215A: Algebraic Topology

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Fall 2023

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Introduction

I have dedicated far too many algorithms and computational resources toward finding an answer to this unknowable thing.

—Neal Shusterman, [Shu18]

1.1 August 24

It begins.

1.1.1 Logistics

Here are the logistical notes.

- The professor is Ian Agol, whose office is Evans 921. Office hours are Tuesdays after class, Monday at 3PM, Wednesday at 9AM, or by appointment.
- There is a `bCourses`.
- Homework will be weekly, and it will make up the entire grade.
- The prerequisites are Math 113 and 202A or equivalent. From point-set topology in particular we will want notions of compactness, connectedness, metric spaces, and a few topologies like the identification topology with respect to a continuous map.

1.1.2 Overview

We will cover chapters 0–3 of [Hat01].

- Chapter 0 consists of "geometric notions." Particularly important are the notion of homotopy and CW complexes.
- Chapter 1 is on fundamental groups.
- Chapter 2 is on homology. This is an abelian extension of fundamental groups.
- Chapter 3 is on cohomology. Poincaré duality relates cohomology with homology.

Chapter 4 is typically covered in Math 215B, on homotopy theory.

Let's talk a bit about the interests of the course. Topology as a whole is interested in "spaces up to deformation." In this class, deformation will mean homotopy mostly, but there are finer notions of interest like homeomorphism. As for the spaces, we will focus on spaces which are locally homogeneous in some sense, like manifolds (which are locally homeomorphic to $\mathbb{R}^n$). These notions come up naturally throughout mathematics; for example, integrals of holomorphic functions are roughly independent of path chosen. Poincaré himself was interested in differential equations, whose configuration spaces could be manifolds.

In this class, we will attach invariants to our topological spaces to be able to understand how to differentiate between our spaces (up to deformation). We focus on the following invariants.

- Fundamental groups and covering spaces. This has a close tie to Galois theory, an analogy made process by the étale fundamental group in algebraic geometry.
- Cohomology. The origins are from complex analysis and Stokes's theorem, but cohomology itself has vast generalizations and manifestations throughout mathematics, leading to the field of homological algebra. However, there are applications to algebraic geometry, number theory, and so on. The most notable application here is the proof of the Weil conjectures.
- Higher homotopy groups. Our approach will not begin with this viewpoint, but it is possible.

1.1.3 Homotopy and Homotopy Type

Let's jump in chapter 0.

Notation 1.1. We set $I := [0,1]$ for convenience.

Definition 1.2 (deformation retract)**.** Fix a subspace A of a topological space X. Then a *deformation retract* is a family of functions $f_\bullet \colon X \times I \to X$ where $f_0 = \mathrm{id}_X$ and $\operatorname{im} f_1 = A$ and $f_t|_A = \mathrm{id}_A$ for all $t \in I$.

Example 1.3 (mapping cylinder)**.** Fix a continuous function $f \colon X \to Y$. Then the *mapping cylinder* M_f is the space $(X \times I) \sqcup Y$ quotiented by $(x,1) \sim f(x)$. Then M_f has a deformation retraction to Y by $f_t(x) := (x,t)$. Visually, we have attached Y to a thickening of X.

Example 1.4. Define $f \colon S^1 \to S^1$ by $f(z) := z^2$. Then M_f has S^1 on one domain side and S^1 covered twice on the target side. With a little deformation, this is a Möbius strip. Approximately speaking, one should cut the cylinder in half and then rearrange. One can see that the Möbius strip deformation retracts to S^1 by squishing the width of the cylinder to the central line.

A deformation retract is a special case of a homotopy. Here is the definition of a homotopy.

Definition 1.5 (homotopy)**.** Two continuous maps $f_0, f_1 \colon X \to Y$ are *homotopic* if and only if there is a continuous function $F_\bullet \colon X \times I \to Y$ such that $F_0 = f_0$ and $F_1 = f_1$. Here, F is called a *homotopy*, and we write $f_0 \sim f_1$.

Example 1.6. A subspace $A \subseteq X$ has a deformation retract if and only if id_X is homotopic to some $r \colon X \to X$ with $\operatorname{im} r = A$ and $r|_A = \mathrm{id}_A$. Indeed, the deformation retract is exactly the needed homotopy.

