setminus S$ is connected. Note r being a smooth retraction makes it a smooth submersion, so $r^{-1}(\{s\})$ is a closed 1-dimensional submanifold such that $r^{-1}(\{s\}) \setminus S$. This is compact and connected, so one can see that $r^{-1}(\{s\})$ is a disjoint union of circles. Even after subtracting out S then, this set will continue to be path-connected.

[2] Notably, even though π itself is a submersion, meaning all values are regular, the map $\pi|_W$ might get some critical values. For example, one can restrict the projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x,y) := y$ to the parabola $\{(x,y) : y = x^2\}$, which now has 0 as a critical value.

Remark 2.140. Fix a compact oriented n-manifold M. Then one can use Sard's theorem to show that each $\alpha \in H_{n-1}(M,\mathbb{Z})$ comes from a bona fide embedded submanifold! The idea is to write

$$H_{n-1}(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z})$$

by Poincaré duality, and $H^1(M;\mathbb{Z})$ is basically homotopy classes of maps $M \to S^1$ by a discussion of the fundamental group. So one finds a map $f\colon M \to S^1$ representing α and brings it back to a submanifold, where the point is that we are allowed to adjust f by a homotopy, allowing us to assume that we actually have an embedded submanifold.

Remark 2.141. In general, an embedded k-submanifold $S \subseteq M$ of the smooth n-manifold M provides a class $[S] \in H_k(M;\mathbb{Z})$. Given two such embedded submanifolds S_1 and S_2 of dimensions k_1 and k_2, respectively, one can perturb them to intersect transversally into $[S_1 \cap S_2] \in H_{k_1+k_2-n}(M;\mathbb{Z})$. As such, we have defined a "cap product"

$$\cap\colon H_{k_1}(M;\mathbb{Z}) \otimes_{\mathbb{Z}} H_{k_2}(M;\mathbb{Z}) \to H_{k_1+k_2-n}(M;\mathbb{Z}).$$

By Poincaré duality, one produces a cup product on cohomology.

Example 2.142. Consider $M := T^2 = S^1 \times S^1$, and let S_1 and S_2 be the embedded circles in M. One sees that $S_1 \cap S_2$ has a single point of intersection, so $[S_1] \cap [S_2]$ is the generator of $H_0(M;\mathbb{Z})$. On the other hand, $[S_1] \cap [S_1] = 0$ because S_1 can be perturbed a little to not intersect with itself at all.

2.11.3 Lie Groups

We now change our topic of discussion to Lie groups.

Definition 2.143 (Lie group)**.** A *Lie group* is a smooth manifold G equipped with a smooth multiplication map $m\colon G \times G \to G$ and smooth inversion map $i\colon G \to G$ making G into a group.

Here are many examples.

Example 2.144. The manifolds $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with addition are Lie groups. Indeed, addition and inversion are both polynomial maps, which are smooth.

Example 2.145. The manifolds $\mathbb{R}^\times$ and $\mathbb{C}^\times$ are Lie groups equipped with multiplication. Multiplication is polynomial, and inversion is rational, both of which are smooth.

Example 2.146. The manifolds $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are Lie groups, where the group operation is matrix multiplication. Indeed, matrix multiplication is a polynomial, and inversion is a rational function, both of which are smooth (where defined).

Example 2.147. There are more matrix groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SU}(n)$, and so on. The main content is that they are cut out by polynomial equations, so they are all embedded submanifolds of some general linear group, where the multiplication and inversion maps are known to be smooth.

It will be helpful to have some notation.

Definition 2.148. Given $g \in G$, we define the *left translation* $L_g \colon G \to G$ and *right translation* $R_g \colon G \to G$ by $L_g(h) := gh$ and $R_g(h) := hg$.

Remark 2.149. The translations are smooth. For example, the left translation is the smooth composite

$$M \stackrel{(g,\mathrm{id})}{\to} M \times M \stackrel{m}{\to} M.$$

Remark 2.150. Let $g_1, g_2 \in M$ be elements, and let $e \in M$ be the identity. Here are some basic identities, checked by just plugging in a test element $x \in M$ and evaluating.

- $L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}$.
- $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$.
- $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$. (Note the reversal!)
- $R_e = L_e = \mathrm{id}_M$.

The last three points show that R_g and L_g are diffeomorphisms with inverses given by $R_{g^{-1}}$ and $L_{g^{-1}}$, respectively.

We want to upgrade our notion of morphism.

Definition 2.151 (homomorphism)**.** A smooth map $f \colon G \to H$ of Lie groups is a *Lie group homomorphism* if and only if it is also a group homomorphism.

Example 2.152. The exponential map $\exp \colon \mathbb{C} \to \mathbb{C}^\times$ is a Lie group homomorphism. Notably, $\exp$ is smooth!

Example 2.153. The determinant map $\det \colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ is smooth (it's the restriction of a polynomial map $\mathbb{R}^{n \times n} \to \mathbb{R}$) and a homomorphism.

"Homogeneity" of groups mean that morphisms must look the same everywhere.

Proposition 2.154. Fix a homomorphism $F \colon G \to H$ of Lie groups. Then F has constant rank.

Proof. To see the aforementioned homogeneity, we compute

$$(F \circ L_g)(x) = F(gx) = F(g)F(x) = L_{F(g)}F(x) = \left(L_{F(g)} \circ F\right)(x).$$

So $F \circ L_g = L_{F(g)} \circ F$. To see our constant rank, we compute the differential. For $g \in G$, we see

$$dF_g \circ (dL_g)_e = d(F \circ L_g)_e = (dL_{F(g)})_{F(e)} \circ dF_e.$$

But $L_\bullet$ is always a diffeomorphism by Remark 2.150, so we conclude that $\operatorname{rank} dF_g = \operatorname{rank} dF_e$ is forced. Thus, the rank is in fact constant. ■

Corollary 2.155. Fix a homomorphism $F \colon G \to H$ of Lie groups. Then $\ker F \subseteq G$ is a closed embedded submanifold.

Proof. The map F is constant rank by Proposition 2.154 above, so $\ker F = F^{-1}(\{e_G\})$ is an embedded submanifold by Theorem 2.84. It is closed by continuity. ■

Example 2.156. Let's actually check that $\mathrm{SL}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is an embedded submanifold. Well, $\mathrm{SL}_n(\mathbb{R})$ is the kernel (i.e., pre-image of the identity) of the map $\det\colon\ \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$, so we are done! One can similarly check that $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ are all embedded submanifolds.

Remark 2.157. By the "global" rank theorem, we see that a homomorphism of Lie groups is an immersion if and only if injective, a submersion if and only if surjective, and bijective if and only if a diffeomorphism.

2.12 March 12

We continue discussing Lie groups. Today will be a little light on proofs.

2.12.1 Lie Subgroups

Here is our definition.

Definition 2.158 (Lie subgroup)**.** Fix a Lie group G. Then a *Lie subgroup* is a subset $H \subseteq G$ which is the image of the injective Lie group homomorphism.

Example 2.159. If $H \subseteq G$ is an embedded submanifold and a subgroup of G, then the embedding $H \subseteq G$ provides the injective Lie group homomorphism making H a Lie subgroup. For example, all the matrix groups in Example 2.156 are Lie subgroups of GL (of suitable dimension).

Remark 2.160. An injective Lie group homomorphism is an immersion by Remark 2.157, so H is an immersed submanifold.

Example 2.161. Consider the Lie group $T := S^1 \times S^1$. Then for $\alpha \in \mathbb{R}$, there is a smooth map $F_\alpha\colon \mathbb{R} \to T$ given by

$$F_\alpha(t) := \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right).$$

There are two cases.

- If $\alpha \in \mathbb{Q}$, then F fails to be injective; one can precisely compute the period k as being the least positive integer so that $e^{2\pi i k} = e^{2\pi i \alpha k} = 1$, which we can see is the denominator of α. So one can define $\widetilde{F}_\alpha$ by restricting to S^1 as

$$\widetilde{F}_\alpha(t) := \left((e^{2\pi i k t}, e^{2\pi i \alpha k t}\right),$$

  and now we see that $\operatorname{im} F_\alpha = \operatorname{im} \widetilde{F}_\alpha$ is a Lie subgroup.
- If $\alpha \notin \mathbb{Q}$, then F is injective, so $\operatorname{im} F$ is a Lie subgroup. Notably, it is dense in T, though we will not show it.

Here's a quick check.

Lemma 2.162. Suppose H is an open Lie subgroup of G. Then H is the union of connected components of G.

Proof. Note that $H \subseteq G$ is a subgroup (it is the image of a group under a homomorphism), so we may partition

$$G = \bigsqcup_{g \in G} gH$$

into cosets. Each gH is open because L_g is a homeomorphism by Remark 2.150, so the complement of H is the union of open subsets of G, so H is also closed. So H is open and closed, and the result follows. ■

Proposition 2.163. Fix a connected Lie group G. Given an open neighborhood $U \subseteq G$ of e, the group G is generated by U.

Proof. Let H be the subgroup generated by U. For example, $U \subseteq H$. Now, for any $g \in H$, we see that $L_g(U)$ is open by Remark 2.150 and lives inside H, so H is open. Thus, Lemma 2.162 tells us that H is the union of connected components of G, so $H = G$ follows because G is connected. ■

This motivates us to work with the identity component of e for disconnected groups.

Definition 2.164 (identity component)**.** Fix a Lie group G. Then the *identity component* $G_\circ$ is the connected component of G containing $e \in G$.

Proposition 2.165. Fix a Lie group G. Then $G_\circ$ is a properly embedded Lie subgroup.

Proof. In fact, we claim that the open submanifold $G_\circ \subseteq G$ is itself a Lie group under the restricted multiplication and inversion. Namely, we must show that $m(G_0 \times G_0) \subseteq G_0$ and $i(G_0) \subseteq G_0$. Well, m and i are continuous maps, so because $G_0 \times G_0$ and G_0 are connected, their images are still connected. To finish, we note that $e = m(e, e)$ and $e = i(e)$ tells us that their images must land in the connected component of e, so $m(G_0 \times G_0) \subseteq G_0$ and $i(G_0) \subseteq G_0$. ■

Example 2.166. Note that $\det\colon \operatorname{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ is surjective, but the target $\mathbb{R}^\times$ is disconnected (it's $\mathbb{R}_{>0} \sqcup \mathbb{R}_{<0}$), so $\operatorname{GL}_n(\mathbb{R})$ must fail to be connected. But the pre-image of $\mathbb{R}_{>0}$ is $\operatorname{GL}_n^+(\mathbb{R})$, consisting of the invertible matrices with positive determinant, and $\operatorname{GL}_n^+(\mathbb{R})$ turns out to be connected, so $\operatorname{GL}_n^+(\mathbb{R})$. We will not show that it is connected here.

Example 2.167. Similarly, one can check that $\operatorname{SO}_n(\mathbb{R})$ is the connected component of the identity in $\operatorname{O}_n(\mathbb{R})$.

We close with the following result.

Proposition 2.168. Fix a Lie subgroup $H \subseteq G$ which is actually an embedded submanifold. Then $H \subseteq G$ is closed.

Sketch. As a sketch, one takes a sequence $\{h_i\}_{i \in \mathbb{Z}^+}$ in H approaching $g \in G$, and we need to check that $g \in H$. One works in a slice chart of g to conclude. ■

Remark 2.169. It turns out that if $H \subseteq G$ is a closed subgroup, then it turns out that H is an embedded Lie subgroup, but we will not show this here.

2.12.2 Group Actions

Groups will be known by their actions. Lie group actions should account for manifold structure, as the following definition establishes.

Definition 2.170 (smooth action)**.** Fix a Lie group G and a manifold M. Then a *smooth left action* G on M is a smooth map $\cdot\colon G \times M \to M$ satisfying the following.

- Associativity: $(g_1 g_2) \cdot p = g_1 \cdot (g_2 \cdot p)$.
- Identity: $e \cdot p = p$.

A right Lie group action is defined analogously on the right via $\cdot\colon M \times G \to M$.

Example 2.171. If M is a countable set, then we recover usual group actions of G on sets.

Example 2.172. Suppose G and H are Lie groups, and H as a right action on G. Then we get a right action of G on H via

$$p \cdot g := g^{-1} \cdot p.$$

(The right-hand side is the right action of p on g^{-1}.)

Example 2.173. Here are some actions of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$.

- Note $\mathrm{GL}_n(\mathbb{R})$ has a smooth left action on $\mathbb{R}^n$ by matrix-vector multiplication.
- Alternatively, one could define $A \cdot v := \left(A^{-1}\right)^\intercal v$ to be a right action.
- There is also a smooth right action by $v \cdot A := A^\intercal v$; notably, $(AB)^\intercal = B^\intercal A^\intercal$.

Example 2.174. Fix a Lie group G. Then here are some ways that the group G could act on itself; they are all composites of multiplication and inversion, so they are smooth.

- G has a smooth right and left action on G by translation.
- G has a smooth left action on G by $g \cdot h := hg^{-1}$.
- G has a smooth left action on G by $g \cdot h := ghg^{-1}$.

Example 2.175. Fix a smooth manifold M. Then $\pi_1(M)$ has a smooth action on the universal cover $\widetilde{M}$ of M by deck transformations. (Note $\pi_1(M)$ is a countable set, which we give the discrete topology, and it becomes a smooth 0-manifold.)

Group actions take on the usual definitions.

Definition 2.176 (orbit, isotropy)**.** Fix a Lie group G with smooth action on the smooth manifold M.

- The *orbit* of $p \in M$ is the set $G \cdot p := \{gp : g \in G\}$. We let $G\backslash M$ denote the set of orbits.
- The *isotropy subgroup* of $p \in M$ is the subgroup

$$G_p := \{g \in G : gp = p\}.$$

Remark 2.177. The orbits $G\backslash M$ of M partition M, by the usual abstract algebra argument.

Definition 2.178 (transitive, free)**.** Fix a Lie group G with smooth action on the smooth manifold M. The action is *transitive* if and only if $G \cdot p = M$ for any $p \in M$. The action is *free* if and only if $G_p = \{e\}$ for all $p \in M$.

Example 2.179. Consider the action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathbb{R}^2$ by matrix-vector multiplication. Here, $\mathrm{SO}_2(\mathbb{R})$ is the set of rotations of $\mathbb{R}^2$. Thus, this action is not transitive (a point in $\mathbb{R}^2$ only gets slid along a circle) and is not free (the isotropy subgroup of 0 is all $\mathrm{SO}_2(\mathbb{R})$).

Example 2.180. Consider the action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ by matrix-vector multiplication. There are two orbits, given by $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$, so the action again is neither free nor transitive.

Example 2.181. Let a Lie subgroup H of G act on the Lie group G by left multiplication. Then the orbits are the right cosets $\{Hg : g \in G\}$.

Example 2.182. Consider the action of the group $\mathrm{GL}_n(\mathbb{C})$ on itself by conjugation. Then the orbits are classified by Jordan normal forms by some linear algebra over algebraically closed fields.

Example 2.183. Consider the action of $\mathrm{SO}_n(\mathbb{R})$ on $\mathrm{GL}_n(\mathbb{R})$ by left multiplication. Then the orbits are given by the cosets, which one can show are in bijection with the group of upper triangular matrices $\mathrm{U}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$. Indeed, for $A \in \mathrm{GL}_n(\mathbb{R})$, one has a unique QR decomposition

$$A = QR$$

where $Q \in \mathrm{SO}_n(\mathbb{R})$ and $R \in \mathrm{U}_n(\mathbb{R})$.

Example 2.184. Algebraic topology informs us that the orbits of the action of $\pi_1(M)$ on the universal cover $\widetilde{M}$ (by deck transformations) are given by points in M.

With group actions on a particular set, we want to understand maps between them.

Definition 2.185 (equivariant)**.** Fix a Lie group G with smooth action on the smooth manifolds M and N. Then a smooth map $F\colon M \to N$ is *G-equivariant* if and only if

$$F(g \cdot m) = g \cdot F(m)$$

for any $g \in G$ and $m \in M$.

Example 2.186. Let V be a vector space. Then the Lie group $\mathrm{GL}(V)$ acts on V by multiplication. One can define an action of $\mathrm{GL}(V)$ on $V \otimes V$ by $g \cdot (v_1 \otimes v_2) := (gv_1 \otimes gv_2)$. Then the diagonal embedding $F\colon V \to V \otimes V$ given by $v \mapsto v \otimes v$ is G-equivariant by construction.

Remark 2.187. Please read some additional properties of equivariant maps.

Studying Lie groups gets interesting when one studies their representations, which are a special kind of group action. We won't say much here, but we can define them.

Definition 2.188. Fix a Lie group G. Then a *representation* of G is a Lie group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ for some finite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$).

Remark 2.189. One can expand out what it means to be a Lie group homomorphism so that a representation simply means that G has a smooth action on V where each g acts by a linear transformation on V.

Example 2.190. The identity map $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is a representation, corresponding to matrix-vector multiplication.

Example 2.191. The map $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ by $A \mapsto \left(A^{-1}\right)^{\intercal}$ is a representation.

Example 2.192. Matrix multiplication defines a smooth linear action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^{n\times n}$, so we get a representation $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^{n\times n})$.

Remark 2.193. It turns out that any compact Lie group G has a faithful (i.e., injective) representation into a finite-dimensional vector space. Roughly speaking, one has G act on $C^\infty(G)$ by $(g\cdot f)(x) := f(x\cdot g)$ and then finds a way to reduce the dimension.

2.12.3 The Groups SO_3 and $\mathrm{SU}(2)$

We spend some time showing how $\mathrm{SO}_3(\mathbb{R})$ and SU_2 relate. Here, SU_2 consists of the 2×2 matrices such that $A^\dagger A = 1_2$ and $\det A = 1$; as such, one can realize SU_2 as a real compact manifold of dimension 3. This group has an action on the space V of Hermitian matrices $H \in \mathbb{C}^{2\times 2}$ (namely, satisfying $H^\dagger = H$) with $\operatorname{tr} H = 0$ by

$$U \cdot H := UHU^\dagger.$$

Namely, one can check that $UHU^\dagger$ remains Hermitian and trace 0 (for example, $\operatorname{tr} UHU^\dagger = \operatorname{tr} HU^\dagger U = \operatorname{tr} H$). Now, one can compute that V has $\mathbb{R}$-basis given by

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \text{and} \qquad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The point is that SU_2 now gets a map ρ to $\mathrm{GL}_3(\mathbb{R})$ because $\dim V = 3$.

Remark 2.194. It turns out that $V \cong T_e \,\mathrm{SU}_2$, but we will not show this here.

It will turn out that $\operatorname{im}\rho \subseteq \mathrm{SO}_3(\mathbb{R})$, which perhaps can be shown by hand, and $\ker\rho = \{\pm 1_2\}$. Now, ρ is a homomorphism and hence of constant rank, and the kernel computation tells us that ρ must now be an immersion, and dimension considerations tell us that ρ must in fact be a local diffeomorphism. This upgrades to a smooth double-covering because it is a smooth local diffeomorphism between manifolds of the same dimension.

Remark 2.195. One can write down any $A \in \mathrm{SU}_2$ as $A = \left[\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right]$. The properties of SU_2 imply that it is determined by $z_{11}, z_{12} \in \mathbb{C}$ which must satisfy $|z_{11}|^2 + |z_{12}|^2 = 1$. So one finds that SU_2 is diffeomorphic to S^3 by this projection, and SO_3 is diffeomorphic to $\mathbb{RP}^3$ by taking the quotient by $\mathbb{Z}/2\mathbb{Z}$.

Remark 2.196. Let's discuss some other complex (irreducible) representations of SU_2.

- There is the trivial representation on $\{0\}$.
- There is the standard matrix-vector multiplication on $\mathbb{C}^2$.
- Taking V as above, we see SU_2 acts on $V \otimes \mathbb{C}$, which can be realized as $\mathrm{Sym}^2(\mathbb{C}^2)$ in some functorial way.
- It turns out that the remaining irreducible representations are all the form $\mathrm{Sym}^{2k}(\mathbb{C}^2)$.

As an aside, we note that these can go down to representations on SO_3.

THEME 3

VECTOR BUNDLES

3.1 March 14

Midterm scores have been released. I did okay.

3.1.1 Vector Fields

We would like to attach a vector field of directions to a manifold M. Intuitively, this is smoothly attaching a vector to each point in M. Here is our definition.