Example 1.7. Suppose $f, g \colon X \to Y$ are equal maps. Then define $h \colon X \times I \to Y$ by $h_t = f = g$ for all t. We see that h is continuous ($h^{-1}(V) = f^{-1}(U) \times I$ for any open $V \subseteq Y$), so it provides a homotopy from f and g.

It should not be surprising that homotopy is an equivalence relation.

Lemma 1.8. Fix topological spaces X and Y. Then $\sim$ is an equivalence relation on continuous functions $X \to Y$.

Proof. We have the following checks.

- Reflexive: this is direct from Example 1.7.
- Symmetric: if $f \sim g$, then we have $F_\bullet \colon X \times I \to Y$ with $F_0 = f$ and $F_1 = g$. We now define $G_\bullet \colon X \times I \to Y$ by $G_t := F_{1-t}$. Then G is continuous by the continuity of $t \mapsto 1 - t$ and F, and $G_0 = g$ and $G_1 = f$, so G witnesses $g \sim f$.
- Transitive: if $f \sim g$ and $g \sim h$, find $F_\bullet \colon X \times I \to Y$ and $G_\bullet \colon X \times I \to Y$ with $F_0 = f$ and $F_1 = g$ and $G_0 = g$ and $G_1 = h$. Then we define $H_\bullet \colon X \times I \to Y$ by

$$H_t := \begin{cases} F_{2t} & \text{if } 0 \leqslant t \leqslant 1/2, \\ G_{2t-1} & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

  Note that this is well-defined at $t = 1/2$ because $F_1 = g = G_0$. Note H will witness $f \sim h$ once we show that it is continuous, which is what we do now.

  By looking locally at F or G, we see that H is continuous at any point not of the form $(x, 1/2)$. Then for any point of the form $(x, 1/2)$ and open subset $V \subseteq Y$ containing $H_{1/2}(x)$, continuity of F gives an open subset $U_F \times (1/2 - \varepsilon, 1/2]$ mapping to V, and continuity of G gives an open subset $U_G \times [1/2, 1/2 + \varepsilon)$ mapping to V, so $(U_F \cap U_G) \times (1/2 - \varepsilon, 1/2 + \varepsilon)$ will suffice. ■

Homotopy also behaves well with composition.

Lemma 1.9. Fix topological spaces X, Y, Z, and let $f_0, f_1 \colon X \to Y$ and $g_0, g_1 \colon Y \to Z$ be homotopic maps. Then $(g_0 \circ f_0) \sim (g_1 \circ f_1)$.

Proof. Fix a homotopy $F_\bullet \colon X \times I \to Y$ with $F_0 = f_0$ and $F_1 = f_1$ and a homotopy $G_\bullet \colon Y \times I \to Z$ with $G_0 = g_0$ and $G_1 = g_1$. Then we define $H_\bullet \colon X \times I \to Z$ by

$$H_t(x) := G_t(F_t(x)).$$

Then $H_0 = g_0 \circ f_0$ and $H_1 = g_1 \circ f_1$, so we will be done if we can show H is continuous. Well, $H_\bullet$ is the composite map

$$X \times I \stackrel{(F, \mathrm{id}_I)}{\to} Y \times I \stackrel{G}{\to} Z,$$

which we can see is the composite of continuous maps. ■

Homotopy allows us to define homotopy equivalence.

Definition 1.10 (homotopy equivalence). A continuous map $f \colon X \to Y$ is a *homotopy equivalence* if and only if there is a continuous map $g \colon Y \to X$ such that $(g \circ f) \sim \mathrm{id}_X$ and $(f \circ g) \sim \mathrm{id}_Y$. We then say that X and Y have the same *homotopy type* and write $X \simeq Y$.

Remark 1.11. It is not enough to merely require $(g \circ f) \sim \mathrm{id}_X$. For example, let $X := \{x\}$ be a point. Then any map $f \colon \{x\} \to Y$ can use the unique map $g \colon Y \to \{x\}$ so that $(g \circ f) = \mathrm{id}_X$.

Here is a quick sanity check.