Definition 3.1 (vector field)**.** Fix a smooth manifold M, and let $\pi\colon TM \to M$ be the canonical projection. Then a *vector field* is a smooth section $X\colon M \to TM$ of π. In particular, X is smooth, and $\pi(X(p)) = p$ for each $p \in M$; i.e., $X(p) \in T_pM$ for each $p \in M$. A *local vector field* is a vector field on an open subset $U \subseteq M$. We let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M.

Example 3.2. The map $X\colon M \to TM$ given by $X(p) := (p, 0)$ is a vector field.

Example 3.3. Fix some index i. Given a smooth chart (U, φ) on M where $\varphi\colon U \to \mathbb{R}^m$, we note that $X_i\colon U \to TU$ defined by

$$X_i(p) := \left.\frac{\partial}{\partial x_i}\right|_p$$

is a local vector field. Recall $\left.\frac{\partial}{\partial x_i}\right|_p$ denotes $d\varphi_p^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right)$.

Remark 3.4. The set $\mathfrak{X}(M)$ is in fact a vector space, where we define $a_1X_1 + a_2X_2$ by

$$(a_1X_1 + a_2X_2)(p) := a_1X_1(p) + a_2X_2(p).$$

Here, the linear combination is legal because it takes place in T_pM. More generally, given a smooth function $f\colon M \to \mathbb{R}$, we see that $fX\colon M \to TM$ defined by $(fX)(p) := f(p)X(p)$ is a smooth section of the projection $TM \to M$ and hence a vector field; one can check smoothness on a smooth chart. Thus, $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

Remark 3.5. Suppose X is a local vector field on the smooth chart (U,φ) of M. Because any T_pU has basis given by the $\frac{\partial}{\partial x_i}\big|_p$, so we can write

$$X(p)=\sum_{i=1}^n f_i(p)\frac{\partial}{\partial x_i}\bigg|_p.$$

Because projecting onto coordinate is smooth, we see that the f_i are smooth if X is. Conversely, if the f_i are all smooth, then their linear combination to X continues to be smooth. Because a function is smooth if and only if it is smooth on a cover of smooth charts, we see that we can check the smoothness of the vector field X on a cover of smooth charts.

3.1.2 Frames

It will be useful to have a notion of "basis" for $\mathfrak{X}(M)$.

Definition 3.6 (frame)**.** Fix an open subset U of a smooth manifold M.

- Local vector fields $X_1,\ldots,X_k$ on U are *linearly independent* if and only if $\{X_1(p),\ldots,X_k(p)\}$ is linearly independent for all $p\in U$.
- Local vector fields $X_1,\ldots,X_k$ on U form a *local frame* if and only if $\{X_1(p),\ldots,X_k(p)\}$ is a basis of T_pM for all $p\in U$.
- A local frame is a *global frame* if all the local vector fields are actually global vector fields.

The point is that a frame is locally a basis (of sorts) for $\mathfrak{X}(U)$, though one cannot in general expect there to be a global frame at all. (Granted, one cannot in general expect there to be a global vector field at all.)

Example 3.7. Let (U,φ) be a smooth chart on the smooth m-manifold M. Then define the local vector field X_i on U by $X_i(p)\coloneqq\frac{\partial}{\partial x_i}\big|_p$. Then $\{X_1,\ldots,X_m\}$ is a local frame on U.

Remark 3.8. Fix an open subset U on the smooth m-manifold M. Given two local frames $\{X_i\}_{i=1}^m$ and $\{Y_i\}_{i=1}^m$ on U, we note that having a basis means that there are smooth functions a_{ij} such that

$$Y_j(p)=\sum_{i=1}^n a_{ij}(p)X_i(p)$$

for all $p\in U$.

Here is a quick result on extending frames.

Proposition 3.9. Fix an open subset U on the smooth m-manifold M. Given $p\in U$ and linearly independent local vector fields $\{X_1,\ldots,X_k\}\subseteq\mathcal{X}(U)$ such that $\{X_1(p),\ldots,X_k(p),v_{k+1},\ldots,v_m\}$ is a full basis of T_pM, one can find an open neighborhood $V\subseteq U$ of p and local vector fields $X_{k+1},\ldots,X_m$ of V such that

$$\{X_1,\ldots,X_m\}$$

is a local frame over V and $X_i(p)=v_i$ for $i>k$.

Proof. Using coordinates and adjusting φ suitably, we may assume that $X_i(p)=\frac{\partial}{\partial x_i}\big|_p$ for $i\le k$. Then define $X_i(q)\coloneqq\frac{\partial}{\partial x_i}\big|_p$ for $i>k$. Now, a set of frames being linearly independent is an open condition (we are asking for some determinant to fail to vanish), so there is an open neighborhood V of U in which the set $\{X_1, \ldots, X_m\}$ is linearly independent and hence a local frame. ■

The existence of frames is nice enough for us to provide an adjective.

Definition 3.10. A smooth manifold M is *parallelizable* if and only if M has a global frame.

Remark 3.11. Fix a Lie group G. Then G is parallelizable. Indeed, fix a basis $\{v_1, \ldots, v_m\}$ of T_eG, and then we can define

$$X_i(g) := (dL_g)_e(v_i).$$

One can check that X_i is in fact smooth because the L_g are diffeomorphisms.

Example 3.12. The manifolds $\mathbb{R}^n$, S^1, $\left(S^1\right)^n$, and $S^3 \cong \mathrm{SU}_2$ are all

3.1.3 Pushforward and Pullback

There is some danger in pushforward because a smooth map $F\colon M \to N$ may fail to be injective, so we might be asking for the vector field F_*X to take multiple directions in N. The correct definition is as follows.

Definition 3.13. Fix a smooth map $F\colon M \to N$ of smooth manifolds. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{Y}(N)$ are *F-related* if and only if $dF_p(X(p)) = Y(p)$ for all $p \in M$.

Here is our result for existence.

Proposition 3.14. Fix a diffeomorphism $F\colon M \to N$ of smooth manifolds.

(a) For any $X \in \mathfrak{X}(M)$, there is a unique F-related vector field $F_*X \in \mathfrak{X}(N)$ such that

$$(F_*X)(q) := dF_{F^{-1}q}X(F^{-1}(q)).$$

(b) For any $Y \in \mathfrak{X}(N)$, there is a unique F-related vector field $F^*X \in \mathfrak{X}(M)$ such that

$$(F^*X)(p) := (dF_p)^{-1}Y(F(p)).$$

Proof. We have defined each of the vector fields on points, and one can see these definitions make them uniquely defined. It remains to show smoothness, which we omit. ■

More generally, a smooth map permits us to understand vector fields between manifolds.

Definition 3.15 (vector field)**.** Fix a smooth map $F\colon M \to N$ of smooth manifolds, and let $\pi_N : TN \to N$ be the projection. Then a *vector field of N along F* is a map $X\colon M \to TN$ such that $\pi_N \circ X = F$; i.e., $X(p) \in T_{F(p)}N$ for each $p \in N$. We let $\mathfrak{X}^F(N)$ denote this set of vector fields.

For example, a vector field of $\mathbb{R}^2$ along a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ is some smooth ways to place vectors along the curve γ.

Remark 3.16. Please read about vector fields and smooth submanifolds.

3.1.4 Lie Bracket

Given a vector field $X \in \mathfrak{X}(M)$ and $v \in T_pM$, we would like to compute a directional derivative $\partial v X$. For example, we might hope to take $\partial_v X$ to be $(dX)_p(v)$, but this lives in $T_{X(p)}(TM)$ because X maps $M \to TM$. Perhaps we want to project this down along $\pi\colon TM \to M$, but the composite $\pi \circ X = \operatorname{id}_M$, so we would just get $v \in T_pM$ back again.

Let's see an example to make explicit what's going on.

Example 3.17. Take $M = \mathbb{R}^2$. Then we have global frames $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ and $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. One should expect that $\partial_{\partial/\partial x_1}\frac{\partial}{\partial x_2} = 0$ because these are independent, but perhaps $\partial_{\partial/\partial r}\frac{\partial}{\partial x} \neq 0$ because these are not so orthogonal.

The point is that we really want to take

Definition 3.18 (Lie bracket)**.** Fix vector fields X and Y on a smooth manifold M. Then there is a unique vector field Z such that

$$Zf = X(Yf) - Y(Xf) = Zf$$

for any $f \in C^\infty(M)$. We write $[X, Y]$ for Z and name it the *Lie bracket*.

The following lemma explains that Z exists.

Lemma 3.19. Fix vector fields X and Y on a smooth manifold M. Then there is a unique vector field Z such that

$$Zf = X(Yf) - Y(Xf) = Zf.$$

Proof. Well, for each $p \in M$, we are asking for $D_p\colon C^\infty(M) \to \mathbb{R}$ defined by

$$D_p(f) \coloneqq (X(Yf) - Y(Xf))(p)$$

to be a derivation and that sending $p \mapsto D_p$ is a smooth section of $TM \to M$. (This also explains that Z is unique provided that it exists.) Linearity of D_p, and checking the Leibniz rule is a matter of writing everything out: note

$$XY(f_1f_2) = X(f_1 \cdot Y(f_2) + f_2 \cdot Y(f_1)) = X(f_1) \cdot Y(f_2) + f_1 \cdot XY(f_2) + X(f_2) \cdot Y(f_1) + f_2 \cdot XY(f_1).$$

Writing this out for YX and then subtracting produces the needed cancellation of the terms $X(f_1) \cdot Y(f_2)$ and $X(f_2) \cdot Y(f_1)$.

It remains to check that $p \mapsto D_p$ is smooth. Well, we work locally on a smooth chart (U, φ) of M. Write $\varphi \coloneqq (x_1, \ldots, x_m)$. Then Remark 3.5 assures us that we get smooth functions $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ such that

$$X = \sum_{i=1}^m f_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad Y = \sum_{i=1}^m g_i \frac{\partial}{\partial x_i}.$$

We now carry out the computation at some smooth function $f \in C^\infty(M)$. For example,

$$\begin{aligned} X(Y(f)) &= X\left(\sum_{j=1}^m g_j \frac{\partial f}{\partial x_j} f\right) \\ &= \sum_{i=1}^m f_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^m g_j \frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i,j=1}^m f_i \frac{\partial g_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i,j=1}^m f_i g_j \frac{\partial f}{\partial x_i \partial x_j}. \end{aligned}$$

A similar computation gives $Y(X(f))$, and then we can compute

$$\begin{aligned} X(Y(f)) - Y(X(f)) &= \sum_{i,j=1}^{m} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \sum_{i,j=1}^{m} g_j \frac{\partial f_i}{\partial x_j} \frac{\partial f}{\partial x_i} \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_i}{\partial x_i} \right) \frac{\partial f}{\partial x_j}. \end{aligned}$$

Thus, noting that partial derivatives commute in Euclidean space

$$XY - YX = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_i}{\partial x_i} \right) \frac{\partial}{\partial x_j}. \tag{3.1}$$

We now see that this smoothly varies as p varies because all the internal functions are smooth, so we are done. ■

3.2 March 19

We continue.

3.2.1 More on the Lie Bracket

Let's compute the Lie bracket in some examples.

Remark 3.20. Intuitively, the Lie bracket amounts to taking the derivative of one vector field with respect to another vector field.

Example 3.21. In $\mathbb{R}^m$, one has $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ whenever i and j are distinct indices. One sees this because partial derivatives commute in Euclidean space or more explicitly from (3.1). As another example computation, we see

$$\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + f_1 \frac{\partial}{\partial x_1}\right] = \frac{\partial}{\partial x_1} \left(\frac{\partial}{\partial x_2} + f_1 \frac{\partial}{\partial x_1} \right) - \left(\frac{\partial}{\partial x_2} + f_1 \frac{\partial}{\partial x_1} \right) \frac{\partial}{\partial x_1}$$

collapses down to $\partial f_1/\partial x_1$ after the dust settles. This makes sense intuitively because we are taking the derivative $\frac{\partial}{\partial x_2} + f_1 \frac{\partial}{\partial x_1}$ with respect to x_1.

We are essentially computing a commutator via the Lie bracket, so we have the following definition.

Definition 3.22 (commute)**.** Fix an m-manifold M. A set $S \subseteq \mathfrak{X}(M)$ of global vector fields *commutes* if and only if $[X, X'] = 0$ for any $X, X' \in S$.

Remark 3.23. Essentially by construction, we see that the Lie bracket is $\mathbb{R}$-linear in both coordinates.

Remark 3.24. Note $[X, Y] = -[Y, X]$ by definition. In particular, $[X, X] = 0$, so any vector field commutes with itself.

Remark 3.25. For $f \in C^\infty(M)$, we see that $[X, fY] = f[X, Y] + (Xf)Y$, essentially by the product rule. Explicitly, we find

$$[X, fY] = X(fY) - fYX = (Xf)Y + fXY - fYX = f[X, Y] + (Xf)Y.$$

Remark 3.26. Another rather explicit computation shows

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

For example, one sees that $[X,[Y,Z]]f = X[Y,Z]f - [Y,Z]Xf = X(YZf - ZYf) - (YZ - ZY)Xf = (XYZ - XZY - YZX - ZYX)f$ and then sums cyclically to make the total vanish.

Remark 3.27. We note that the Lie bracket does not depend on diffeomorphism class. Namely, if $F\colon M \to N$ is a diffeomorphism, and X_1 and X_2 is related to Y_1 and Y_2, then we find that $[X_1,X_2]$ and $[Y_1,Y_2]$ continue to be F-related. For example, one can show that

$$F_*[X_1,X_2]=[F_*X_1,F_*X_2],$$

though we will not write this out. This is a matter of working sufficiently locally everywhere and checking.

Let's do a quick computation, for fun. Suppose we have two coordinate charts $(x_1,\ldots,x_m)$ and $(y_1,\ldots,y_m)$ on some open chart U of a manifold. Let's compute the Lie brackets of $A,B\in\mathfrak{X}(M)$ via both coordinate charts. Well, we will write

$$A:=\sum_{i=1}^m a_i\frac{\partial}{\partial x_i}\qquad\text{and}\qquad B:=\sum_{i=1}^m b_i\frac{\partial}{\partial x_i}.$$

Using change of coordinates, we may write

$$A=\sum_{i=1}^m a_i\sum_{j=1}^m \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}=\sum_{i=1}^m \widetilde{a}_j\frac{\partial}{\partial y_j}$$

where $\widetilde{a}_j$ collects terms as is necessary. We similarly write $\widetilde{b}_j$ so that $B=\sum_j \widetilde{b}_j\frac{\partial}{\partial y_j}$, and then we find that

$$[A,B]=\sum_{k=1}^m\left(\sum_{i=1}^m a_i\frac{\partial b_k}{\partial x_i}-b_i\frac{\partial a_k}{\partial x_i}\right)\frac{\partial}{\partial x_k}=\sum_{\ell=1}^m\left(\sum_{j=1}^m \widetilde{a}_j\frac{\partial\widetilde{b}_k}{\partial y_j}-\widetilde{b}_j\frac{\partial\widetilde{a}_k}{\partial y_j}\right)\frac{\partial}{\partial y_\ell}.$$

We know that these must be the same vector field, so taking the $\partial/\partial y_\ell$ coordinate reveals

$$\sum_{j=1}^m \widetilde{a}_j\frac{\partial\widetilde{b}_k}{\partial y_j}-\widetilde{b}_j\frac{\partial\widetilde{a}_k}{\partial y_j}=\sum_{k=1}^m\left(\sum_{i=1}^m a_i\frac{\partial b_k}{\partial x_i}-b_i\frac{\partial a_k}{\partial x_i}\right)\frac{\partial}{y_\ell}\partial x_k.$$

3.2.2 Lie Algebras on Lie Groups

On Euclidean space, we have a good notion of how to translate vectors around, which is able to produce lots of nice global vector fields like $\partial/\partial x_\bullet$. What is good about Euclidean space is that we have access to a group structure to translate vectors around, so a similar story will work on other Lie groups. To make sense of this, we have the following definition.

Definition 3.28 (invariant)**.** Fix a Lie group G. Then a vector field $Y\in\mathfrak{X}(G)$ is *left-invariant* if and only if $(L_g)_*X = X$ for all $g\in M$. In other words, for any $g'\in G$, we are asking for

$$(dL_g)_{g'}X(g')=X(gg').$$

We let $\operatorname{Lie} G$ denote the vector space of left-invariant vector fields.

For example, we are asking for $X(g)=(dL_g)_eX(e)$, so X will be completely determined by $X(e)$. This is codified in the following lemma.

Lemma 3.29. Fix a Lie group G. Then $\operatorname{Lie} G \cong T_eG$, where the isomorphism sends X to $X(e)$.

Proof. This map is certainly linear. To see that it is injective, suppose $X(e) = 0$, and we want to show that X itself vanishes. Well, for any $g \in G$, we see

$$X(g) = (dL_g)_e X(e) = 0,$$

so $X = 0$. Lastly, we want to check that the map is surjective. Well, for $v \in T_eG$, define $X: G \to TG$ by

$$X(g) \coloneqq (dL_g)_e X(e).$$

A direct computation shows that this definition is left-invariant, so it really just remains to show that X is smooth, which is a check that we will omit. The main point is that Xf is smooth for any smooth function f by writing out everything explicitly, which is enough upon trying enough test functions f on X. ■

Remark 3.30. The above lemma verifies that every Lie group has a global frame: let $v_1, \ldots, v_n$ be a basis of T_eG, and then Lemma 3.29 provides vector fields $X_1, \ldots, X_n$ such that $X_i(e) \coloneqq v_i$ for each i. Then we see that $\{X_1, \ldots, X_n\}$ is global frame because translating by L_g for any g preserves being a basis from e to g.

We now note that our Lie group structure descends.

Lemma 3.31. Fix vector fields $X, Y \in \operatorname{Lie} G$ where G is a Lie group. Then $[X, Y] \in \operatorname{Lie} G$.

Proof. We are tasked with showing that $[X, Y]$ is left-invariant. Well, L_g is a diffeomorphism, so

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y],$$

so we are done. ■

This information is now packaged into a Lie algebra.

Definition 3.32 (Lie algebra)**.** A *Lie algebra* is a vector space V equipped with a Lie bracket $[\cdot, \cdot]$ which is bilinear, antisymmetric, and satisfying the Jacobi identity.

Example 3.33. Given a Lie group G, we have shown above that $\operatorname{Lie} G$ becomes a Lie algebra.

Example 3.34. Take the Lie group $G \coloneqq \mathbb{R}^n$. Then $\operatorname{Lie} G \cong T_0\mathbb{R}^n \cong \mathbb{R}^n$. Notably, T_eG has basis given by $\partial/\partial x_i$, all of which commute with each other, so the Lie bracket vanishes on $\operatorname{Lie} G$.

Exercise 3.35. Take the Lie group $G \coloneqq \operatorname{GL}_n(\mathbb{R})$. We compute the Lie bracket.

Proof. We take coordinates given by $\partial/\partial x_{ij}$, and we will go ahead and compute our left-invariant vector fields. Notably, our left action is given by $L_XY = XY$ where $X, Y \in \operatorname{GL}_n(\mathbb{R})$, which is linear in Y. Now, suppose $A \coloneqq \sum_{i,j} A_{ij} \frac{\partial}{\partial x_{ij}}$ is left-invariant. Then for $X \in G$ we have

$$A(X) = (dL_X)_e(A(e)) \stackrel{*}{=} XA(e) = \sum_{i,k=0}^{n} X_{ij}A_{jk}(e) \frac{\partial}{\partial x_{ik}}\bigg|_X,$$

where $\overset{*}{=}$ is just because L_X is the linear map given by multiplication by X, which goes down to the tangent space. The moral of the story is that any $A(e) \in M_n(\mathbb{R})$ produces the left-invariant vector field $A(X) := XA(e)$.

This allows us to compute the Lie bracket: fix $A, B \in \operatorname{Lie} G$. Then, at some $X \in \mathrm{GL}_n(\mathbb{R})$, using coordinates as before, we see

$$\begin{aligned}[A,B](X) &= \sum_{r,s=0}^{n}\left(\sum_{i,j=0}^{n} A_{ij}(X)\frac{\partial B_{rs}}{\partial x_{ij}}(X) - B_{ij}(X)\frac{\partial A_{rs}(X)}{\partial x_{ij}}\right)\frac{\partial}{\partial x_{rs}} \\ &= \left(X_{ki}A_{ij}(e)\frac{\partial X_{tr}B_{rs}(e)}{\partial x_{ij}} - X_{ki}B_{ij}(e)\frac{\partial X_{tr}A_{rs}(e)}{\partial x_{ij}}\right)\frac{\partial}{\partial x_{rs}} \\ &= \left(X_{ki}A_{ij}(e)B_{rs}(e)\frac{\partial X_{tr}}{\partial x_{ij}} - X_{ki}B_{ij}(e)A_{rs}(e)\frac{\partial X_{tr}}{\partial x_{ij}}\right)\frac{\partial}{\partial x_{rs}}.\end{aligned}$$

In total, after computing these derivatives, one sees that $[A,B] = AB - BA$. ■

3.3 March 21

Today we begin studying integration along curves.

3.3.1 Trajectories

We are going to want to understand trajectories.

Definition 3.36 (trajectory)**.** Fix a vector field $V \in \mathfrak{X}(M)$. A smooth curve $\gamma\colon (a,b) \to M$ is a *trajectory* or *integral curve* of V if and only if

$$\gamma' = V \circ \gamma.$$

Namely, the tangent vector along $\gamma(t)$ is the same as the one given by $V(\gamma(t))$.

Remark 3.37. Suppose we are in a local chart (U, φ) where $\varphi = (x_1, \ldots, x_n)$. On one hand, we may write

$$V = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i}$$

for smooth functions $V_1, \ldots, V_n$. On the other hand, a curve γ with $\gamma(t) \in U$ will have

$$\gamma'(t) = \sum_{i=1}^{n} \gamma_i'(t) \frac{\partial}{\partial x_i}\bigg|_{\gamma(t)},$$

so having a trajectory amounts to solving the system

$$\gamma_i'(t) = V_i(t) \text{ for } i \in \{1, \ldots, n\}.$$

Example 3.38. Take $M := \mathbb{R}^2$ so that we may identify $TM \cong \mathbb{R}^2 \times \mathbb{R}^2$. Then $V(x) := (x, x)$ can be solved directly for trajectories γ. Namely, we are asking for $\gamma'(t) = \gamma(t)$, so our curve must look like $\gamma(t) = ve^t$ where $v \in \mathbb{R}^2$ is some vector.

Example 3.39. If we replace $M := \mathbb{R}^2$ with $M := B(0,1)$, then the same vector field $V(x) := (x, x)$ will have basically the same trajectories, just perhaps limited in time.

Example 3.40. Identify $M := \mathbb{C}$ with $\mathbb{R}^2$, and consider the vector field $V(x) := (x, ix)$. Then our trajectories look like $\gamma(t) = ve^{it}$ by solving the system in the usual way.

Remark 3.41. Here is a quick aside: if γ is a trajectory of V, and $t_0 \in \mathbb{R}$, then the function $\gamma_{t_0}(t) := \gamma(t+t_0)$ is also a trajectory of V. This is simply because $\gamma'_{t_0}(t) = \gamma'(t+t_0)$.

We would like for trajectories to exist and be unique. This is basically checked locally on charts, and then we will be able to glue along charts by some uniqueness.

The following lemma is proven by working on charts.

Lemma 3.42. Fix a smooth manifold M, a vector field $V \in \mathfrak{X}(M)$, and some $p \in M$.

(a) Existence: we are granted open neighborhoods $U_0 \subseteq M$ and $U \subseteq M$ for which there is $\varepsilon > 0$ and a smooth map $\theta\colon (-\varepsilon, \varepsilon) \times U_0 \to U$ such that any $q \in U_0$ makes

$$\gamma_q(t) := \theta(t, q)$$

a trajectory of V with $\gamma_q(0) = q$.

(b) Uniqueness: for any other trajectory $\widetilde{\gamma}\colon (a, b) \to U$ of V with $\widetilde{\gamma}(0) = q$, we have $\widetilde{\gamma} = \gamma_q$ on $(-\varepsilon, \varepsilon) \cap (a, b)$.

Proof. This simply holds by passing to the chart U_0, where existence and uniqueness for systems of ordinary differential equations holds by general theory. ■

We now glue together uniqueness.

Lemma 3.43. Fix a smooth manifold M, and suppose that we have two trajectories $\gamma_1\colon (a_1, b_1) \to M$ and $\gamma_2\colon (a_2, b_2) \to M$ of the same vector field V. If $\gamma_1(t) = \gamma_2(t)$ for any t, then $\gamma_1 = \gamma_2$ on $(a_1, b_1) \cap (a_2, b_2)$.

Proof. Let I be the set of $t \in (a_1, b_1) \cap (a_2, b_2)$ where $\gamma_1(t) = \gamma_2(t)$. By definition, we see that I is a closed subset of this interval, but by working in charts, we see that any point in I has an open neighborhood in I (by Lemma 3.42), so I is also closed in the interval. Lastly, I is nonempty by hypothesis, so connectivity forces I to be the full interval. ■

As such, we can glue together trajectories.

Corollary 3.44. Fix a smooth manifold M, and suppose that we have two trajectories $\gamma_1\colon (a_1, b_1) \to M$ and $\gamma_2\colon (a_2, b_2) \to M$ of the same vector field V. If $\gamma_1(t_0) = \gamma_2(t_0)$ for any t, then

$$\widetilde{\gamma}(t) := \begin{cases} \gamma_1(t) & \text{if } t \in (a_1, b_1), \\ \gamma_2(t) & \text{if } t \in (a_2, b_2), \end{cases}$$

is also a trajectory of V.

Proof. The function $\widetilde{\gamma}$ is well-defined by Lemma 3.43, and it is a smooth trajectory because its restrictions to (a_1, b_1) and (a_2, b_2) is ■

To be more precise about our gluing, we will require a maximality notion.

Definition 3.45 (maximal trajectory)**.** Fix a vector field V on a smooth manifold M. Given $p \in M$, there is a *maximal trajectory* $\gamma\colon (a,b) \to M$ of V such that $\gamma(0) = p$ in the following sense: any other trajectory $\widetilde{\gamma}\colon (\widetilde{a},\widetilde{b}) \to M$ of V satisfying $\widetilde{\gamma}(0) = p$ has $(\widetilde{a},\widetilde{b}) \subseteq (a,b)$.

Remark 3.46. Let's show that these maximal trajectories exist. Indeed, consider the collection Γ_p of all trajectories $\gamma\colon U_\gamma \to M$ of V such $\gamma(0) = p$. Then let U be the union of all the U_γ, and the uniqueness result of Lemma 3.43 allows us to define a trajectory $\widetilde{\gamma}\colon U \to M$ by saying $\widetilde{\gamma}(q) = \gamma(q)$ whenever $q \in U_\gamma$. (Namely, the lemma shows that this $\widetilde{\gamma}$ is well-defined in that it does not depend on the choice of γ used to set $\gamma(q)$. That γ is a smooth trajectory can be done because $\widetilde{\gamma}|_{U_\gamma} = \gamma$ for each γ.) This $\widetilde{\gamma}$ is maximal by construction: U contains U_γ for each γ!

We would like for our maximal trajectories to always be defined over $\mathbb{R}$, but the following example shows that this is not always the case.

Example 3.47. Take $M := \mathbb{R}$, and let V be the vector field $V(x_0) := x_0^2 \frac{\partial}{\partial x}\big|_{x_0}$. As such, we are trying to solve the ordinary differential equation $\gamma' = \gamma^2$, and we will also enforce $\gamma(0) = 1$. Then solving produces $\gamma(t) = 1/(1-t)$, so we see that the maximal trajectory must be $(-\infty, 1)$. Namely, any other trajectory $\gamma_0\colon (a,b) \to \mathbb{R}$ must agree with γ on $(a,b) \cap (-\infty,1)$, but if $b > 1$, then we note $\gamma_0(1)$ must be the limit of $\gamma(t)$ as $t \to 1^-$ by continuity, which does not exist.

This is somehow a problem of the vector field, so we will produce a definition to fix it.

Definition 3.48 (complete)**.** A vector field V on a smooth manifold M is *complete* if and only if any $p \in M$ has a trajectory $\gamma_p\colon \mathbb{R} \to M$ of V such that $\gamma_p(0) = p$. Note that γ_p is then the maximal trajectory of p.

The following lemma explains how a vector field might fail to be complete.

Lemma 3.49 (Escape)**.** Fix a vector field V on a smooth manifold M. Suppose that $\gamma\colon (a,b) \to M$ is a maximal trajectory of V for some point $p \in M$. If $b < \infty$, then any compact $K \subseteq M$ and $t_0 \in (a,b)$ will have $\gamma([t_0,b)) \not\subseteq K$.

Sketch. Suppose for the sake of contradiction that $\gamma([t_0,b))$ lands fully inside K. Then the point is that we should be able to extend the trajectory to (an open neighborhood of) $\gamma(b)$, violating the maximality of γ. ■

Corollary 3.50. Fix a vector field V on a smooth manifold M. If the support of V is compact, then V is complete.

Proof. We proceed by contraposition, whereupon we get the result from Lemma 3.49. Indeed, the maximal trajectory of some $p \in M$ must always stay inside the support of V, which is compact by hypothesis. ■

3.3.2 Flows

We now glue together our maximal trajectories to see how vector fields flow.

Definition 3.51 (flow)**.** Fix a complete vector field V on a smooth manifold M. For each p, let $\gamma_p\colon \mathbb{R} \to M$ be the maximal trajectory so that $\theta_p(0) = p$. Then the *flow* of V is the corresponding function $\theta\colon \mathbb{R}\times M \to M$ given by $\theta(t,p) := \gamma_p(t)$.

Remark 3.52. By construction, we see $\theta_0 = \mathrm{id}_M$ for each p.

Remark 3.53. One can check that $\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}$. This is simply how maximal trajectories work: by the uniqueness of trajectories forces

$$\gamma_p(t_1 + t_2) = \gamma_{\gamma_p(t_2)}(t_1)$$

because both sides define a trajectory of V giving the same point at $t_1 = 0$. As such, we see that $\mathbb{R}$ has been given an action on M; for example, it follows that $\theta_t \colon M \to M$ is a diffeomorphism with inverse given by θ_{-t}.

Remark 3.54. The previous remark establishes that θ is fully smooth. Indeed, by smooth variation of solutions, one sees that θ is smooth in some open neighborhood of $(0, p)$ for any $p \in M$, and then we get smoothness in general because

$$\theta_t = \underbrace{\theta_{t/N} \circ \cdots \theta_{t/N}}_{N}$$

for any $N > 0$, so sending N large is able to take the smoothness local at $(0, p)$ to smoothness everywhere.

Here's an application, for fun.

Proposition 3.55. Fix a connected smooth manifold M and points $p, q \in M$. Then there is a compact subset K containing p and q and a diffeomorphism $f \colon M \to M$ such that $f(p) = q$ and $f|_{M\setminus K} = \mathrm{id}_{M\setminus K}$.

Proof. Let G be the group of diffeomorphisms $M \to M$ which fix some the complement of some compact subset K; note that G is indeed a group. (The interesting check is closure under composition, where the point is that the union of the two compact subsets whose complements are fixed will work.) We will show that the action of G on M is transitive; because M is connected, it is enough to show that the orbits of G are open (because the orbits partition M).

In other words, for any $p \in M$, it suffices to show that there is an open neighborhood U of p contained in the orbit. We may assume that U is a regular coordinate ball $B(0, 1/2) \subseteq B(0, 1)$ where (U, φ) is a chart where $\varphi(p) = 0$. Then we will show that any $q \in \varphi^{-1}(B(0, 1/2))$ has a diffeomorphism f sending p to q but fixing $M \setminus \varphi^{-1}(B(0, 3/4))$.

Well, choose a vector field V to be supported on $\varphi^{-1}(B(0, 3/4))$ (which is complete by Corollary 3.50) but on $B(0, 1/2)$ is just given by having all tangent vectors point with unit length from p to q. Then a maximal trajectory for V will be able to have $\gamma(p) = q$, so using the flow of V as in Remark 3.53 will complete the proof. ■

3.4 April 2

Welcome back from spring break.

3.4.1 The Flowout Theorem

We begin with a remark.

Remark 3.56. Fix a vector field V on M. When V fails to be complete, there is not much we can say. We do know that each $p \in M$ has some maximal open neighborhood $D_p \subseteq \mathbb{R}$ such that we have a trajectory $\theta_t(p)$ defined on D^p, so we can glue these together into

$$\mathcal{D}_V := \bigcup_{p \in M} D_p \times \{p\}$$

as a subset of $\mathbb{R} \times M$. The point is that we can glue these together into a big flow-like trajectory.

Flowouts essentially allow us to work in a small neighborhood of a submanifold. Here is our result.

Theorem 3.57 (Flowout)**.** Let S be a smooth k-submanifold of the smooth n-manifold M, and let $V \in \mathfrak{X}(M)$ be a vector field such that $V_p \notin T_pS$ for all $p \in S$. Set $\Gamma := (\mathbb{R} \times S) \cap \mathcal{D}_V$, and define $\Phi := \theta|_\Omega$ to be a flow of V.

(a) Φ is an immersion.

(b) Φ relates $\frac{\partial}{/}\partial t$ and V.

(c) For some smooth $\delta \colon S \to \mathbb{R}_{>0}$, set

$$\Omega_\delta := \{(t,p) \in \Omega : |t| < \delta(p)\}.$$

Then $\Phi|_{\Omega_\delta}$ is injective.

(d) In the context of (c), further suppose that $k = n-1$. Then $\Phi|_{\Omega_\delta}$ is a diffeomorphism onto an open submanifold of M.

Sketch. Note (b) is basically by the definition of being a trajectory. For (a), we note that it follows for points $p \in S$ because $V_p \notin T_pS$ by hypothesis and then use the $\mathbb{R}$-action from the flow to extend the immersive property to other points. For (c), one can use k-slice charts to check it on locally; a clever argument with partition of unity makes the injectivity global. Lastly, (d) follows from (a) and (c) because we have established that we have an injective local diffeomorphism on Ω_δ. ■

Computations with vector fields are aided by the following nice form.

Lemma 3.58. Fix a vector field V on the smooth n-manifold M. Then each $p \in M$ with V_p
$ne0$ has a smooth chart $(U, (x_1, \ldots, x_n))$ such that

$$V|_U = \frac{\partial}{\partial x_1}.$$

In fact, given an $(n-1)$-submanifold $S \subseteq M$ and $p \in S$ with $V_p \notin T_pS$, we may assume that the smooth chart is a local slice chart for S cut out by $x_1 = 0$.

Proof. The first claim follows from the second claim by just taking any chart $(U, (x_1, \ldots, x_n))$ and defining the needed S by $x_1 = 0$. For the second claim, use Theorem 3.57. ■

3.4.2 The Lie Derivative

Given two vector fields V and W, we would like to compute something like $\partial_V W$. It is not so obvious, however, how to interpret these derivatives when not working in Euclidean space where we have an obvious identification between points and differentials.

Flows will allow us to take derivatives by moving points (and thus their tangent spaces) around.

Definition 3.59 (Lie derivative)**.** Fix vector fields V and W of a smooth manifold M. Given $p \in M$, we define the *Lie derivative* of W with respect to V at p as

$$(\mathcal{L}_V W)_p := \frac{d}{dt}(d\theta_t)_p^{-1}(W_{\theta_t(p)})\bigg|_{t=0} \in T_pM,$$

where $\theta\colon \mathcal{D} \to M$ is the flow of V.

The point of this definition is that we should want to take the derivative with respect to the "direction" of V just by looking at how W changes along the flow θ of V. This is essentially what the definition above is doing; note that the limit in the derivative makes sense because we only ever have vectors in T_pM.

Remark 3.60. On smooth functions $f \in C^\infty(M)$, the analogous computation is

$$(\mathcal{L}_V f)_p = \frac{d}{dt}(d\theta_t)_p^{-1} = \frac{d}{dt}(d\theta_t)_p^{-1} f\bigg|_{t=0} = \frac{d}{dt} f(\theta_t(p))\bigg|_{t=0} = df_p(V_p) = V_p(f).$$

So our notion of derivative makes some sense.

Remark 3.61. One can see that $\mathcal{L}_V W$ assembles into a vector field on $\mathfrak{X}$. This amounts to checking that everything is sufficiently smooth on charts, which can be done by writing everything out. For example, one can use Lemma 3.58 to make V particularly nice, allowing for easy computation of the flow, whereupon we see that we are basically differentiating W along a direction in Euclidean space, which is legal because V and W started out as smooth anyway.

In fact, the Lie derivative agrees with our bracket!

Proposition 3.62. Fix vector fields V and W of a smooth manifold M. Then $\mathcal{L}_V W = [V, W]$.

Proof. We check this at individual points $p \in M$. Note that the question is local, so we will repeatedly shrink M without much comment. The hardest case is when $V_p \neq 0$. In this case, we use Lemma 3.58 in order to replace M with $\mathbb{R}^n$ so that $V = \frac{\partial}{/}\partial x_1$. Then the flow $\theta_t(x)$ is just

$$\theta_t(x) := (t + x_1, x_2, \ldots, x_n).$$

In this case, we expand out our Lie derivative to find

$$(\mathcal{L}_V W)_p = \sum_j \frac{\partial W_j}{\partial x_1} \frac{\partial}{\partial x_1},$$

and a direct computation shows

$$[V, W] = \sum_{i,j} \left(V_i \frac{\partial W_j}{\partial x_i} - W_i \frac{\partial V_j}{x_i} \right) \frac{\partial}{\partial x_j},$$

which indeed agree because $V_i = 1_{i=0}$.

We now handle the other cases. If V vanishes in a neighborhood of p, then everything in sight vanishes (namely, the flow is constant), so there is nothing to do; the previous two cases are dense in M, so we are done. ■

Example 3.63. Consider the vector field $V = r\partial r$ on Euclidean space $\mathbb{R}^2$, where we defined V using polar coordinates (r, α). Then this vector field is complete with flow given by $\theta_t(p) := e^t p$. With $W := \partial/\partial\alpha$, we expect to have $\mathcal{L}_V W$ because the two vector fields are essentially perpendicular, and indeed we can compute

$$[V, W] = [r\partial r, \partial\alpha] = r[\partial r, \partial\alpha] - \partial_\alpha(r)\partial r,$$

which vanishes.

Remark 3.64. Here are a few more properties of the Lie derivative.

- By the antisymmetry of the Lie bracket, we see $\mathcal{L}_V W = -\mathcal{L}_W V$.
- The Jacobi identity grants the "product rule"

$$\mathcal{L}_V[W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X].$$

Alternatively, one can write $\mathcal{L}_{[V,W]}X = \mathcal{L}_V\mathcal{L}_W X - \mathcal{L}_W\mathcal{L}_V X$.

3.4.3 Commuting Vector Fields

We are now able to provide a better understanding of commuting vector fields.

Proposition 3.65. Fix vector fields V and W on a smooth manifold M. Then the following are equivalent.

(a) V and W commute: $[V, W] = 0$.

(b) W is invariant under the flow of V: $(\theta_s^V)_* W = W$ for all s, where θ_s^V is the flow of V.

(c) V is invariant under the flow of W: $(\theta_t^W)_* V = V$ for all t, where θ_t^W is the flow of W.

(d) The flows of V and W commute: one has $\theta_s^V \circ \theta_t^W = \theta_t^W \circ \theta_s^V$.

Proof. Note (b) implies that $\mathcal{L}_V W = 0$ because it tells us that $(d\theta_s)_p^{-1}(W_{\theta_s(p)})$ is constant in s by the invariance. So (b) implies (a); an analogous argument shows (c) implies (a).

Next up, to use (d), we see that

$$\frac{d}{dt}(\theta_s^V \circ \theta_t^W)(p)\bigg|_{t=0} = (d\theta_s^V)_p(W_p)$$

by definition of the flow θ_t^W, and

$$\frac{d}{dt}(\theta_t^W \circ \theta_s^V)(p)\bigg|_{t=0} = W(\theta_s^V(p))$$

again by using the definition of the flow θ_t^W. Comparing our two equalities, we see that we have achieved (b); a symmetric argument is able to show that (d) implies (c).

Lastly, we have to show (a) implies (b)–(d); we focus on (b) and (d) and get (c) by symmetry. Fix a smooth chart $(U, (x_1, \ldots, x_n))$ of M. Assuming $V_p \neq 0$ for now, we may assume that $V|_U = \partial/\partial x_1$. Now writing $W = \sum_j W_j \frac{\partial}{\partial x_j}$, we note that commuting $[V, W] = 0$ implies $\mathcal{L}_V W = 0$, which by a direct computation expanding the Lie derivative forces

$$\sum_j \frac{\partial}{\partial x_1} W_j = 0.$$

This implies (b) and (d) locally at p. We should also consider the case where p is outside the support of V, but then everything in sight vanishes; a continuity argument now achieves the result for all $p \in M$. ■

We can now expand to many commuting vector fields.