Lemma 1.12. Ignoring size issues, homotopy equivalence provides an equivalence relation on topological spaces.

Proof. We have the following checks. Fix topological spaces X, Y, Z.

- Reflexive: we show $X \simeq X$. Indeed, use the maps $\mathrm{id}_X, \mathrm{id}_X \colon X \to X$ so that $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$ is homotopic to id_X by Example 1.7.
- Symmetric: we show $X \simeq Y$ implies $Y \simeq X$. Indeed, let $f \colon X \to Y$ and $g \colon Y \to X$ be the promised maps so that $(f \circ g) \sim \mathrm{id}_Y$ and $(g \circ f) \sim \mathrm{id}_X$. Reading these data in reverse tell us that $Y \simeq X$.
- Transitive: suppose $X \simeq Y$ and $Y \simeq Z$, and we show $X \simeq Z$. Thus, we have maps $f \colon X \to Y$ and $g \colon Y \to X$ and $f' \colon Y \to Z$ and $g' \colon Z \to Y$ such that $(f \circ g) \sim \mathrm{id}_Y$ and $(g \circ f) \sim \mathrm{id}_X$ and $(f' \circ g') \sim \mathrm{id}_Z$ and $(g' \circ f') \sim \mathrm{id}_Y$. We now claim that $(f' \circ f) \colon X \to Z$ and $(g \circ g') \colon Z \to X$ are the desired maps to witness $X \simeq Z$. Well, using Lemma 1.9, we compute

  $$(f' \circ f) \circ (g \circ g') = f' \circ (f \circ g) \circ g' \sim f' \circ \mathrm{id}_Y \circ g' = f' \circ g' \sim \mathrm{id}_Z,$$

  and similar for the other direction. ■

Remark 1.13. One can check directly that $\sim$ is an equivalence relation on spaces. The main check here is that one can compose homotopies.

We will often find that our algebraic invariants are only able to detect homotopy equivalence, which is why homotopy equivalence will be so important to us.

Example 1.14. Example 1.4 shows that the Möbius strip is homotopic to S^1.

More generally, one can show that a deformation retract is a homotopy equivalence.

Lemma 1.15. Fix a subspace A of a topological space X. Then a deformation retract witnesses a homotopy between the inclusion $i \colon A \hookrightarrow X$ and the identity $\mathrm{id}_X \colon X \to X$. In particular, it follows that i is a homotopy equivalence.

Proof. This is a matter of unraveling the definitions. Fix a deformation retract $f_\bullet \colon X \times I \to X$, and let $r := f_1$ so that $\operatorname{im} r = A$. We now claim that i and r are inverse homotopy equivalences.

- We show that $(r \circ i) \sim \mathrm{id}_A$. Indeed, $r(i(a)) = a$ for any $a \in A$ by hypothesis on r, so in fact $r \circ i = \mathrm{id}_A$.
- We show that $(i \circ r) \sim \mathrm{id}_X$. The relevant homotopy is just $f_\bullet$: we have $f_0 = \mathrm{id}_X$ and $f_1 = (i \circ r)$, so $\mathrm{id}_X \sim (i \circ r)$ by Lemma 1.8. ■

Example 1.16 (dunce cap)**.** Take the disc D^2 and glue the edges together as follows: mark three points A, B, and C, and glue AB to AC to CB (in those orientations). Then the resulting space is homotopic to a point.

We have a special name for being homotopic to a point.

Definition 1.17 (contractible)**.** A topological space X is *contractible* if and only if it is homotopic to a point.

These notions allow us to define a homotopy category, whose objects are homotopy classes of topological spaces and morphisms are continuous maps. In some sense, our algebraic invariants are trying to distinguish between objects in this category. It turns out that this category is not concrete, meaning that there is no way to realize its objects as sets reasonably. Approximately speaking, this means that there can be no canonical representing topological space for each homotopy class, but topologists try anyway.

Remark 1.18. There are a number of results called "topological rigidity" theorems which give homeomorphism $X \cong Y$ given merely $X \simeq Y$ and some extra hypotheses. For example, this holds for closed surfaces by a classification result.

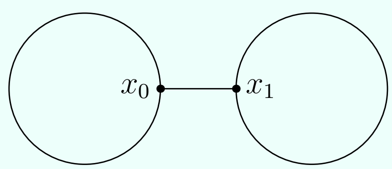
Example 1.19. Attach two S^1s by a line to make a space X, and attach them along an edge to make a space Y. These spaces are homotopic, but they are not homeomorphic (removing a point from X may disconnect it, but this is not the case for Y).

1.1.4 CW Complexes

Here is our definition.

Definition 1.20 (CW complex)**.** Let X^0 be a discrete set of points, and define X^n inductively by $X^{n+1} := X^n \cup \{e_\alpha^{n+1}\}$, where $\varphi_\alpha : \partial e_\alpha^{n+1} \to X^n$ is a homeomorphism telling us how to union. Here, e_α^{n+1} is a copy of the n-ball B^n, so the φ_α are explaining how to identify the edges.

Example 1.21. Here is a CW complex.



Namely, $X^0 = \{x_0, x_1\}$, and X^1 is the edges.

Example 1.22. Take a point $\{*\}$ for X^0, and define φ_n to be some loop based on $\{*\}$. Then the resulting space is some infinite union of circles intersecting at $\{*\}$. Notably, this space is not compact and in fact should not even be embedded into the plane or $\mathbb{R}^3$ because such an embedding is unlikely to be a homeomorphism.

Example 1.23. The sphere $S^n := D^n/\partial D^n$ is a CW structure with only two cells: it is $e^0 \cup e^n$. Notably, the CW structure here has $X^0 = X^1 = \cdots = X^{n-1}$.

Example 1.24. Alternatively, one can define S^n inductively as follows: take S^0 to be two points, and define S^{n+1} to be S^n as an equator unioned with two $(n+1)$-cells making hemispheres attached to the equator. One can then define S^∞ to be the union of all the $S^\bullet$ where we identify $S^n \hookrightarrow S^{n+1}$ via the equator. This is a CW complex of infinite dimension. It turns out that S^∞ is contractible, though S^n is not for any finite n.

Example 1.25. Define real projective space $\mathbb{RP}^n$ as the set of vectors $x \in \mathbb{R}^{n+1}\backslash\{0\}$ where we identify x with λx for any $\lambda \in \mathbb{R}^\times$. Notably, by setting the last coordinate equal to 0, we expect to get $\mathbb{RP}^{n-1}$. But if the last coordinate is equal to zero, we can scale it uniquely to 1, and then the remaining coordinates may vary arbitrarily. In total, we find

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \mathbb{R}^{n-1}.$$

Thus, we get the cell structure $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

Remark 1.26. The CW structure is not unique. For example, one can separate out edges by putting a point in the middle of them.

One can show that the CW complex is compact if and only if it has finitely many cells.

1.2 August 29

Last time we discussed homotopies, homotopy equivalence, and CW complexes. To review, the goal of algebraic topology is to define (algebraic) invariants of topological spaces and then perhaps figure out when two spaces are equivalent (for suitable definition of equivalent). In theory, our invariants would be able to entirely classify some subset of spaces we are looking at, but it is rather rare. To execute this plan, we need a source of spaces (mostly CW complexes and ways to combine them) and then methods to tell if spaces are equivalent.

1.2.1 Operations on Spaces

Let's discuss how to make new spaces from old ones. Thankfully, our operations will send CW complexes to CW complexes, though there is something to check.

Definition 1.27 (product)**.** Fix CW complexes X and Y. Then we form the *product* $X \times Y$ (at the level of CW complexes) using as $(n+m)$-cells $e_\alpha^m \times f_\beta^n$ where e_α^m is an m-cell of X and f_β^m is an n-cell of Y. Notably, the n-skeleton is

$$(X \times Y)^n = \bigcup_{k+\ell=n} X^k \times Y^\ell,$$

and one can attach in the obvious way. This produces a CW structure.

Remark 1.28. It is possible that $X \times Y$ with its CW structure need not be the same as the product topology. There is an example in the appendix of [Hat01], but we won't care so much for this course.

Definition 1.29 (subcomplex)**.** Fix a CW complex X. Then a *subcomplex* is a closed subspace $A \subseteq X$ which is a union of cells of X and also a CW complex.