Theorem 3.66. Fix an open submanifold $W \subseteq M$, and let $V_1, \ldots, V_k$ be linearly independent commuting vector fields of W. Then any $p \in W$ has a smooth chart $(U, (x_1, \ldots, x_n))$ with $V_i = \partial/\partial x_i$ for each i.

Sketch. Everything is local, so we may replace M with W. By shrinking M to a regular coordinate ball, we may assume that the $V_\bullet$s are complete. (Namely, being local means that we can replace them with compactly supported counterparts which agree in an open neighborhood p, but then we can shrink M to this open neighborhood.) For simplicity, we will take $k = n$, but it has no impact on the actual logic of the argument.

Now, the point is that we should be able to read off the needed coordinate functions $x_\bullet$ by following the flows of V. Indeed, one just checks that the map

$$(t_1, \ldots, t_n) \mapsto (\theta^{V_1}_{s_1} \circ \cdots \circ \theta^{V_n}_{s_n})$$

is a local diffeomorphism, which is enough for our purposes. ■

Remark 3.67. If $k = n$, then we are noting that commuting local frames are actually local coordinate frames on (perhaps) a smaller open neighborhood.

3.5 April 4

Today we talk about vector bundles.

3.5.1 Vector Bundles

A vector bundle attaches a vector space to each point on our manifold in a way that is "locally" the trivial way to put a vector space (namely, the same vector space everywhere). Here is our definition.

Definition 3.68 (vector bundle). Fix a smooth n-manifold M, possibly with boundary. A *real smooth vector bundle of rank k* is a smooth surjective map $\pi\colon E \to M$ of smooth manifolds, where $\dim E = n + k$, satisfying the following: each $p \in M$ has an open neighborhood $U \subseteq M$ and a diffeomorphism $\varphi\colon \pi^{-1}U \to U \times \mathbb{R}^k$ where $\varphi|_{\pi^{-1}(\{q\})}\colon \pi^{-1}(\{q\}) \to \{q\} \times \mathbb{R}^k$ is an isomorphism of vector spaces and the following diagram commute.

$$\begin{array}{ccccc} U \times \mathbb{R}^k & \xrightarrow{\varphi^{-1}} & \pi^{-1}U & \subseteq & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & = & U & \subseteq & M \end{array}$$

Here, E is the *total space*. For $q \in M$, we set $E_q := \pi^{-1}(\{q\})$.

The point is that φ is supposed to be a local trivialization of E.

Example 3.69. The projection $\pi\colon TM \to M$ is a vector bundle of rank $n = \dim M$. Indeed, for any $p \in M$, choose a smooth chart $(U, (x_1, \ldots, x_n))$ of p, and then we know there exists a local frame $V_1, \ldots, V_n \in \mathfrak{X}(U)$ where $V_i := \partial/\partial x_i$. Then we have a trivialization Φ on U given by

$$\Phi\colon \sum_{i=1}^n v_i V_i \mapsto \sum_{i=1}^n v_i e_i.$$

Example 3.70. There is a projection map $\pi\colon M \times \mathbb{R}^k \to M$, which gives a vector bundle of rank k. In particular, π is globally trivialized by the identity map $M \times \mathbb{R}^k \to M \times \mathbb{R}^k$.

Example 3.71 (Möbius strip)**.** Define E to be the quotient of $\mathbb{R}^2$ by the equivalence relation $\sim$ where $(x, y) \sim (x', y')$ if and only if there is an integer n with $(x', y') = (x + n, (-1)^n y)$. Then there is a projection $E \to \mathbb{R}/\mathbb{Z}$ given by projection onto the first coordinate.

As with any kind of projection, we have a notion of section.

Definition 3.72 (section)**.** Let $\pi\colon E \to M$ be a smooth vector bundle over the smooth manifold M. Then a *local section* defined on some open subset $U \subseteq M$ is a map $\sigma\colon U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. The map σ is a *global section* if $U = M$. We let $\Gamma(E)$ denote the space of smooth global sections $M \to E$.

Example 3.73. Let $\pi\colon E \to M$ be the trivial vector bundle $E := M \times \mathbb{R}^k$. Then any smooth map $f\colon M \to \mathbb{R}^k$ defines a global section $\sigma\colon M \to E$ given by $\sigma(p) := (p, f(p))$. In fact, it is not hard to see that this defines all global sections because any global section $\sigma\colon M \to E$ must take the form $\sigma(p) = (p, \mathrm{pr}_2\, \sigma(p))$, so we may just take $f := \mathrm{pr}_2 \circ \sigma$. For example, $C^\infty(M)$ is in natural bijection with $\Gamma(M \times \mathbb{R})$.

Our notion of frame now generalizes.

Definition 3.74 (frame)**.** Fix a vector bundle $\pi\colon E \to M$ of rank k on the smooth manifold M. Then a *smooth local frame* of π on the open subset $U \subseteq M$ is a collection of local sections $\{\sigma_1, \ldots, \sigma_k\}$ on U such that $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ is a basis for E_p for all $p \in U$.

It is notable that frames relate to trivializations.

Remark 3.75. Using the trivialization of the vector bundle, we see that any vector bundle $\pi\colon E \to M$ has a smooth local frame around any $p \in M$. In fact, all local frames arise this way.

- Explicitly, we take a local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ produces the smooth local frame $\{\sigma_1, \ldots, \sigma_k\}$ given by $\sigma_i(p) := \Phi^{-1}(p, e_i)$, which is a basis at each point because $\{e_1, \ldots, e_k\}$ is a basis of $\mathbb{R}^k$, and Φ is providing a vector space isomorphism at the fiber.
- Conversely, one takes a local frame $\{\sigma_1, \ldots, \sigma_k\}$ on U and defines $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ by

$$\Phi\left(\sum_{i=1}^k v_i(p)\sigma_i(p)\right) := (p, v_1(p), \ldots, v_i(p))\,.$$

  This map is smooth basically by the smoothness of the $\sigma_\bullet$s, and it is providing a vector space isomorphism at the fibers because the $\sigma_\bullet(p)$ are supposed to form a basis.

We are now prepared to make the following definition.

Definition 3.76 (trivial)**.** A vector bundle $\pi\colon E \to M$ is *trivial* if and only if there is a smooth global trivialization.

In light of the previous remark, being trivial is equivalent to having a smooth global frame.

Remark 3.77. As we saw with vector fields, one has difficulty defining directional derivatives or Lie derivatives or Lie brackets directly on global sections $\sigma \in \Gamma(E)$. One needs to make some extra choice about where to go along a given direction.

3.5.2 Constructing Vector Bundles

We quickly make a remark on computations. It may often be the case that we have two different smooth local frames that we want to compare. Explicitly, let $\pi\colon E \to M$ be our vector bundle of rank k, and let $\{\sigma_1, \ldots, \sigma_k\}$ and $\{\sigma'_1, \ldots, \sigma'_k\}$ be smooth local frames on the open subset $U \subseteq M$. It will be helpful to have a change of basis matrix at each point $p \in U$ between our two bases $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ and $\{\sigma'_1(p), \ldots, \sigma'_k(p)\}$ of the fiber E_p. Namely, one has

$$\sigma'_j(p) = \sum_{i=1}^{k} a_{ji}(p)\sigma_i(p).$$

By taking projections suitably, we see that the functions $a_{ji}(p)$ are all smooth functions in p. Explicitly, one can see this because these local frames give rise to local trivializations, and the coefficients of this matrix are essentially projections of the composite of the trivializations, which must be smooth.

Example 3.78. Consider the tangent bundle $\pi\colon T\mathbb{R}^2 \to \mathbb{R}^2$, and note that we have two local frames given by $\{\partial/\partial x, \partial/\partial y\}$ and $\{\partial/\partial r, \partial/\partial\theta\}$. One can compute explicitly that

$$\frac{\partial}{\partial r} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} \qquad \text{and} \qquad \frac{\partial}{\partial\theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

allowing us to write down a change-of-basis matrix.

Being able to change bases provides us with the following idea to construct a vector bundle: just specify a trivializing open cover, explain how to transition between two trivializations on overlaps, and then this will give a vector bundle.

Lemma 3.79. Fix a smooth manifold M, possibly with boundary, and let k be a nonnegative integer. Fix the following data.

(i) We have k-dimensional vector spaces $\{E_p\}_{p\in M}$, define $E := \bigsqcup_{p\in M} E_p$, equipped with the standard projection $\pi\colon E \to M$.

(ii) We have an open cover $\{U_\alpha\}_{\alpha\in\kappa}$ on M and local "frames" $\{\sigma_{\alpha 1}, \ldots, \sigma_{\alpha k}\}$ for each $\alpha \in \kappa$ providing a basis for E_p at each $p \in U_\alpha$.

(iii) For any $\alpha, \beta \in \kappa$, our change-of-basis equations

$$\sigma_{\beta j} = \sum_{i=1}^{k} a_{\beta\alpha, ji}\sigma_{\alpha i}$$

makes the functions $a_{\beta\alpha,ji}$ into smooth functions.

Then there is a unique smooth manifold structure on E such that $\pi\colon E \to M$ becomes a vector bundle, and the $\{\sigma_{\alpha 1}, \ldots, \sigma_{\alpha k}\}$ become actual local frames.

Proof. We omit the proof but make one or two comments gesturing in the direction of a proof. One can use the analogous result for smooth manifolds to at least provide a smooth structure for E. Then one finds that the functions $\sigma_{\alpha i}$ are all smooth, so these bases will produce local trivializations for E, making E into a vector bundle. Lastly, the previous sentence doubles as a check that $\{\sigma_{\alpha 1}, \ldots, \sigma_{\alpha k}\}$ is in fact a local frame. ■

Let's use this result to construct some vector bundles.

Example 3.80 (Whitney sum)**.** Fix two vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ of ranks k and k', respectively. Then one can define the *Whitney sum* $\widetilde{E}$ of E and E' with fibers given by

$$\widetilde{E}_p := E_p \oplus E'_p,$$

which of course provides a projection $\widetilde{\pi}\colon \widetilde{E} \to M$. Let's explain how to do this via Lemma 3.79. By shrinking open neighborhoods as necessary, any point $p \in M$ has an open neighborhood $U_p \subseteq M$ where E and E' have local frames given by $\{\sigma_{p1}, \ldots, \sigma_{pk}\}$ and $\{\sigma'_{p1}, \ldots, \sigma'_{pk'}\}$, respectively. Then we will want $\{\sigma_{p1}, \ldots, \sigma_{pk}, \sigma'_{p1}, \ldots, \sigma'_{pk'}\}$ to provide the local frames on U_p of $\widetilde{E}$.

So we need to examine our change-of-basis matrices between the frames on U_p and U_q. Now, the fact that E and E' are already vector bundles provides us with smooth coefficient functions $a_{p,ji}$ and $a'_{p,ji}$ such that

$$\sigma_{qj} = \sum_{i=1}^{k} a_{p,ji}\sigma_{pi} \qquad \text{and} \qquad \sigma'_{qj} = \sum_{i=1}^{k'} a'_{p,ji}\sigma'_{pi},$$

Concatenating these two change-of-basis matrices, we provide a change-of-basis matrix from the local frame on U_p to the local frame on U_q, and the coefficients are now smooth by construction.

Example 3.81. In basically the same way, one can define a tensor product $\widetilde{E}$ of vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ of ranks k and k', respectively. In short, we take $\widetilde{E}_p := E_p \otimes E'_p$, and for our local frames, over a trivializing open subset $U \subseteq M$, we can take local frames $\{\sigma_1, \ldots, \sigma_k\}$ of E and $\{\sigma'_1, \ldots, \sigma'_{k'}\}$ and turn them into a local frame

$$\{\sigma_i \otimes \sigma'_j\}_{1 \le i \le k, 1 \le j \le k'}.$$

Again, the fact that E and E' are vector bundles to see that change-of-basis maps between the σs and the σ's are smooth, so some algebra shows the same is true of the above proposed local frames.

Example 3.82. One can also take duals. Let $\pi\colon E \to M$ be a vector bundle. Then we define the dual bundle E^* by $E^*_p := E^*_p$, and we propose local frames to be $\{\sigma_1^*, \ldots, \sigma_k^*\}$ whenever $\{\sigma_1, \ldots, \sigma_k\}$ is a local frame on some trivializing open subset $U \subseteq M$. The change-of-basis matrix for these dual bases will end up being the inverse transpose of the change-of-basis matrix for any original basis, so the change-of-basis matrix will succeed in having smooth coordinates, as needed.

3.6 April 9

The sub is back. Today we continue discussing vector bundles.

3.6.1 Bundle Homomorphisms

Any reasonable object has a notion of morphisms between them. Here are the morphisms of vector bundles.

Definition 3.83 (bundle homomorphism)**.** Fix two smooth vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M'$. Then a *bundle homomorphism* $(F, f)\colon \pi \to \pi'$ is the data of a smooth map $f\colon M \to M'$ and smooth map $F\colon E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

commutes, and the restricted maps $F\colon E_p \to E'_{f(p)}$ are linear. If $f = \mathrm{id}_M$, we say that F is a *bundle homomorphism over* M; we denote the set of all bundle homomorphisms $E \to E'$ over M as $\mathrm{Hom}_M(E, E')$.

Remark 3.84. Note the commutativity of the diagram implies that F does in fact map $E_p = \pi^{-1}(\{p\})$ to $E'_{f(p)} = (\pi')^{-1}(\{f(p)\})$.

Remark 3.85. We note that the function f is uniquely determined by F. Indeed, suppose we have two functions f_1 and f_2 such that $\pi' \circ F = f_i \circ \pi$ for each i. Well, π is surjective, so $f_1 \circ \pi = f_2 \circ \pi$ implies that $f_1 = f_2$.

Remark 3.86. Suppose that we are given some smooth F for which some function f exists with $\pi' \circ F = f \circ \pi$. Then note that π is a smooth surjective submersions, so to check that f_0 is smooth, it is enough to check that $f \circ \pi$ is smooth. But this is $\pi' \circ F$, so we get our smoothness from the smoothness of F.

The point is that the bundle homomorphism is uniquely given by the data of F.

Remark 3.87. The set $\mathrm{Hom}_M(E, E')$ is in bijection with global sections $\Gamma(E^* \otimes E')$. The main point is that, on fibers, we see

$$(E^* \otimes E')_p = E_p^* \otimes E'_p = \mathrm{Hom}(E_p, E'_p),$$

where the last map is by $\varphi \otimes v' \mapsto (v \mapsto \varphi(v)v')$, which is checked to be an isomorphism by hand. Thus, a global section $M \to E^* \otimes E'$ provides a family of linear maps $E_p \to E'_p$, which can be checked to be smooth. Conversely, a bundle homomorphism $F\colon E \to E'$ provides maps $F_p\colon E_p \to E'_p$ on fibers, which then provides an element of $(E^* \otimes E')_p$ as above, which will in total assemble into a smooth section by some examination on charts (where the question is about trivial vector bundles on Euclidean spaces).

Here is a cute application.

Lemma 3.88. Fix a smooth manifold M. Then the following are equivalent.

(a) M is parallelizable.

(b) M has a global frame for TM.

(c) The vector bundle TM is trivial.

(d) There is a bundle isomorphism $TM \to M \times \mathbb{R}^k$.

Proof. We already know that (a) and (b) are equivalent. Further, (c) and (d) are equivalent by definition of "trivial." Lastly, (b) and (c) are equivalent because the global frame provides equivalent data to the isomorphism in (d). ■

Notation 3.89. Given a bundle homomorphism $F\colon E \to E'$ of vector bundles over M, we induce a map $\Gamma(F)\colon \Gamma(E) \to \Gamma(E')$ by sending a global section $\sigma\colon M \to E$ to the global section $(F \circ \sigma)\colon M \to E'$.

Remark 3.90. The fact that $F \circ \sigma$ is actually a vector bundle follows because F is a homomorphism of vector bundles over M. Anyway, the above notation turns global sections Γ into a functor, which we won't bother to check. For example, we note that $\Gamma(F)$ is a $C^\infty(M)$-linear map, essentially because composition is linear.

The point of introducing $\Gamma(F)$ is that we are able to detect bundle homomorphisms purely on the level of our functions.

Lemma 3.91. Fix smooth vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ on the smooth manifold M. Then Γ provides a bijection between $\operatorname{Hom}_M(E, E')$ and $\operatorname{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(E'))$.

Sketch. We begin with the injectivity check. Suppose $F_1, F_2 \in \operatorname{Hom}_M(E, E')$ satisfy $\Gamma(F_1) = \Gamma(F_2)$. Then we want to check that $F_1(v) = F_2(v)$ for any $v \in E$. Say $p := \pi(v)$, and we at least know that $F_1(v), F_2(v) \in E'_p$ because our homomorphisms are over M. Now, choose some section $\sigma\colon M \to E$ such that $\sigma(p) = v$, which exists by some sort of partition of unity argument. Then $F_1 \circ \sigma = F_2 \circ \sigma$ implies $F_1(v) = F_2(v)$.

We now turn to the surjectivity check. Suppose we are given some $\mathcal{F}\colon \Gamma(E) \to \Gamma(E')$ which is $C^\infty(M)$-linear. For any $v \in E$, set $p := \pi(v)$, and we can find some smooth section $\sigma\colon M \to E$ such that $\sigma(p) = v$. Then $\mathcal{F}(\sigma)(p) \in E'$, and we can check using the linearity that we must have $\mathcal{F}(\sigma)(p) \in E'_p$. So we define $F(v) := \mathcal{F}(\sigma)(p)$. It then remains to check that F does not depend on the choice of σ (this is true essentially by the $C^\infty(M)$-linearity requiring that the output of $\mathcal{F}$ cannot really adjust too much) and that F is smooth (which can be checked locally, where the discussion becomes explicit as trivial bundles on Euclidean spaces, so by providing local frames to everything, F basically becomes a matrix whose coefficients are smooth functions by hypothesis on F!). ■

Example 3.92. There is a canonical isomorphism $E \to E^{**}$. Well, on fibers, there is a natural isomorphism $E_p \to E_p^{**}$ given by $v \mapsto (\varphi \mapsto \varphi(v))$. Checking on charts (where the discussion becomes checking some linear map of trivial vector bundles on Euclidean spaces), we see that we are basically sending a global frame to its double dual global frame identically, which is certainly smooth.

3.6.2 Subbundles

A special kind of bundle homomorphism is given by a subbundle. Here is our definition.

Definition 3.93 (subbundle)**.** Fix a smooth vector bundle $\pi\colon E \to M$ on a smooth manifold M. Then a submanifold $D \subseteq E$ is a *subbundle* if and only if $D_p = E_p \circ D$ is a linear subspace for all $p \in M$, and $\pi|_D\colon D \to M$ is a vector bundle with these vector subspaces.

The main point is that we have some injection $D \to E$ of vector bundles.

Example 3.94. Given some linearly independent global sections $\sigma_1, \ldots, \sigma_r \in \Gamma(E)$ for a vector bundle $\pi\colon E \to M$, we see that

$$D := \bigcup_{p \in M} \operatorname{span}\{\sigma_1(p), \ldots, \sigma_r(p)\}$$

is a vector subbundle of E of rank r. Indeed, by construction, all fibers have the correct dimension, and by working locally on charts, we can extend our linearly independent sections to a local frame, whereupon we can build a defining function by asking for the newly added local sections to vanish.

Example 3.95. Fix a bundle homomorphism $F\colon E \to E'$ over M. Assume that F has constant rank. Then

$$\ker F := \bigcup_{p\in M} \ker F|_{E_p} \qquad \text{and} \qquad \operatorname{im} F := \bigcup_{p\in M} \operatorname{im} F|_{E_p}$$

are subbundles. For example, $\ker F$ is a submanifold because it is the pre-image of F of the image of the zero global section $M \subseteq E$. For the image, one fixes a local frame and then uses continuity of F to show that this local frame will preserve its rank in a neighborhood, from which we are able to use the previous example.

Example 3.96. The natural trace homomorphism $\operatorname{tr}\colon V \otimes V^* \to \mathbb{R}$ for an $\mathbb{R}$-vector space V (given by choosing a basis, identifying $V \otimes V^* \cong \operatorname{Hom}_{\mathbb{R}}(V, V)$, and then computing the trace in the usual matrix way) extends to a natural bundle homomorphism

$$\operatorname{tr}\colon E \otimes E^* \to (\mathbb{R} \times M)$$

for any vector bundle $\pi\colon E \to M$. One can show that $E \otimes E^*$ now decomposes into $\ker\operatorname{tr}$ and the span of the global section associated to $\operatorname{id}_E \in \operatorname{Hom}_M(E, E)$.

3.6.3 The Cotangent Bundle

We begin with a general discussion of $\mathbb{R}$-vector spaces V. A basis $\{e_1, \ldots, e_n\}$ of V gives rise to a dual basis $\{e_1^*, \ldots, e_n^*\}$ of V^*, where $e_i^*(e_j) := 1_{i=j}$ by definition. Notably, if we expand $v \in V$ and $w^* \in V^*$ in our basis as $v = \sum_{i=1}^n v_i e_i$ and $w^* = \sum_{i=1}^n w_i e_i^*$, and we find that

$$w^*(v) = \sum_{i=1}^n w_i v_i$$

by definition of e_i^* and linearity.

Remark 3.97. The vector e_1^* is not determined by e_1 alone, as can be seen from its construction. Instead, the full dual basis of V^* is determined by the full basis of V.

Anyway, here is our definition.

Definition 3.98 (cotangent bundle)**.** Fix a smooth manifold M. The *cotangent bundle* T^*M is the dual of the tangent bundle TM.

Remark 3.99. Given $\sigma \in \Gamma(TM)$ and $\tau^* \in \Gamma(T^*M)$, then we produce a smooth function $\tau^*(\sigma) \in C^\infty(M)$. Explicitly, one has

$$\tau^*(\sigma)(p) := \tau_p^*(\sigma_p),$$

which can be checked to be smooth locally on charts in the usual way.

Definition 3.100 (covector field)**.** Fix a smooth manifold M. A *smooth covector field* is a smooth section of the canonical projection $T * M \to M$. We let $\mathfrak{X}^*(M)$ denote the collection of covector fields.

As usual, one can check smoothness of such a section locally on charts.

3.7 April 11

Today we continue talking about the cotangent bundle.

3.7.1 Differentials of a Function

By way of motivation, recall that there is a notion of the gradient ∇f of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$. As we currently understand it, this is a special case of the total derivative df, which we can express as

$$df = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\frac{\partial}{\partial x_i}.$$

However, note that if we change coordinates to $y_\bullet$, then we get

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} = \sum_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}.$$

The problem is that this does not look like $\partial f/\partial y_j$ anywhere.

As such, the correct thing to do is to think of the gradient not as a column vector in $T\mathbb{R}^n$ but as a row covector in $T^*\mathbb{R}^n$. Namely, we would like to define our gradient as

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i,$$

where "$dx_\bullet$" is the dual basis for $\partial/\partial x_i$.

Remark 3.101. Here is another piece of motivation: we might want to imagine a chain rule

$$\frac{d}{dt}f(\gamma(t)) = df(\gamma'(t)),$$

but then df is acting as a map sending column vectors to scalars, so it should really be a row vector in the dual space.

Anyway, here is our definition.

Definition 3.102. Fix a smooth manifold M without boundary. For an open subset $U \subseteq M$ and $f \in C^\infty(M)$, define the *covector* $df_p \in T_p^*M$ by

$$df_p(v) := v(f)$$

for all $v \in T_pM$.

Remark 3.103. Note that the data of df assembles to a "covector field."

Example 3.104. Let's see how this works on local coordinates $(x_1, \ldots, x_n)$ of some open subset $U \subseteq M$. Set $v := \sum_i v_i\partial/\partial x_i$ to be some vector field. Then

$$df(v) = v(f) = \sum_i v_i\frac{\partial f}{\partial x_i}.$$

On the other hand, we could compute

$$\left(\sum_j \frac{\partial}{f}\partial x_j dx_j\right)\left(\sum_i V_i\frac{\partial}{\partial x_i}\right)$$

gives the same answer by direct expansion.

Example 3.105. One sees that $dx_1, \ldots, dx_n$ is a local frame for the vector bundle T^*M on the chart (U, φ) where $\varphi = (x_1, \ldots, x_n)$.

We pick up the following computational lemmas.

Lemma 3.106. Fix an open subset U of a smooth manifold M. Then let $\sigma \in \mathfrak{X}(U)$ be a local vector field, and let $\omega \in \mathfrak{X}^*(U)$ be a local covector field, and let $(x_1, \ldots, x_n)$ be a chart on U. Then

$$\sigma = \sum_i dx_i(\sigma) \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \omega = \sum_j \omega\left(\frac{\partial}{\partial x_j}\right) dx_j.$$

Proof. Simply plug in the basis everywhere. For example, $\sigma(dx_i)$ on both sides reveals what the coordinate of σ should be at the basis vector $\partial/\partial x_i$, which produces the left equality. The right equality is proven similarly. ∎

Lemma 3.107. If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are smooth charts on an open subset U of a smooth manifold M, then

$$\frac{\partial}{\partial y_j} = \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i} \qquad \text{and} \qquad dy_j = \frac{\partial y_j}{\partial x_i} dx_i.$$

Proof. The left equality is already known as our chain rule. The right equality holds by using the previous lemma and noting that

$$dy_j \left(\frac{\partial}{\partial x_i}\right) = \frac{\partial y_j}{\partial x_i}$$

by definition of dy_j. ∎

Remark 3.108. If $\gamma\colon (a, b) \to M$ is a smooth curve, then we see

$$\frac{d}{dt} f(\gamma'(t)) = \gamma'(t)(f) = df\left(\frac{d\gamma}{dt}\right).$$

We then expect that

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b df\left(\frac{d\gamma}{dt}\right) dt$$

as suggested by the notation. We will be able to make more sense of this later.

3.7.2 Pullback

As with differentials, we want to be able to pull back differentials.

Definition 3.109 (pullback)**.** Fix smooth manifolds N and M and a smooth map $F\colon N \to M$. Given $p \in N$ and $\omega \in T_p^*M$ and a smooth map $F\colon N \to M$, we define the *pullback* by

$$(F^*\omega)_p(v) := \omega_{F(p)}(dF_p(v)).$$

This is in fact a linear functional on T_pN and hence provides an element $F^*\omega \in T_p^*N$.

Remark 3.110. One can check that $\omega \in \mathfrak{X}^*(M)$ gets pulled back smoothly to $F^*\omega \in \mathfrak{X}^*(N)$. The smoothness check can be done on charts, as usual.

Remark 3.111. If $f \in C^\infty(M)$, then $F^*(df) = d(f \circ F)$. This can be checked directly: for $p \in N$ and $v \in T_pN$, we compute

$$F^*(df)_p(v) = df_{F(p)}(dF_p(v)) = d(f \circ F)_p(v),$$

as desired.

Remark 3.112. Let's explain a computation. Suppose M has a smooth chart (U, φ) with coordinates $\varphi = (x_1, \ldots, x_m)$. Given a smooth map $F\colon N \to M$ as above and $\omega := \sum_i \omega_i dx_i$ where $\omega_i \in C^\infty(M)$, we compute

$$F^*\omega = F^*\left(\sum_i \omega_i dx_i\right) = \sum_i (\omega_i \circ F) F^* dx_i = \sum_i (\omega_i \circ F) d(x_i \circ F),$$

and here $(x_i \circ F)$ is the ith component $F_i := x_i \circ F$ of F. We can then give local coordinates $(y_1, \ldots, y_n)$ of N and find that

$$F^*\omega = \sum_{i,j} (\omega_i \circ F) \frac{\partial F_i}{\partial y_j} dy_j.$$

Remark 3.113. Given an embedding $i\colon S \to M$ of smooth manifolds, then for $\omega \in \mathfrak{X}^*(M)$, we have $i^*\omega \neq \omega|_S$. Indeed, $\omega|_S$ is defined for all $v \in T_pM$ for $p \in S$, but $i^*\omega$ is only defined for $v \in T_pS$ for $p \in S$.

3.7.3 Line Integrals

We are finally able to integrate. Viewing $[a, b]$ as a 1-manifold with boundary, we let $t := \mathrm{id}_{[a,b]}$ be a coordinate function, and then we get the covector $dt \in \mathfrak{X}^*([a, b])$. Then for any $\omega \in \mathfrak{X}^*([a, b])$ the fact that dt is a global frame (all on its own), we are able to write

$$\omega = g\, dt.$$

We are now able to provide the following definition.

Definition 3.114. Fix a closed interval $[a, b] \subseteq \mathbb{R}$ and some $w \in \mathfrak{X}^*([a, b])$. Choosing $t := \mathrm{id}_{[a,b]}$, we define

$$\int_{[a,b]} \omega := \int_a^b g(t)\, dt,$$

where $\omega = g(t)\, dt$.

As the notation suggests, we would like for our definition to be independent of t.

Proposition 3.115. Fix an increasing diffeomorphism $\varphi\colon [c, d] \to [a, b]$. For $\omega \in \mathfrak{X}^*([a, b])$, we have

$$\int_{[a,b]} \omega = \int_{[c,d]} \varphi^*\omega.$$

Proof. We only sketch the proof. The point is that one can write out $\varphi^*\omega$ and then use a u-substitution. ■

We now move on to integrating on manifolds.

Definition 3.116. Fix a smooth manifold M. Choose a path $\gamma\colon [a,b] \to M$ and covector $\omega \in \mathfrak{X}^*(M)$. Then we define

$$\int_\gamma \omega := \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt.$$

Remark 3.117. Note that

$$\int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt = \int_a^b (\gamma^*\omega)\left(\frac{\partial}{\partial}t\right)\,dt = \int_{[a,b]} \gamma^*\omega,$$

so this definition is independent of the choice of t. Explicitly, replacing γ with $\gamma \circ \varphi$ for some increasing diffeomorphism $\varphi\colon [c,d] \to [a,b]$ implies that

$$\int_{[c,d]} (\gamma \circ \varphi)^*\omega = \int_{[c,d]} \varphi^*(\gamma^*\omega) = \int_{[a,b]} \gamma^*\omega,$$

where the last equality is by Proposition 3.115.

Here is our main theorem.

Theorem 3.118 (Fundamental theorem of line integrals)**.** Fix a smooth manifold M. Given $f \in C^\infty(M)$ and a smooth path $\gamma\colon [a,b] \to M$, we have

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Proof. Unwinding definitions, we find

$$\int_\gamma df = \int_{[a,b]} \gamma^*\,df = \int_{[a,b]} d(f \circ \gamma) = \int_a^b \frac{\partial(f\circ\gamma)}{\partial t}\,dt.$$

Then the fundamental theorem of calculus tells us that this is $f(\gamma(b)) - f(\gamma(a))$. ■

Remark 3.119. This result even holds if γ is only piecewise smooth, as can be seen by breaking up γ into its smooth pieces and summing.

3.7.4 Conservative Vector Fields

As with multivariable calculus, we will want to give some adjectives to covector fields.

Definition 3.120 (exact, conservative)**.** Fix a smooth manifold M.

- We say $\omega \in T^*M$ is *exact* if and only if there exists $f \in C^\infty(M)$ such that $\omega = df$.
- We say $\omega \in T^*M$ is *conservative* if and only if

  $$\int_\gamma \omega = 0$$

  for any piecewise smooth closed curve $\gamma\colon [a,b] \to M$.

Example 3.121. If ω is exact, then Theorem 3.118 tells us that ω is conservative.

In fact, this example has a converse.

Proposition 3.122. Fix a smooth manifold M. Then $\omega \in T^*M$ is conservative if and only if it is exact.

Proof. We know that exact implies conservative by Theorem 3.118. In the other direction, suppose ω is conservative, and we must construct $f \in C^\infty(M)$ with $\omega = df$. We will do this by integrating.

Let $\{M_i\}$ denote the set of connected components of M, and select some $p_i \in M_i$. We begin by setting $f(p_i) := 0$, and then for any $q \in M_i$, we may choose some smooth path $\gamma\colon [a,b] \to M$ such that $\gamma(a) = p_i$ and $\gamma(b) = q$, and then we set

$$f(q) := \int_\gamma \omega.$$

Because ω is conservative, we can concatenate paths to see that f is well-defined. Smoothness can be checked on charts, and then again on coordinates we can check that $\omega = df$. We omit the details of these checks. ■

Example 3.123. Take $\omega := x\,dx + y\,dy$ on $\mathbb{R}^2$. Then one can check that ω is conservative by hand. On the other hand, we can see that $\omega = df$ where $f(x,y) = \frac{1}{2}\left(x^2+y^2\right)$.

Example 3.124. Take $\omega := x\,dy - y\,dx$ on $\mathbb{R}^2$. This is not conservative.

- For example, parameterize the unit circle counter-clockwise by $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ by $\gamma(t) := (\cos t, \sin t)$. Then one can compute $\int_\gamma \omega = 2\pi$ by some explicit integration.
- Alternatively, we can check that ω fails to be exact. Suppose $\omega = df$. Then f needs to satisfy $\partial f/\partial x = -y$ and $\partial f/\partial y = x$, but then

$$\frac{\partial^2 f}{\partial x \partial y} = -1 \neq 1 = \frac{\partial^2 f}{\partial y \partial x}.$$

3.8 April 16

Today we discuss tensor products.

3.8.1 Closed Covector Fields

Here is our definition.

Definition 3.125 (closed)**.** Fix a covector field ω on a smooth manifold M. Then ω is *closed* if and only if any smooth chart (U,φ) where $\varphi = (x_1,\ldots,x_n)$, our expansion $\omega = \sum_i \omega_i\,dx_i$ has

$$\frac{\partial \omega_j}{\partial x_i} = \frac{\partial \omega_i}{x_j}$$

for any i and j.

Example 3.126. Exact forms are closed. Indeed, say $\omega = df$ where $f \in C^\infty(M)$. Then for any smooth chart (U, φ) where $\varphi = (x_1, \ldots, x_n)$ makes

$$\omega = df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, dx_i,$$

so we compute

$$\frac{\partial \omega_j}{\partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial \omega_i}{\partial x_j}.$$

Remark 3.127. There are closed forms which fail to be exact. For example, set $\omega := x\,dy - y\,dx$ on $\mathbb{R}^2$, and pull it back to $i^*\omega$ where $i\colon S^1 \to \mathbb{R}^2$ is the inclusion. Because this is a 1-manifold, this is vacuously closed by the definition. However,

$$\int_\gamma i^*\omega = 2\pi$$

for a path $\gamma\colon [0,1] \to S^1$ going around S^1 by a direct computation (see Example 3.124), so $i^*\omega$ fails to be conservative and hence fails to be exact.

We quickly remove the "any smooth chart" part of the definition.

Lemma 3.128. Fix a covector field ω on a smooth manifold M. Then the following are equivalent.

(i) ω is closed.

(ii) Each $p \in M$ has some smooth chart (U, φ) such that $\varphi = (x_1, \ldots, x_n)$ provides coordinates where $\omega = \sum_i \omega_i \, dx_i$ and

$$\frac{\partial \omega_j}{\partial x_i} = \frac{\partial \omega_i}{x_j}$$

for any i and j.

(iii) For any local vector fields $X, Y \in \mathfrak{X}(U)$ on an open subset U, we have

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]).$$

Proof. Note (i) implies (ii) with no content. To see (iii) implies (i), we work locally on charts. Choose a smooth chart (U, φ) where $\varphi = (x_1, \ldots, x_n)$. Then for distinct indices i and j, we take $X := \partial/\partial x_i$ and $Y := \partial/\partial x_j$ and compute

$$X(\omega(Y)) - Y(\omega(X)) = \frac{\partial}{\partial x_i}\omega\left(\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\omega\left(\frac{\partial}{\partial x_i}\right) = \omega\left(\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]\right) = \omega(0) = 0.$$

Now, writing $\omega = \sum_i \omega_i \, dx_i$, we see that the left-hand side now reads

$$\frac{\partial}{\partial x_i}\omega_j - \frac{\partial}{\partial x_j}\omega_i = 0,$$

as needed.

It remains to show that (ii) implies (iii). Well, we can verify the last equality at points, so choose local vector fields $X, Y \in \mathfrak{X}(U)$ and some point $p \in U$ to verify the equality around, and then we shrink U around p so that we have a smooth chart (U, φ) where $\varphi = (x_1, \ldots, x_n)$. Then we can expand $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and

$Y = \sum_j Y_j \frac{\partial}{\partial x_j}$ and $\omega = \sum_i \omega_i \, dx_i$, so we are now able to compute

$$\begin{aligned} X(\omega(Y)) &= \sum_{i,j} X_i \frac{\partial}{\partial x_i}(\omega_j Y_j) \\ &= \sum_{i,j} X_i Y_j \frac{\partial \omega_j}{\partial x_i} + X_i \omega_j \frac{\partial Y_j}{\partial x_i} \\ Y(\omega(X)) &= \sum_{i,j} Y_i \frac{\partial}{\partial x_i}(\omega_j X_j) \\ &= \sum_{i,j} Y_i X_j \frac{\partial \omega_j}{\partial x_i} + Y_i \omega_j \frac{\partial X_j}{\partial x_i}. \end{aligned}$$

We now see that subtracting makes the left terms of the sum cancel, so we are left with $\omega([X,Y])$, as required. ■

To close up our discussion of closed covector fields, we do note that simple spaces will be able to show that closed implies exact.

Proposition 3.129. Fix an open star-like open subset $U \subseteq \mathbb{R}^n$. Then any closed local covector field $\omega \in \mathfrak{X}^*(U)$ is exact.

Here, "star-like" means that there is a point $p \in U$ such that the line segment connecting p to any other $p' \in U$ is contained in U.

Proof. By translating, we may as well assume that U is star-like with the "center point" just the origin $0 \in \mathbb{R}^n$. Now, one has a global frame $\omega = \sum_i \omega_i \, dx_i$. Imitating the proof of Proposition 3.122, we define our function $f \colon U \to \mathbb{R}$ by

$$f(x) := \int_{\gamma_x} \omega$$

where γ_x is the straight-line line segment $\gamma_x(t) := tx$ from 0 to x. As such, we see that

$$f(x) = \sum_{i=1}^n \int_0^1 \omega_i(tx) x_i \, dt$$

We omit the check that f is smooth, which can be seen basically because each ω_i is smooth, and integration preserves smoothness (because one can integrate under the integral sign). We would like to check that $\omega = df$, so for any x_j, we compute

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x) &= \int_0^1 \sum_{i=1}^n \frac{\partial \omega_i}{\partial x_j}(tx) \cdot tx_i + \omega_j(tx) \, dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial \omega_j}{\partial x_i}(tx) \cdot tx_i + \omega_j(tx) \, dt \\ &= \int_0^1 \frac{\partial \omega_j}{\partial t}(tx) \cdot t + \omega_j(tx) \, dt \\ &= \int_0^1 \frac{\partial}{\partial t}(t\omega_j(tx)) \, dt \\ &= t\omega_j(tx) \bigg|_{t=0}^{t=1} \\ &= \omega_j(x), \end{aligned}$$

as required. ■

Remark 3.130. A more direct modification of the proof of Proposition 3.122 shows that any closed covector field on a simply connected manifold is exact. We won't bother to write this out; the main point is to check that the definition is well-defined up to the choice of path from a given basepoint to a given point in the (simply connected!) manifold.

3.8.2 Tensors

We begin by discussing tensor products on vector spaces, which we will upgrade to vector bundles.

Definition 3.131 (tensor)**.** Fix a finite-dimensional $\mathbb{R}$-vector space V and nonnegative integers k and ℓ. Then we define the space of *tensors* as

$$T^{(k,\ell)}V := V^{\otimes k} \otimes (V^*)^{\otimes \ell}.$$

By way of convention, a *covariant tensor* is an element of $T^{(0,\ell)}V$ for some ℓ, and a *contravariant tensor* is an element of $T^{(k,0)}V$ for some k.

By the universal property of the tensor product (and the identification $V \simeq V^{**}$), we can think about an element of $T^{(k,\ell)}$ as a multilinear map $(V^*)^k \times V^\ell \to \mathbb{R}$.

Example 3.132. Here are some special cases.

- By convention, $T^{(0,0)}V = \mathbb{R}$.
- $T^{(1,0)}V = V$.
- $T^{(0,1)}V = V^*$.
- $T^{(1,1)}V = \operatorname{End} V$. One simply sends $v \otimes v^*$ to the linear map $V \to V$ given by $w \mapsto v^*(w)v$. Certainly this map is linear, and one can show that it is an isomorphism by working on a basis.