Definition 1.30 (quotient)**.** Fix a subcomplex A of a CW complex X. Then X/A is also a CW complex. Here, the definition of X/A is somewhat technical: its cells are the cells of $X\backslash A$ and then a 1-cell from A, and one attaches in the obvious way (inductively) via the quotient map $X^{n-1} \to X^{n-1}/A^{n-1}$.

Definition 1.31 (suspension)**.** Fix a CW complex X. Then the *suspension* is the quotient

$$SX := \frac{X \times I}{\{(0,x) \sim (0,x') \text{ and } (1,x) \sim (1,x')\}}.$$

Example 1.32. Take $X = S^0$, which is two points. Then $X \times I$ is two lines, and we then identify the endpoints of the two lines accordingly to produce a circle S^1. More generally, $SS^n = S^{n+1}$ essentially by just gluing two S^ns onto the equator of S^{n+1}.

Definition 1.33 (join)**.** Fix CW complexes X and Y. Then the *join* $X * Y$ is the product $X \times Y \times I$ (as CW complexes) modded out by the equivalence relation identifying $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$.

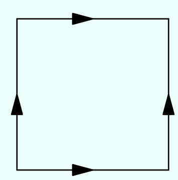
Example 1.34 (simplex)**.** Consider $X = Y = I = \Delta^1$. Then $X * Y$ is the cube modded out by crushing Y on one end and crushing X on the other end, forming a tetrahedron, which is Δ^3. More generally, $\Delta^n * \Delta^m = \Delta^{n+m+1}$.

Example 1.35. One has $S^0 * S^0 = S^1$, and more generally $S^n * X = SX$. Essentially, we are gluing two copies of X onto an equator, which is the suspension.

Definition 1.36 (wedge product)**.** Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then we form the *wedge product* $X \vee Y$ as $X \sqcup Y$ identifying $x_0 \sim y_0$.

Definition 1.37 (smash product)**.** Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then the *smash product* is $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is embedded into $X \times Y$ as $x \mapsto (x, y_0)$ and $y \mapsto (y, x_0)$.

Example 1.38. One can check that $S^1 \times S^1$ is a torus. To form the smash product, we are crushing the boundary of the square as follows.



More generally, $S^m \wedge S^n = S^{m+n}$.

Definition 1.39 (attach)**.** Fix a subcomplex A of a CW complex X_1 and a map $f\colon A \to X_0$ to another CW complex X_0. Then $X_0 \sqcup_f X_1$ is the space $X_0 \sqcup X_1$ modded out by the equivalence relation $a \sim f(a)$ for all $a \in A$.

Example 1.40. An attaching map $\varphi_\alpha\colon \partial D^n \to X^{n-1}$ of a CW complex are attachments $X^{n-1} \sqcup_{\varphi_\alpha} D^n$ in the above sense.

1.2.2 Homotopy Extension

We are going to, over time, prove the following results. To begin, quotients preserve homotopy type.

Proposition 1.41. Fix a subcomplex A of a CW complex X. If A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

Example 1.42. Fix a connected graph X, which is a one-dimensional CW complex. Fix a spanning tree $T \subseteq X$, which is contractible (any tree can be contracted one edge at a time), so $X \to X/T$ is a homotopy equivalence. Then X/T becomes a wedge of loops corresponding (roughly) to the number of "independent" cycles. Notably, this collapsing is far from canonical, essentially unique up to choosing the spanning tree and then an order of edges. In some sense, because the homotopy group of a wedge of loops is a free group, we are able to study automorphisms of the free group in this way.

Proposition 1.43. Fix a subcomplex A of a CW complex X_1. Given homotopic maps $f, g\colon A \to X_0$, then $X_0 \sqcup_f X_1 = X_0 \sqcup_g X_1$.

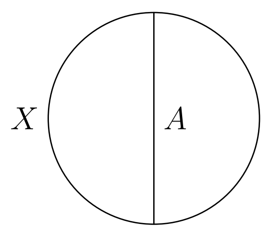
The idea of the above result is that if we can move the attaching maps f and g around, we should not really be adjusting the homotopy type.