Remark 3.133. A basis $\{e_1, \ldots, e_n\}$ of V produces a dual basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of V^*. This produces a basis of $T^{(k,\ell)}$ of tensors of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_\ell}.$$

Remark 3.134. As usual, there are permutation morphisms $T^{(k,\ell)}V \to T^{(k,\ell)}V$ given by permuting the factors in $V \otimes \cdots \otimes V$ or in $V^* \otimes \cdots \otimes V^*$.

Remark 3.135. There is a canonical isomorphism $T^{(k,\ell)}V \otimes T^{(k',\ell')}V \to T^{(k+k',\ell+\ell')}V$, which then comes from a bilinear map

$$T^{(k,\ell)}V \times T^{(k',\ell')}V \to T^{(k+k',\ell+\ell')}V$$

given by "attaching" tensors.

Remark 3.136. There is a trace map $\operatorname{tr}\colon T^{(1,1)}V \to \mathbb{R}$; for example, on the basis $\{e_1, \ldots, e_n\}$ of V, this is given by

$$\sum_{i,j=1}^n t_{ij} e_i \otimes \varepsilon_j \mapsto \sum_{i=1}^n t_{ii}.$$

This is basis-free, which can be checked directly or seen because trace is the map $v \otimes v^* \mapsto v^*(v)$ on pure tensors of $V \otimes V^*$.

Remark 3.137. More generally, there is a "contraction" map $C_{ij}\colon T^{(k,\ell)}V$ where $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,\ell\}$ given by

$$(v_1 \otimes \cdots \otimes v_k) \otimes (v_1^* \otimes \cdots \otimes v_\ell^*) \mapsto v_j^*(v_i)(v_1 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_k) \otimes (v_1^* \otimes \cdots \otimes \widehat{v}_j \otimes \cdots \otimes v_\ell^*)$$

on pure tensors. One can expand this out on a basis as before, in particular finding that there is a "diagonal sum" hiding.

Example 3.138. Given $A \in T^{(0,k)}V$, we can produce the multilinear map $V^k \to \mathbb{R}$ as the repeated contraction

$$(v_1,\ldots,v_k) \mapsto C_{11}\cdots C_{kk}(v_1\cdots v_kA).$$

By expanding everything out on the basis, this is the usual identification of A with a multilinear map $V^k \to \mathbb{R}$.

We now upgrade to tensor products for vector bundles.

Definition 3.139 (tensor bundle)**.** Fix a smooth manifold M and nonnegative integers k and ℓ. Then given a vector bundle V on M, we define the *tensor bundle* as

$$T^{(k,\ell)}V := V^{\otimes k} \otimes (V^*)^{\otimes \ell}.$$

A global section of $T^{(k,\ell)}TM$ is called a *tensor field*; a global section of $T^{(0,\ell)}TM$ is called a *covariant tensor field*.

Example 3.140. As before, we find that $T^{(0,0)}V = M \times \mathbb{R}$ and $T^{(1,0)}V = V$ and $T^{(0,1)}V = V^*$.

Remark 3.141. Given a smooth chart (U,φ) with $\varphi = (x_1,\ldots,x_n)$, one can provide a local frame for TM and T^*M as $\{\partial/\partial x_1,\ldots,\partial/\partial x_n\}$ and $\{dx_1,\ldots,dx_n\}$, respectively. Thus, we get a local frame of $T^{(k,\ell)}TM$ by

$$\frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_k}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_\ell}.$$

Something similar works using trivializing open subsets of a more general vector bundle.

Remark 3.142. A covariant tensor field $A \in \Gamma\left(T^{(0,\ell)}TM\right)$ can be viewed as a multilinear form $\mathfrak{X}(M)^\ell \to C^\infty(M)$ given by

$$(X_1,\ldots,X_\ell) \mapsto C_{11}\cdots C_{kk}(X_1\cdots X_\ell A).$$

One can expand this out on coordinates in the typical way.

3.9 April 18

Today we discuss Riemannian metrics.

3.9.1 More on Tensors

Let's discuss covariant tensor fields more explicitly. Quickly, we note that tensors have some notion of $C^\infty(M)$-multilinearity. Explicitly, a smooth covariant tensor field $A \in \Gamma\left(T^{(0,\ell)}TM\right)$ amounts to $C^\infty(M)$-multilinear map

$$\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

by sending $(X_1, \ldots, X_\ell) \mapsto A(X_1, \ldots, X_\ell)$ via the usual identification $(V \otimes \cdots \otimes V)^* = V^* \otimes \cdots \otimes V^*$. This turns out to characterize our tensors.

Proposition 3.143. Fix a smooth manifold M. Given a $C^\infty(M)$-multilinear map $\mathcal{A}\colon \mathfrak{X}(M)^\ell \to C^\infty(M)$, then there is a unique covariant tensor field $A \in \Gamma\left(T^{(0,\ell)}TM\right)$ such that $\mathcal{A}$ comes from A.

Sketch. Let's be brief.

1. We begin by using the $C^\infty(M)$-multilinearity to show that $\mathcal{A}$ is "local." Explicitly, if $(X_1, \ldots, X_\ell)$ and $(X_1', \ldots, X_\ell')$ agree locally in a neighborhood of some point $p \in U$, then

$$\mathcal{A}(X_1, \ldots, X_\ell) = \mathcal{A}(X_1', \ldots, X_\ell').$$

   The point is to use smooth cutoff functions to compare these two values.

2. Next up, we can show that $\mathcal{A}(X_1, \ldots, X_\ell)$ only depends on the data of $(X_1(p), \ldots, X_\ell(p))$ by expanding out locally; here, one uses the $C^\infty(M)$-multilinearity more crucially to compare $X_\bullet$ and $X_\bullet'$. This constructs the (rough) section $A\colon M \to T^{(0,\ell)}TM$.

3. Then one shows that A is in fact smooth by working in local coordinates. ∎

Next up, we discuss how change of coordinates happens to a tensor field $A \in \Gamma\left(T^{(k,\ell)}TM\right)$. Well, suppose we have some point p contained in the two smooth charts (U, φ) and (V, ψ) with $\varphi = (x_1, \ldots, x_n)$ and $\psi = (y_1, \ldots, y_n)$. In these local coordinates, one can write

$$A = \sum A^{i_1, \ldots, i_k}_{j_1, \ldots, j_\ell} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes dx_{j_\ell},$$

and then the change of coordinates formulae for the individual differentials and covectors extends via the tensor product. Explicitly,

As another remark, we discuss pullbacks of covariant tensor fields.

Definition 3.144 (pullback)**.** Fix a smooth manifold M and a covariant tensor field $A \in \Gamma\left(T^{(0,\ell)}TM\right)$. For a smooth map $F\colon N \to M$, we define the *pullback covariant tensor field* F^*A by

$$(F^*A)(v_1, \ldots, v_\ell) \coloneqq A_{F(p)}(dF_p(v_1), \ldots, dF_p(v_\ell)).$$

We won't bother to check that this map is smooth, but it is; roughly speaking, we are taking the composite of the smooth functions A and dF.

One can also take Lie derivatives.

Definition 3.145 (Lie derivative)**.** Fix a smooth manifold M and a smooth covariant tensor field $A \in \Gamma\left(T^{(0,\ell)}TM\right)$. For a vector field $V \in \mathfrak{X}(M)$, let $\theta_\bullet$ be the flow of V, and we define the *Lie derivative*

$$\mathcal{L}_v A \coloneqq \frac{d}{dt}\theta_t^* A\bigg|_{t=0}.$$

Explicitly, we see

$$(\mathcal{L}_V A)_p(v_1, \ldots, v_\ell) = \frac{d}{dt} A_{\theta_t(p)}((d\theta_t)_p(v_1), \ldots, (d\theta_t)_p(v_\ell))\bigg|_{t=0}.$$

The intuition here is exactly the same as what was done for just covector fields.

Remark 3.146. Suppose that we have managed to get $V = \partial/\partial x_1$ locally (which is always doable), where $A = \sum A_{i_1,\ldots,i_\ell}\, dx_{i_1} \otimes \cdots \otimes dx_{i_\ell}$. Then the flow is constantly moving in the x_1 direction, so we see that

$$\mathcal{L}_V A = \sum \frac{A_{i_1,\ldots,i_\ell}}{\partial x_1}\, dx_{i_1} \otimes \cdots \otimes dx_{i_\ell}.$$

Note $\mathcal{L}_V A \in \Gamma\left(T^{(0,\ell)}TM\right)$ by some $C^\infty(M)$-multilinearity check.

Remark 3.147. Viewing $f \in C^\infty(M)$ as a covariant tensor field in $\Gamma\left(T^{(0,0)}TM\right)$, we see that $(\mathcal{L}_V f)_p = V_p(f)$. For example, one can track through the previous remark to see this.

Remark 3.148. One has a "Leibniz rule": for $X_1, \ldots, X_\ell \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_V(A(X_1,\ldots,X_\ell)) = \mathcal{L}_V A(X_1,\ldots,X_\ell) + A(\mathcal{L}_V X_1, X_2, \ldots, X_\ell) + \cdots + A(X_1,\ldots,\mathcal{L}_V X_\ell).$$

3.9.2 Riemannian Metrics

As motivation, we note that the length of a curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is computed as

$$\ell(\gamma) = \int_a^b |\gamma'(t)|^2\, dt.$$

Here, $|\gamma'(t)|^2$ is a norm of the derivative, so if we want to generalize this notion to a manifold, we need a notion of a norm on our tangent spaces. It turns out that norms (with enough structure, namely a parallelogram law) must come from bilinear forms, so we may as well ask for our tangent spaces to have a bilinear form. In Euclidean space, this is easy because $T\mathbb{R}^n$ has a global frame, so we may just identify all $T_p\mathbb{R}^n$s with $\mathbb{R}^n$ and then use the standard inner product on $\mathbb{R}^n$, but in general it may not be so easy to produce a good inner product everywhere.

A good choice of inner product everywhere is essentially the data of a Riemannian metric.

Definition 3.149 (Riemannian metric)**.** Fix a smooth manifold M. A *Riemannian metric* on M is a smooth covariant tensor field $g \in \Gamma\left(T^{(0,2)}TM\right)$ such that each $p \in M$ makes $g_p\colon T_pM \times T_pM \to \mathbb{R}$ induce a symmetric positive-definite inner product on T_pM. We will write $\langle\cdot,\cdot\rangle_g := g(\cdot,\cdot)$ and $|\cdot|_g := \sqrt{\langle\cdot,\cdot\rangle}$. (We will suppress the g from our notation as much as possible.) A *Riemannian manifold* is a pair (M,g) of a smooth manifold M equipped with a Riemannian metric g.

Remark 3.150. Let's explain how this looks on a smooth chart (U,φ) where $\varphi = (x_1,\ldots,x_n)$. Then g can be expanded out as on coordinates as

$$g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j,$$

so being symmetric and positive-definite corresponds to the same adjectives on the matrix $G := \{g_{ij}\}$. Now, if we have some differentials $u = \sum_i u_i \frac{\partial}{\partial x_i}\big|_p$ and $v = \sum_j v_j \frac{\partial}{\partial x_j}\big|_p$, we see

$$g_p(u,v) = \sum_{i,j} u_i v_j g_{ij}(p).$$

Example 3.151. On $\mathbb{R}^n$, our standard Riemannian metric is given by

$$\sum_{i=1}^{n} dx_i \otimes dx_i.$$

We would like to show that Riemannian metrics exist in general. It will be helpful to have a little freedom in our construction, as follows.

Lemma 3.152. Fix an immersion $F\colon N \to M$ of smooth manifolds. If g is a Riemannian metric on M, then F^*g is a Riemannian metric on N.

Proof. Certainly F^*g is smooth because it is the pullback of something smooth, so we only need to do the other checks. At each $p \in N$, we compute

$$(F^*g)_p(u,v) = g_{F(p)}(dF_p(u), dF_p(v))$$

for any $u, v \in T_pM$. This is certainly symmetric and bilinear because $g_{F(p)}$ is. Additionally, this is certainly nonnegative because $g_{F(p)}$, and it is positive-definite because $(F^*g)_p(u,v)$ implies $dF_p(u) = dF_p(v) = 0$ (because g is positive-definite), which implies $u = v = 0$ because F is an immersion! ■

Example 3.153. For any embedded submanifold $S \subseteq M$, let $i\colon S \to M$ be the embedding. Then if g is a Riemannian metric on M, we see that i^*g is a Riemannian metric on S. For example, any smooth manifold can be embedded into Euclidean space by the Whitney embedding theorem 2.119, so any smooth manifold

Remark 3.154. One can avoid using Theorem 2.119 to show that Riemannian metrics exist. Indeed, one can more directly note that any smooth chart produces a "local" Riemannian metric, which extends a smooth covariant tensor field which is symmetric and bilinear but perhaps only positive-definite in a neighborhood. Then one can glue these local almost Riemannian metrics together via partition of unity. The main check here is that we can take convex linear combinations of positive-definite forms to get another positive-definite form, which can be checked directly (and indeed, is just linear algebra on the tangent spaces).

Remark 3.155. We have in fact produced some new structure: it is possible to prove two Riemannian metrics g_1 and g_2 on a smooth manifold M which are distinct, and worse, it is possible for there to be no diffeomorphism $F\colon M \to M$ such that $g_1 = F^*g_2$!

3.9.3 Metrics from Riemannian Metrics

We now provide a notion of distance for our manifolds.

Definition 3.156 (length). Fix a Riemannian manifold (M, g). Given a (piecewise) C^1 curve $\gamma\colon [a,b] \to M$, we define the *length* of γ to be

$$\ell_g(\gamma) := \int_a^b |\gamma'(t)|_g^2 \, dt.$$

We will suppress the g from our notation as much as possible.

Remark 3.157. One can check that $\ell_g(\gamma)$ is independent of reparameterization by the usual arguments. Namely, if $\varphi\colon [c,d] \to [a,b]$ is an increasing piecewise C^1 path, then one can show that $\ell_g(\gamma) = \ell_g(\gamma \circ \varphi)$. Indeed, by breaking up into intervals, it suffices to handle the smooth case, and then an argument with some u-substitution grants the equality.

Definition 3.158. Fix a Riemannian manifold (M, g). Then we define

$$d_g(p,q) := \inf\left\{\ell_g(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ path } [a,b] \to M, \gamma(a) = p, \gamma(b) = q\right\}.$$

Remark 3.159. The infimum in this definition need not be achieved. For example, in $M := \mathbb{R}^2 \setminus \{(0,0)\}$, there is no path achieving the smallest possible distance between $(-1,0)$ and $(1,0)$. It turns out that this minimum is in fact always achieved as long as M is complete as a metric space; these minimal curves are called "geodesics."

And here are our checks.

Theorem 3.160. Fix a Riemannian manifold (M, g). Then d_g is a metric on M, and it induces the topology on M.

Proof. Let's quickly discuss some of these checks.

- $d_g(x,x) = 0$ holds by using the constant path.
- $d_g(x,y) > 0$ for $x \neq y$ holds because $x \neq y$ requires any piecewise C^1 path $\gamma\colon [a,b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$ to have $|\gamma'(t)|^2$ positive on a set of positive measure.
- Symmetry holds by taking any path in one direction and reversing it to get a path in the opposite direction.
- The triangle inequality $d(x,y) + d(y,z) \geq d(x,z)$ is achieved by taking any path from x to y and path from y to z and attaching them (piecewise!) to produce a path from x to z.

So we have a metric. It remains to show that d_g induces the topology on M. This is a local question on M, so it suffices to work locally in a chart, meaning that we may assume that $M = \mathbb{R}^n$. But now it turns out that any two Riemannian metrics on $\mathbb{R}^n$ induces the same topology by some bounding argument, so we are done because the standard Riemannian metric on $\mathbb{R}^n$ does induce the correct topology. ■

Remark 3.161. One can "recover" the Riemannian metric from the length of curves. Morally, one can take the derivative of length in a direction at a point $p \in M$ to recover the norm on T_pM induced by g, which recovers the inner product on T_pM.

3.9.4 More on Riemannian Metrics

Quickly, recall that a choice of inner product $\langle\cdot,\cdot\rangle$ on a finite-dimensional vector space V defines an isomorphism $(\cdot)^\flat\colon V \to V^*$ via $v \mapsto \langle v, \cdot\rangle$; we let the inverse isomorphism be $(\cdot)^\sharp$. This extends smoothly to provide a $C^\infty(M)$-linear isomorphism between $\mathfrak{X}(M) \to \mathfrak{X}^*(M)$ for any smooth manifold M, which is basically the following result.

Proposition 3.162. Fix a Riemannian manifold (M, g). There is a vector bundle isomorphism TM and T^*M.

Proof. We define our vector bundle isomorphism $(\cdot)^\flat\colon TM \to T^*M$ by

$$v^\flat(w) := g_p(v,w)$$

for any $p \in M$ and $v, w \in T_pM$. One can check that this is smooth by expanding out everything in terms of coordinates. Further, one can directly check that we have defined a homomorphism of vector bundles, so it remains to check that we have a diffeomorphism, which again can be checked locally on coordinates (because we are just doing some linear maps everywhere). ■

As an aside, we note that it is not always possible to choose coordinates on a Riemannian manifold (M,g) so that g is locally $\sum_i dx_i \otimes dx_i$. I didn't really follow the discussion on curvature in class.

3.10 April 23

We begin class continuing our discussion of Riemannian manifolds.

3.10.1 Curvature and Connections

Warning 3.163. The following content is unlikely to be on the exam.

Fix a Riemannian manifold (M,g). While we're here, we note that there is a notion of curvature for manifolds, which is basically a tensor $\operatorname{Rm}_g \in T^{(0,4)}TM$, which on vectors of the form (v,w,v,w) in some T_pM outputs the curvature of the manifold with respect to the plane spanned by v and w.

Remark 3.164. Suppose (M,g) is a compact Riemannian manifold. Suppose that $\operatorname{Rm}_g(v,w,v,w)$ is a constant K when v and w are orthonormal.

- If $K=0$, then M is a torus $\mathbb{R}^n/\Gamma$.
- If $K>0$, then M is a quotient of the n-sphere S^n.
- If $K<0$, then M is a quotient of hyperbolic space $\mathbb{H}^n$.

We now quickly discuss connections. Fix some $W \in \mathfrak{X}(M)$ and $v \in T_pM$. We would like a notion of Lie derivative $(D_vW)_p \in T_pM$.

Remark 3.165. One attempt at this was to take the Lie derivative with respect to some $V \in \mathfrak{X}(M)$, where V extends v. However, $\mathcal{L}_VW = [V,W]$ will depend on the choice of extension V of v! The point is that V determines a flow, and we are really taking a derivative via this flow.

However, now that we have a Riemannian metric, we will be able to define $(D_vW)_p$ in a way that does not depend on the extension.

As usual, one can choose a curve $\gamma\colon(-\varepsilon,\varepsilon)\to M$ with $\gamma(0)=p$ and $\gamma'(0)=v$. The central problem is to compare $\gamma'(0)=T_pM$ with $\gamma'(t)\in T_{\gamma(t)}M$ for some small t. For this, one wants to define a "parallel transport" to move around tangent spaces which preserves the inner product (i.e., preserves lengths and angles) and is "torsion-free" in the sense that it does not move around local frames. Letting $\mathcal{P}$ denote this parallel transport, we could then define our directional derivative as

$$\nabla_V W := \lim_{t\to 0}\frac{\mathcal{P}^{-1}_{\gamma(t)}(W_{\gamma(t)}) - W_p}{t}.$$

Instead, we will codify what our directional derivative is, and this turns out to provide our parallel transport.

Definition 3.166 (Levi–Civita connection)**.** Fix a Riemannian manifold (M, g). Then the *Levi–Civita connection* is the unique map

$$\nabla \colon \Gamma(TM) \to \Gamma(T^*M \otimes TM)$$

satisfying the following.

- Linearity and Leibniz rule:

$$\nabla_V(f_1 W_1 + W_2) = f_1 \nabla_V W_1 + V(f_1)W + \nabla_V W_2.$$

- Preserves the inner product g:

$$V(g(X, Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y).$$

- Torsion-free:

$$\nabla_X Y - \nabla_Y = [X, Y].$$

Remark 3.167. One can use the Levi–Civita connection ∇ in order to define curvature. Professor Chen wrote out the formula, but I didn't really follow its construction.

Remark 3.168. Doing parallel transport around a loop has no need to be the identity (e.g., imagine going around a loop of the sphere, but any reasonable way to parallel transport will have a problem at some poles). So parallel transport can send closed loops $\gamma \colon [a, b] \to M$ with $p := \gamma(a) = \gamma(b)$ will induce a map $T_pM \to T_pM$; in fact, the action must preserve the inner product. The point is that we see that smooth loops give an orthogonal group action on $\mathrm{O}(T_pM)$. Curvature, roughly speaking, is the failure of small loops to fix these tangent spaces.

3.10.2 Alternating Forms

We now shift gears and start discussing differential forms.

Definition 3.169 (alternating)**.** Fix a finite-dimensional (real) vector space V. An *alternating k-form* is a functional $\alpha \in (V^*)^{\otimes k} \cong \left(V^{\otimes k}\right)^*$ such that

$$\alpha(v_{\sigma 1}, \ldots, v_{\sigma k}) = \operatorname{sgn}(\sigma)\alpha(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in V$ and $\sigma \in S_k$.

Remark 3.170. This definition does not immediately imply that everything should vanish because the map $\operatorname{sgn} \colon S_k \to \{\pm 1\}$ is a homomorphism. Namely, permuting by σ and then permuting by τ will have the same effect on the sign as permuting by $\tau\sigma$. We let $\wedge^k(V^*)$ denote the space of alternating k-forms.

Lemma 3.171. Fix a finite-dimensional (real) vector space V and some functional $\alpha \in (V^*)^{\otimes k} \cong \left(V^{\otimes k}\right)^*$. Then the following are equivalent.

(i) $\alpha \in \wedge^k(V^*)$.

(ii) $\alpha(v_1, \ldots, v_k) = 0$ if $\{v_1, \ldots, v_k\}$ is linearly dependent.

(iii) $\alpha(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for any distinct i and j.

Proof. Certainly (a) implies (c) by using the transposition σ swapping i and j, and then (c) implies (a) by building up arbitrary permutations from transpositions. Then (c) implies (b) by using linearity to reduce to the equality case, and (b) implies (c) because having equal vectors requires linear dependence. ■

Remark 3.172. There is a projection operator $\operatorname{Alt}\colon (V^*)^{\otimes k} \to \wedge^k(V^*)$ given by

$$(\operatorname{Alt}\alpha)(v_1,\ldots,v_k) := \frac{1}{k!}\sum_{\sigma\in}$$

One can show that Alt does actually output to $\wedge^k(V^*)$ by using our group action, so $\operatorname{Alt}\alpha = \alpha$ implies $\alpha \in \wedge^k(V^*)$. Conversely, one can check that $\alpha \in \wedge^k(V^*)$ implies $\operatorname{Alt}\alpha = \alpha$ again by a computation with the group action.

Example 3.173. Here are some small computations. Given V the basis $\{e_1,\ldots,e_n\}$ so that V^* has dual basis $\{\varepsilon_1,\ldots,\varepsilon_n\}$.

- We have $\wedge^0(V^*) = \mathbb{R}$ because all constants are alternating.
- We have $\wedge^1(V^*) = V^*$ because all functionals are alternating. (Namely, for these two computations, there is nothing to check.)
- One can check that $\alpha := \sum_{i,j} a_{ij}(\varepsilon_i \otimes \varepsilon_j) \in \wedge^2(V^*)$ if and only if $\alpha(e_i,e_j) = -\alpha(e_j,e_i)$ for all i and j (by extending this identity linearly to all V), which is equivalent to $a_{ij} = -a_{ji}$ for all i and j.

Example 3.174. There is an element $\det \in \wedge^n(\mathbb{R}^n)$ given by

$$\det(v_1,\ldots,v_n) := \det\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

To provide a simple basis for $\wedge^k(V^*)$, we would some basic elements in there.

Definition 3.175 (elementary alternating tensor)**.** Fix a real vector space V with basis $\{e_1,\ldots,e_n\}$ so that we have a dual basis $\{\varepsilon_1,\ldots,\varepsilon_n\}$ on V^*. Given a sequence $I = \{i_1,\ldots,i_k\}$ of k elements in $\{1,\ldots,n\}$, we define $\varepsilon^I\colon V^k \to \mathbb{R}$ by

$$\varepsilon_I(v_1,\ldots,v_k) := \det\begin{bmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{bmatrix}.$$

One can check that ε^I is multilinear (because $\det$ is multilinear), and in fact ε^I is alternating (again, because $\det$ is alternating).

Example 3.176. On $\mathbb{R}^n$, we have $\varepsilon_I = \det$ when $I = \{1,\ldots,n\}$.

Many of our elementary alternating tensors are the same as each other, so we want a way to declare them the same.

Notation 3.177. Given two sequences $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$, we define

$$\delta_{I,J} := \det \begin{bmatrix} 1_{i_1=j_1} & \cdots & 1_{i_1=j_k} \\ \vdots & \ddots & \vdots \\ 1_{i_k=j_1} & \cdots & 1_{i_k=j_k} \end{bmatrix}.$$

Remark 3.178. One sees that $\delta_{I,J} \neq 0$ if and only if I nor J have any repeated indices (for then we would see two of the same row or column) and each element of I lies in J, meaning that I is a permutation of J.

And here is our proposition.

Proposition 3.179. Fix an n-dimensional real vector space V with basis $\{e_1, \ldots, e_n\}$ so that V^* has dual basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $\wedge^k(V^*)$ has basis given by ε^I where I is a strictly increasing sequence in $\{1, \ldots, n\}$.

Proof. This is some long computation in linear algebra, so we omit the proof. Essentially, we want to show that $\alpha \in \wedge^k(V^*)$ is uniquely a sum of the given ε^I. Well, by being multilinear, α is uniquely determined by its values $\alpha(e_{i_1}, \ldots, e_{i_k})$ where $\{i_1, \ldots, i_k\}$ is some sequence in $\{1, \ldots, n\}$. By being alternating, we may assume that these indices are strictly increasing, but we are now free to set the values $\alpha(e_{i_1}, \ldots, e_{i_k})$. ■

Remark 3.180. Computing the size of our basis, we see that

$$\dim \wedge^k(V^*) = \begin{cases} \binom{n}{k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Example 3.181. Note $\dim \wedge^n(V^*) = \binom{n}{n} = 1$ when $\dim V = n$, so $\wedge^n(V^*)$ is spanned by the single "signed volume form" $\det$.

Remark 3.182. For $\omega \in \wedge^n(V^*)$ and $T \colon V \to V$, one can show that

$$\omega(Tv_1, \ldots, Tv_n) \stackrel{?}{=} (\det T)\omega(v_1, \ldots, v_n).$$

Because $\dim \wedge^n(V^*) = 1$, we may write ω as $c \det$ for some $c \in \mathbb{R}$. After removing this c, we are trying to show

$$\det \begin{bmatrix} | & & | \\ Tv_1 & \cdots & Tv_n \\ | & & | \end{bmatrix} = (\det T) \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

This is exactly the content of $\det(AB) = (\det A)(\det B)$ for matrices A and B.

3.10.3 Some Products

Here is our definition.

Definition 3.183 (wedge product)**.** Fix a finite-dimensional (real) vector space V. For nonnegative integers k and ℓ, we define $\wedge\colon\ \wedge^k(V^*)\times\wedge^k(V^*)\to\wedge^{k+\ell}(V^*)$ by

$$\omega\wedge\eta:=\frac{(k+\ell)!}{k!\ell!}\operatorname{Alt}(\omega\otimes\eta).$$

Example 3.184. For $\omega,\eta\in\wedge^1(V^*)$, we can compute

$$\omega\wedge\eta=(\omega\otimes\eta-\eta\otimes\omega).$$

Remark 3.185. By expanding out the Alt, one finds that

$$(\omega\wedge\eta)(v_1,\ldots,v_{k+\ell})=\frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}\omega(v_{\sigma 1},\ldots,v_{\sigma k})\eta(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)}).$$

Lemma 3.186. Fix a basis $\{e_1,\ldots,e_n\}$ of a vector space V so that there is a dual basis $\{\varepsilon_1,\ldots,\varepsilon_n\}$ of V^*. Then for sequences of indices $I=\{i_1,\ldots,i_k\}$ and $J=\{j_1,\ldots,j_\ell\}$, we have

$$\varepsilon_I\wedge\varepsilon_J=\varepsilon_{I\sqcup J}.$$

Proof. Direct computation with the definitions. For example, one can use Remark 3.185 to compute the value of $(\varepsilon_I\wedge\varepsilon_J)(e_{i_1},\ldots,e_{i_{k+\ell}})$ on strictly increasing sequences $\{i_1,\ldots,i_{k+\ell}\}$ to verify the equality. ■

Remark 3.187. One can show that $\wedge$ distributes over addition and is associative. It is anti-commutative in the sense that

$$\omega\wedge\eta=(-1)^{k\ell}(\eta\wedge\omega)$$

where $\omega\in\wedge^k(V^*)$ and $\eta\in\wedge^k(V^*)$.

Having a product structure now provides a ring.

Definition 3.188 (exterior algebra)**.** Fix a finite-dimensional vector space V. Then we define

$$\wedge^*(V^*):=\bigoplus_{k=0}^n\wedge^k(V^*)$$

to be an anti-commutative graded $\mathbb{R}$-algebra when equipped with the wedge product.

Remark 3.189. One can compute that

$$\dim\wedge^*(V^*)=\sum_{k=0}^n\dim\wedge^k(V^*)=\sum_{k=0}^n\binom{n}{k}=2^n.$$

While we're here, we note that there is a notion of "interior" multiplication.

Definition 3.190. Fix a vector space V. Given $v\in V$, there is a map $\iota_v\colon\ \wedge^k(V^*)\to\wedge^{k-1}(V^*)$ given by

$$\iota_v(w)(v_2,\ldots,v_k):=w(v,v_2,\ldots,v_k).$$

We may write $\iota_v(w)$ as $v\lrcorner w$.

Remark 3.191. One can show that $\iota_v \circ \iota_v = 0$ and $\iota_v(\omega \wedge \eta) = \iota_v(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_v(\eta)$.

3.11 April 25

Today we talk about differential forms.

Remark 3.192. The final exam will cover chapters 1 through 16, though there will be basically nothing on chapter 15 other than the statement that some manifolds have orientations.

Here is basically everything we will need to know about orientations.

Definition 3.193. Fix a smooth manifold M. An *orientation* on M is a minimal smooth atlas $\mathcal{A}$ such that the determinants of the transition maps are positive.

It turns out that a smooth manifold has an orientation if and only if it has a nowhere-vanishing volume form.

3.11.1 Differential Forms

Here is our definition.

Definition 3.194 (differential form)**.** Fix a smooth manifold M, possibly with boundary. Then a *differential form* ω is a global section of the vector bundle $\Omega^k(M) := \wedge^k T^* M$.

Remark 3.195. Here, $\wedge^k E$ for a vector bundle E on M is defined using the usual construction. For example, we can construct it as an "alternating" subbundle of $E^{\otimes k}$ cut out by the requirements of being alternating. Notably, a smooth chart (U, φ) with $\varphi = (x_1, \ldots, x_n)$ gives rise to a local frame given by the sections of the form

$$dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $I = \{i_1, \ldots, i_k\}$ is an increasing sequence, so $\wedge^k E$ will have rank $\binom{n}{k}$ where $n = \operatorname{rank} E$. This allows us to write any $\omega \in \Omega^k(M)$ as

$$\omega = \sum_I \omega_I dx_I,$$

where the sum varies over increasing sequences $I \subseteq \{1, \ldots, n\}$.

Example 3.196. Note $\Omega^0(M) = C^\infty(M)$ because we are asking for global sections of the trivial line bundle $M \times \mathbb{R}$.

Example 3.197. Note $\Omega^1(M) = \Gamma(T^* M) = \mathfrak{X}^*(M)$.

Example 3.198. As usual, let's do the usual computation on change of coordinates. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two systems of coordinates about some $p \in M$. Then we can compute

$$dy_1 \wedge \cdots \wedge dy_n = \left(\sum_{i_1} \frac{\partial y_1}{\partial x_{i_1}} dx_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n} \frac{\partial y_n}{\partial x_{i_n}} dx_{i_n}\right) = \sum_{i_1,\ldots,i_n} \frac{\partial y_1}{\partial x_{i_1}} \cdots \frac{\partial y_n}{\partial x_{i_n}} dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

Notably, after rearranging the coordinates to get back to $dx_1 \wedge \cdots \wedge dx_n$, we get

$$\det \left[\frac{\partial y_j}{\partial x_i}\right]_{i,j} dx_1 \wedge \cdots \wedge dx_n$$

by recalling the definition of $\det$ as some large sum over signed permutations.

Remark 3.199. There is also a notion of pullback: given a smooth map $F\colon M \to N$ and some $\omega \in \Omega^k(N)$, we can define $F^*\omega$ as a covariant k-tensor field at least by

$$(F^*\omega)_p(v_1, \ldots, v_k) \coloneqq \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))$$

for $p \in M$ and $v_1, \ldots, v_k \in T_pM$. (This is the usual pullback for covariant k-tensor fields.) But we now see from this definition that $F^*\omega$ is alternating, so we get to define our pullback as going to $\Omega^k(M)$.

Remark 3.200. There are some basic properties of the pullback that one should read about. For example, one can show by hand that $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$.

3.11.2 The Exterior Derivative

We like exact covectors, but exactness is not a local property: only being closed is exact. So perhaps we would like to understand obstructions to exactness.

Namely, for some $\omega \in \mathfrak{X}^*(M)$, we write $\omega = \sum_i \omega_i dx_i$ where $(x_1, \ldots, x_n)$ are some local coordinates. Then we can define

$$d\omega \coloneqq \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\right) dx_i \wedge dx_j.$$

It is not totally clear that this is independent of the choice of coordinates, but one can in fact check this by hand, and then we see $d\omega$ actually glues together into a smooth covariant 2-tensor field, and we can see by the above construction that $d\omega \in \Omega^2(M)$. The point is that ω is closed if and only if $d\omega = 0$; for example, for $f \in \Omega^0(M)$, we have $d(df) = 0$.

Next, we would like to define a similar map $d\colon \Omega^2(M) \to \Omega^3(M)$ and maybe even $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ in general.

Definition 3.201 (exterior derivative)**.** Fix an open subset of Euclidean space $U \subseteq \mathbb{R}^n$. Given $\omega \in \Omega^k(U)$, we write $\omega = \sum_I \omega_I dx_I$, and we define the *exterior derivative* $d\omega \in \Omega^{k+1}(U)$ by

$$d\omega \coloneqq \sum_{j_1<\cdots<j_k} \sum_{i=1}^n \frac{\partial \omega_{j_1\cdots j_k}}{\partial x_i} dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

One can check that this definition glues to a map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ for an arbitrary smooth manifold M, possibly with boundary.

Example 3.202. For $\omega \in \Omega^1(M)$ given by $\omega = \sum_i \omega_i \, dx_i$ locally, we can compute

$$d\omega = \sum_{i,j} \frac{\partial \omega_j}{\partial x_i} \, dx_i \wedge dx_j,$$

which agrees with our earlier definition.

Remark 3.203. As a multilinear map, we can compute that

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_i, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}). \end{aligned}$$

Remark 3.204 (Cartan's magic formula)**.** One can compute that $\mathcal{L}_V \omega = V \lrcorner d\omega + d(V \lrcorner \omega)$.

Remark 3.205. One can check that d is $\mathbb{R}$-linear by hand, and we can see $d \circ d = 0$ by a length computation. Another by-hand computation shows that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Everything is natural, so we also get $F^*(d\omega) = d(F^*\omega)$.

Example 3.206. Work in $\mathbb{R}^3$, and let's compute $d(u \, dx_1 \wedge dx_3)$ where u is some smooth function. Then

$$\begin{aligned} d(u \, dx_1 \wedge dx_3) &= du \wedge (dx_1 \wedge dx_3) + (-1)^0 u \, d(dx_1 \wedge dx_3) \\ &= du \wedge dx_1 \wedge dx_3 + u(d(dx_1) \wedge dx_3 - dx_1 \wedge (d(dx_3))) \\ &= du \wedge dx_1 \wedge dx_3. \end{aligned}$$

Writing $du = \sum_i \frac{du_i}{dx_i} \, dx_i$, we then see that only the $i = 2$ term may contribute, so we are left with $-\frac{\partial u_2}{\partial x_2} \, dx_1 \wedge dx_2 \wedge dx_3$.

One may be interested in a more coordinate-free definition of the exterior derivative. At the very least, we will be able to note that it is unique from some of our listed properties.

Theorem 3.207. Fix a smooth manifold M, possibly with boundary. Then there is a unique family of maps $d \colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ satisfying the following conditions.

(i) Linear: d is $\mathbb{R}$-linear.

(ii) Product rule: for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) Complex: $d \circ d = 0$.

(iv) Degree 0: for $f \in C^\infty(M)$, the 1-form $df \in \Omega^1(M)$ is the usual differential of a function.

Proof. We already know existence. We won't show uniqueness. ∎

Let's do some more examples. We work with $\mathbb{R}^3$.

- We know $\Omega^0\left(\mathbb{R}^3\right) = C^\infty\left(\mathbb{R}^3\right)$.
- We may identify $\Omega^1\left(\mathbb{R}^3\right) = \mathfrak{X}^*\left(\mathbb{R}^3\right)$ with $\mathfrak{X}\left(\mathbb{R}^3\right)$ by using the standard Riemannian metric (explicitly, we send $dx_i \in \mathfrak{X}^*\left(\mathbb{R}^3\right)$ to $\frac{\partial}{\partial x_i} \in \mathfrak{X}\left(\mathbb{R}^3\right)$).
- Continuing, we may identify $\Omega^2\left(\mathbb{R}^3\right)$ with $\mathfrak{X}\left(\mathbb{R}^3\right)$ again by using the global frame of $\wedge^2 T^*\mathbb{R}^3$: we send $dx_i \wedge dx_{i+1}$ with $\frac{\partial}{\partial x_{i+2}}$, where indices are taken $\pmod 3$. More canonically, we can take $X \in \mathfrak{X}\left(\mathbb{R}^3\right)$ to $\iota_X(dx_1 \wedge dx_2 \wedge dx_3) \in \Omega^2\left(\mathbb{R}^3\right)$.
- Lastly, $\Omega^3\left(\mathbb{R}^3\right)$ has global frame given by $dx_1 \wedge dx_2 \wedge dx_3$, so this space is isomorphic to $C^\infty\left(\mathbb{R}^3\right)$ by sending a smooth function u to $u\, dx_1 \wedge dx_2 \wedge dx_3$.

The point of doing all this is that it turns out that the following diagram commutes.

$$\begin{array}{ccccccc}
C^\infty(M) & \xrightarrow{\text{grad}} & \mathfrak{X}\left(\mathbb{R}^3\right) & \xrightarrow{\text{curl}} & \mathfrak{X}\left(\mathbb{R}^3\right) & \xrightarrow{\text{div}} & C^\infty(M) \\
\| & & \| & & \| & & \| \\
\Omega^0\left(\mathbb{R}^3\right) & \xrightarrow{d} & \Omega^1\left(\mathbb{R}^3\right) & \xrightarrow{d} & \Omega^2\left(\mathbb{R}^3\right) & \xrightarrow{d} & \Omega^3\left(\mathbb{R}^3\right)
\end{array}$$

Here, the vertical maps are the identifications we just described. For example, we discover that $\operatorname{curl}\circ\operatorname{grad} = 0$ and $\operatorname{div} \circ \operatorname{curl} = 0$.

We also note that we can see some pairing: given a Riemannian manifold (M, g), one has a global "volume" form given by

$$dV_g := \sqrt{\deg(g_{ij})}dx_1 \wedge \cdots \wedge dx_n$$

for any local choice of coordinates $(x_1, \ldots, x_n)$. Then there is a unique map $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ such that

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_g dV - g$$

In particular, we are seeing that $\vee$ somehow produces a perfect pairing.

Remark 3.208. It turns out that our Laplacian operator Δf for $f \in C^\infty\left(\mathbb{R}^3\right)$ given by $*d*df$. One can compute this operator as $\operatorname{div}\circ\operatorname{grad}$ where the content becomes that our $*$ operator also commutes with the vertical isomorphisms.

Remark 3.209. Our discussion of the exterior derivative also has applications for $\mathbb{R}^4$: an element of $\Omega^2\left(\mathbb{R}^4\right)$ can be viewed as a smooth map from $\mathbb{R}^4$ to the space of antisymmetric 4×4 matrices (by using the standard global frame, as usual). Professor Chen gave some discussion of Maxwell's equations; basically, it turns out that one can compress everything into two short equations on a single element $\omega \in \Omega^2\left(\mathbb{R}^4\right)$.

3.12 April 30

Today we discuss Stokes's theorem. Here are some notes about the final. There will be more information about the final later today; the format will be similar to the midterm, though not as long as two midterms. Stokes's theorem will be used on at least one problem on the final.

3.12.1 Integration

We begin by discussing integration on $\mathbb{R}^n$. Given a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, we would like to integrate f on $\mathbb{R}^n$ by summing up the value of f on very small boxes. Imagining these small boxes in one coordinate at a time, this amounts to some kind of iterated integration. The direction here should be signed, so we are essentially integrating against the n-form $dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^n)$. This definition then generalizes for arbitrary $\omega \in \Omega^n(\mathbb{R}^n)$.

Definition 3.210. Fix a compactly supported n-form $\omega \in \Omega^n(\mathbb{R}^n)$. Because $\Omega^n(\mathbb{R}^n)$ is a line bundle with global frame given by $dx_1 \wedge \cdots \wedge dx_n$, we may write $\omega := f\, dx_1 \wedge \cdots \wedge dx_n$ for some smooth $f\colon \mathbb{R}^n \to \mathbb{R}$, and then we define

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f\, dx_1 \cdots dx_n.$$

We would like to show that this definition does not depend on our choice of diffeomorphism.

Proposition 3.211. Fix a diffeomorphism $G\colon U \to V$ of connected open subsets of $\mathbb{R}^n$. Given a compactly supported n-form $\omega \in \Omega^n(V)$, we have

$$\int_U G^*\omega = \pm \int_V \omega,$$

where we take the $+$ sign if G preserves orientation, and we take the $-$ sign if G reverses orientation.

Proof. This amounts to change of variables for our iterated integrals. Write $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$. Because pullback commutes with the exterior derivative, we see that

$$G^*\omega = (f \circ G)(G^* dx_1) \wedge \cdots \wedge (G^* dx_n).$$

Now, by a direct expansion of G on coordinates as we did in Example 3.198, we see that this is

$$G^*\omega = (f \circ G)(\det dG)\, dx_1 \wedge \cdots \wedge dx_n,$$

so

$$\int_U G^*\omega = \int_U (f \circ G)(\det dG)\,(dx_1 \wedge \cdots \wedge dx_n).$$

Adjusting our coordinates, we note that our Jacobian is $|\det dG|$, so we are done upon noting that "preserving orientation" just means that $\det dG$ is positive. ■

We are now ready to integrate on (oriented) manifolds.

Definition 3.212. Fix a smooth manifold M, possibly with boundary, and fix a compactly supported n-form $\omega \in \Omega^n(M)$. Choose a collection of positively oriented charts $\{(U_i, \varphi_i)\}_{i=1}^N$ covering the support of ω, and let $\{\psi_i\}_{i=1}^N$ be a smooth partition of unity subordinate to $\{U_i\}_{i=1}^N$. Then we define

$$\int_M \omega := \sum_{i=1}^N \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (\psi_i \omega)$$

The point is that $\omega = \sum_{i=1}^N \psi_i \omega$, so we are summing up our integral in the various pieces.

One can check that this does not depend on the choice of charts covering $\operatorname{supp}\omega$; this basically follows straight from the proposition.

3.12.2 Stokes's Theorem

We are now ready to state Stokes's theorem.

Theorem 3.213 (Stokes)**.** Fix some compactly supported $(n-1)$-form ω on a smooth n-manifold M with boundary ∂M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

For this to make sense, we need to orient ∂M, presumably in a way that agrees with the orientation on M. Essentially, an orientation can be thought of as a non-vanishing choice of n-form $\omega \in \Omega^n(M)$. Then we take an outward-pointing tangent vector v on M at any boundary point, and we define our orientation as $v \lrcorner \omega$. Anyway, here are some applications.

Example 3.214. Take $M = [a, b]$ and $\omega = f(x)$ for some smooth f. Then Theorem 3.213 tells us

$$\int_{[a,b]} f'(x)\, dx = f(b) - f(a),$$

where we get the right-hand side by keeping track of the orientation on $\partial M = \{a, b\}$.

Example 3.215. Choose some compact oriented domain $M \subseteq \mathbb{R}^3$. Given a vector field $X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + X_3 \frac{\partial}{\partial x_3}$ on $\mathbb{R}^3$, we may want to compute the flux through M. Then we define

$$\omega := X_1\, dx_2 \wedge dx_3 - X_2\, dx_1 \wedge dx_3 + X_3\, dx_1 \wedge dx_2.$$

(This is simply $X \lrcorner (dx_1 \wedge dx_2 \wedge dx_3)$.) One can compute that $d\omega = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$, which is precisely $\operatorname{div} X$. Now, Theorem 3.213 tells us

$$\int_M (\operatorname{div} X) = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} (X \lrcorner (dx_1 \wedge dx_2 \wedge dx_3)).$$

Now, we decompose X into $X_\parallel + X_\perp$, where $X_\parallel$ is tangent to the boundary, and $X_\perp$ is perpendicular to the boundary. The tangent part vanishes in the interior product because we will get linearly dependent differential forms in our exterior product, so we are allowed to replace X with $X_\perp$. But now $X_\perp$ has length $X \cdot v$ where v is some unit normal vector, so we get to

$$\int_M (\operatorname{div} X) = \int_{\partial M} (X \cdot v)\, dA,$$

where dA is the surface area measure.

Example 3.216. If M has no boundary, then we are just saying that $\int_M d\omega = 0$.

Example 3.217. If ω is a closed form, then $d\omega$ vanishes, so we are just saying that $\int_{\partial M} \omega = 0$.

Example 3.218. Given an oriented Riemannian manifold (M, g), we can define the Laplacian as $\Delta f := \operatorname{div} \operatorname{grad} f$. Then Theorem 3.213 grants $\int_M \Delta f = 0$ of M is a smooth manifold without boundary because the divergence can be realized as an exterior derivative.

Let's see another application.

Theorem 3.219 (Green)**.** Fix a compact domain $D \subseteq \mathbb{R}^2$, and let $P, Q \in C^\infty(D)$ be smooth functions. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial D} P\, dx + Q\, dy.$$

Proof. Apply Theorem 3.213 to $\omega := P\, dx + Q\, dy$ as a smooth 1-form on ∂D, and we can compute that $d\omega$ is as required. ■

Anyway, let's go ahead and prove Theorem 3.213.

Proof of Theorem 3.213. We only know how to integrate via partition of unity, so we have to fix some partition of unity. Choose positively oriented charts $\{(U_i, \varphi_i)\}_{i=1}^N$ covering $\operatorname{supp} \omega$, and choose a smooth partition of unity $\{\psi_i\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$ so that

$$\omega = \sum_{i=1}^N \psi_i \omega.$$

Now, by linearity of the conclusion Theorem 3.213, it suffices to show the statement for $\psi_i\omega$ for each i.

By possibly shrinking the $U_\bullet$s, we may assume that φ_i sends each U_i to either $\mathbb{R}^n$ or $\mathbb{H}^n$. Thus, it suffices to prove Theorem 3.213 in the two cases $M = \mathbb{R}^n$ and $M = \mathbb{H}^n$.

- Take $M = \mathbb{H}^n$. This will basically be a direct computation. We write

  $$\omega = \sum_{i=1}^n \omega_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

  for smooth functions $\omega_1, \ldots, \omega_n \colon \mathbb{H}^n \to \mathbb{R}$. In this case, we find that

  $$d\omega = \left(\sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n,$$

  so

  $$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^{n-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x_i}\, dx_1 \cdots dx_{n-1}\, dx_n,$$

  where we take R large enough so that $\operatorname{supp}\omega$ is contained in $(-R, R)^n \times [0, R)$. Now, for each $i \neq n$, we note that the corresponding integral contains the integral

  $$\int_{-R}^R \frac{\partial \omega_i}{\partial x_i}\, dx_i = \omega_i \bigg|_{-R}^R = 0,$$

  which vanishes by considerations of $\operatorname{supp}\omega$. So we only have to care about the $i = n$ term, leaving us with

  $$\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1}.$$

  It remains to compute $\int_{\partial\mathbb{H}^n} \omega$. Well, $\mathbb{H}^n$ has an outward pointing tangent vector given by $-e_n$ uniformly, so the coordinates $(x_1, \ldots, x_{n-1})$ will be positively oriented for n even and negatively oriented for n odd, so a computation shows

  $$\int_{\partial\mathbb{H}^n} \omega = \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \ldots dx_{n-1}$$

  to be equal to the above integral by a consideration of the orientation.

- In the case where $M = \mathbb{R}^n$, the same computation for $\int_{\mathbb{R}^n} d\omega$ shows that it vanishes because this time even the $i = n$ term will vanish. And of course ∂M is empty, so $\int_{\partial M} \omega = 0$ as well. ■

3.13 May 2

Today we talk a little about de Rham cohomology.

3.13.1 De Rham Cohomology

We begin by setting some notation. Let M be a smooth manifold with boundary. Then we recall that we have built a chain

$$0 \to \Omega^0(M) \overset{d}{\to} \Omega^1(M) \overset{d}{\to} \Omega^2(M) \overset{d}{\to} \cdots,$$

where d denotes the exterior derivative. Notably, $d^2 = 0$, so we have a cochain complex $(\Omega^\bullet(M), d)$.

Definition 3.220 (de Rham cohomology)**.** Fix a smooth manifold M. Then we define the *de Rham cohomology* of M as the cohomology of the cochain complex $(\Omega^\bullet(M), d)$. More explicitly, we define the *closed p-forms* as

$$Z^p(M) \coloneqq \ker\left(d\colon \Omega^p(M) \to \Omega^{p+1}(M)\right),$$

and

$$B^p(M) \coloneqq \operatorname{im}\left(d\colon \Omega^{p-1}(M) \to \Omega^p(M)\right),$$

so our de Rham cohomology is $H^\bullet_{\mathrm{dR}}(M) \coloneqq Z^p(M)/B^p(M)$. We will suppress the dR from our notation as much as possible.

Example 3.221. We have that $H^0(M)$ consists of functions $f \in \Omega^0(M) = C^\infty(M)$ that vanish under the derivative. Thus, $H^0(M)$ consists of the locally constant functions on M, which we see is $\mathbb{R}^{\pi_0(M)}$.

Remark 3.222. Directly from the definitions, we see that our cohomology are $\mathbb{R}$-vector spaces.

Because our cochain complex has some notion of functoriality, our cohomology does as well. More precisely, let $F\colon M \to N$ be a smooth map. Then pullback makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0(N) & \xrightarrow{d_N} & \Omega^1(N) & \xrightarrow{d_N} & \Omega^2(N) & \xrightarrow{d_N} & \cdots \\ & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\ 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d_M} & \Omega^1(M) & \xrightarrow{d_M} & \Omega^2(M) & \xrightarrow{d_M} & \cdots \end{array}$$

commute, which one can check (via some diagram-chasing) then produces a map $H^\bullet_{\mathrm{dR}}(F)\colon H^\bullet_{\mathrm{dR}}(N) \to H^\bullet_{\mathrm{dR}}(M)$. Explicitly, we take a class $[\omega] \in H^p(N)$ represented by $\omega \in Z^p(N)$ to the class $[F^*\omega] \in H^p(M)$. Let's go ahead and check that this is well-defined. To begin, we note that ω being closed implies $d\omega = 0$, so $F^*(d\omega) = d(F^*\omega) = 0$, so $F^*\omega \in Z^p(M)$. Then we want to check that $[\omega] = [\omega']$ implies $[F^*\omega] = [F^*\omega']$. Well, write $\omega = \omega' + d\eta$ where $\eta \in \Omega^{p-1}(N)$; then

$$F^*\omega = F^*\omega' + F^*d\eta = F^*\omega + dF^*\eta,$$

so $[F^*\omega] = [F^*\omega']$ follows.

Remark 3.223. Functoriality of the pullback assures us that $H^\bullet_{\mathrm{dR}}$ is a functor. For example, one sees $H^\bullet(\mathrm{id}_M) = \mathrm{id}_{H^\bullet(M)}$.

3.13.2 Topology of Manifolds

We expect de Rham cohomology to produce a reasonable cohomology theory, so it should be topological in nature. Let's see some of this topological invariance.

Proposition 3.224. Suppose $F, G\colon M \to N$ are smooth maps of smooth manifolds which are homotopic. Then $H^\bullet_{\mathrm{dR}}(F) = H^\bullet_{\mathrm{dR}}(G)$.

Proof. Let $H_\bullet\colon F \simeq G$ witness the homotopy with $H_0 = F$ and $H_1 = G$, which we may assume is smooth by an argument with Whitney approximation. To see this homotopy, define $i_t\colon M \to M \times [0,1]$ by $i_t(p) := (p,t)$; in particular, $H \circ i_0 = F$ and $H \circ i_1 = G$.

Now, given $\omega \in \Omega^p(N)$, we would like to show that $[F_*\omega] = [G_*\omega]$ when $\omega \in Z^p(N)$. In other words, we want to show that

$$G_*\omega - F_*\omega = i_1^* H^*\omega - i_0^* H^*\omega$$

is in the image of $d\colon \Omega^{p-1}(M) \to \Omega^p(M)$. Set $\eta := H^*\omega$ to be an element of $\Omega^p(M \times [0,1])$. Thus, we want to compute

$$i_1^*\eta - i_2^*\eta = \int_0^1 \left(\frac{d}{dt} i_t^*\omega \right) \, dt,$$

where the derivative and integration is happening on the level of coordinates. (Namely, we should note that everything can be computed locally and glued together later, so fix some $p \in M$, place it in a chart, and expand everything out on coordinates.) We now use Cartan's magic formula to compute

$$\frac{d}{dt} i_t^*\eta = i_t^* \left(\mathcal{L}_{\partial/\partial t}\eta \right) = i_t^* \left(\frac{\partial}{\partial t} \lrcorner d\eta + d \left(\frac{\partial}{\partial t} \lrcorner \eta \right) \right) = i_t^* \left(\frac{\partial}{\partial s} \lrcorner d\eta \right) + d i_t^* \left(\frac{\partial}{\partial s} \lrcorner \eta \right).$$

We are now ready to define our chain homotopy $h\colon \Omega^p(N) \to \Omega^{p-1}(M)$ as

$$h(\omega) := \int_0^1 i_t^* \left(\frac{\partial}{\partial s} \lrcorner H^*\omega \right) \, dt.$$

Thus, we see that $G^*\omega - F^*\omega = h(d\omega) + dh(\omega)$ for any ω. Having a chain homotopy means that $H^\bullet(F) = H^\bullet(G)$. Let's see this more explicitly: given $[\omega] \in H^p(N)$ where $\omega \in Z^p(N)$, we see that

$$G^*\omega - F^*\omega = h(\underbrace{d\omega}_{0}) + dh(\omega) \in B^p(M),$$

as required. ■

Example 3.225. If two smooth manifolds are homotopy equivalent (for example, if they are homeomorphic), then they have isomorphic cohomology groups. Namely, let $F\colon M \to N$ and $G\colon N \to M$ witness the homotopy equivalence, and then the above proposition implies that $H^\bullet_{\mathrm{dR}}(F)$ and $H^\bullet_{\mathrm{dR}}(G)$ are inverse maps on our cohomology.

Example 3.226. If M is a contractible manifold, then it is homotopic to a point, so $H^\bullet(M) = H^\bullet(\{*\})$ which vanishes in positive degree. For example, if $U \subseteq \mathbb{R}^n$ is star-shaped, then

$$H^p(U) = \begin{cases} \mathbb{R} & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

Here is another application.

Proposition 3.227. Fix a smooth connected manifold M, and fix a basepoint $q \in M$. Then the map $\Phi\colon H^1(M) \to \operatorname{Hom}(\pi_1(M,q), \mathbb{R})$ defined by

$$\Phi([\omega])([\gamma]) := \int_\gamma \omega.$$

Proof. To begin, we check that the map is well-defined.

- We note that any $[\gamma] \in \pi_1(M, q)$ is homotopic to a piecewise smooth map with the same basepoint, so we can find a piecewise smooth representative to integrate over, so the integral at least makes sense.
- If $[\omega] = [\omega']$, then write $\omega - \omega' = df$ for some $f \in C^\infty(M)$, and then we see that
$$\int_\gamma \omega - \int_\gamma \omega' = \int_\gamma (\omega - \omega') = \int_\gamma df = f(q) - f(q) = 0$$
by Theorem 3.118.
- If $[\gamma] = [\gamma']$, then let $H_\bullet : \gamma \simeq \gamma'$ witness the homotopy, where $H_0 = \gamma$ and $H_1 = \gamma'$. We may assume that everything in sight is smooth by the usual Whitney approximation arguments, so we may integrate
$$\int_{I \times I} d(H^* \omega) = \int_{I \times I} H^* d\omega = 0$$
because $d\omega = 0$. On the other hand, Theorem 3.213 tells us that this integral is $\int_\gamma \omega - \int_{\gamma'} \omega$, so we finish.
- The map $[\gamma] \mapsto \int_\gamma \omega$ is a homomorphism because concatenating paths will add the integrals together, by the definition of our integral.
- The map Φ itself is $\mathbb{R}$-linear because integration is $\mathbb{R}$-linear on the differential form.

Lastly, we should check that Φ is injective. Well, if $\int_\gamma \omega = 0$ for all γ based at q, then we know ω is conservative (we can get rid of the basepoint by just drawing some smooth map connecting p to q), so ω is exact by Proposition 3.122, so $[\omega] = 0$. ■

Example 3.228. Fix a smooth connected manifold M. If $\pi_1(M)$ is torsion, then we see that $H^1(M) = 0$.

Remark 3.229. It turns out that this map is an isomorphism, but it requires some more work to see.

No discussion relating de Rham cohomology to topology would be complete without at least stating the de Rham theorem.

Theorem 3.230 (de Rham). Fix a smooth manifold M. There is a natural isomorphism $\theta \colon H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{sing}}(M; \mathbb{R})$ given by
$$\theta_{[\omega]}(\sigma) := \int_{\Delta^p} \sigma^* \omega,$$
where $\omega \colon \Delta^p \to M$ is a smooth embedding from the p-simplex to M.

Here, $H^\bullet_{\mathrm{sing}}$ refers to singular cohomology, which we will not define. As such, we will of course not attempt a proof.

3.13.3 The Mayer–Vietoris Sequence

We would like to compute cohomology in some cases, but so far we only know how to compute cohomology of small neighborhoods. Fitting everything we've done so far in this course, one way to imagine doing this is to break up a manifold into charts and then attempt to glue them back together. This will be attempted inductively, so we will work with covers of just two open subsets.

Theorem 3.231 (Mayer–Vietoris)**.** Fix open subsets $U, V \subseteq M$ which cover a smooth manifold M. Then there is a long exact sequence as follows.

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \xrightarrow{\delta^0} H^1(M) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V) \longrightarrow \cdots$$

Here, the horizontal maps are induced by the inclusions $U \cap V \subseteq U, V \subseteq M$, and the diagonal map is the boundary map, and it is harder to define.

Let's see an example computation.

Example 3.232. We claim $H^p(S^n) = \mathbb{R}^{1_{p \in \{0,n\}}}$ for $n \geq 0$. We proceed by induction. For $n = 0$, there is nothing is to do. For $n > 0$, we begin by noting that $H^0(S^n) = \mathbb{R}$ because there is still only one component. For $p = 1$, we use Proposition 3.227: if $n \geq 2$, then we trivialize immediately, and if $n = 1$, then our dimension becomes bounded below by 1, and being oriented shows that the group is nontrivial. So we may focus on $p \geq 2$. Now, let U be an open collar neighborhood of the top hemisphere, and let V be an open collar neighborhood of the bottom hemisphere. Then U and V are each contractible, so they have vanishing cohomology in higher degrees. We now note that any $p \geq 2$ has the exact sequence

$$H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to H^{p+1}(S^n) \to H^{p+1}(U) \oplus H^{p+1}(V).$$

The two ends trivialize, and $U \cap V$ is homotopic to S^{n-1}, so we finish by induction.

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List of Definitions