To prove these results, we want access to the homotopy extension property.

Definition 1.44 (homotopy extension property)**.** Fix a subspace A of a topological space X. Then the pair (X, A) has the *homotopy extension property* if and only if all $F_0\colon X \to Y$ and small homotopy $f_\bullet\colon A \times I \to Y$ with $F_0|_A = f_0$, then there is an extended homotopy $F_\bullet\colon X \times I \to Y$ where $F_t|_A = f_t$ for all $t \in I$.

It will turn out that a subcomplex A of a CW complex A makes (X, A) have the homotopy extension property, but this will take some work to prove.

By way of example, make Y the following "theta graph," and the left edge is X, and A is the middle interval.



Here, $A \subseteq X$ is going to have the homotopy extension property. For example, one can contract A to a point and imagine dragging neighborhoods of $A \cap X$ in X (and in fact all of Y) along for the ride.

One way to think about the homotopy extension property is that we have a map $X \cup (A \times I) \to Y$ (by taking the union F_0 and $f_\bullet$), and we and to extend it to a full map $X \times I \to Y$. With this in mind, we would thus like to have to retract $r\colon (X \times I) \to (X \cup (A \times I))$ and then composing. By taking $Y = X \times I$, one sees that having such a retraction r is in fact equivalent to the homotopy extension property.

So we want to find the retraction $r\colon (X \times I) \to (X \cup (A \times I))$.

Lemma 1.45. Fix a subspace A of a topological space X. Then (X, A) has the homotopy extension property if and only if A has a "mapping cylinder neighborhood." In other words, there is a space B and map $f\colon B \to A$ such that M_f is homeomorphic to a neighborhood of A.

Approximately speaking, what's going on here is that the mapping cylinder allows us some squishing region through which to extend homotopies. Then the above criteria can be checked for CW pairs (X, A) by tracking through attachments. Namely, a reparameterization of the attaching map has mapping cylinder which has the property needed above. Rigorously, one inducts on the n-skeleton of a CW complex X, using the homotopy extension property for cells of X not in A (and not caring about cells already in A).

Read Hatcher

THEME 2

THE FUNDAMENTAL GROUP

2.1 August 31

We now shift gears and talk about our first algebraic invariant: the fundamental group.

2.1.1 The Fundamental Group

Let's start with an example.

Example 2.1. Fix a loop $\gamma\colon S^1 \to (\mathbb{C}\backslash\{0\})$ which is continuously differentiable. Then complex analysis tells us that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz$$

counts the number of times that γ "winds" around the integer. We might call this the "linking number" of γ. Notably, one can check that continuously varying γ does not adjust the linking number, so this linking number is homotopy invariant.

The fundamental group is a generalization of this notion.

Definition 2.2 (fundamental group)**.** Let X be a topological space, and fix a basepoint $x_0 \in X$. Then the *fundamental group* $\pi_1(X, x_0)$ is the set of homotopy equivalence classes

$$\pi_1(X, x_0) := \{[f] \text{ such that } f\colon I \to X \text{ has } f(0) = f(1) = x_0\}.$$

We will give $\pi_1(X, x_0)$ a group structure below.

Remark 2.3. There is also a $\pi_0(X)$, which consists of homotopy classes of points $[x]$ for $x \in X$, where $[x]$ denotes the path-connected component of X. If we let $\Omega(X, x_0)$ denote the topological space of loops $f\colon I \to X$ such that $f(0) = f(1) = x_0$, then we find $\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$.

Remark 2.4. If we don't want to care about basepoints, one can look at $C\left(S^1, X\right)$, which is the set of maps $S^1 \to X$. This can be given a topology via the compact-open topology. Approximately speaking, these will correspond to conjugacy classes in $\pi_1(X, x_0)$ provided that X is path-connected. For example, the homotopy class of a constant loop $S^1 \to X$ consists of the contractible loops in X; note there is something to check here in that one wants to know that a contractible loop (not relative to the basepoints) is in fact contractible relative to the basepoint.

Example 2.5. Let $X = \{x_0\}$ be a point. Then $\pi_1(X, x_0) = 1$ because there is only path $I \to X$.

Example 2.6. Let X be a convex subset of $\mathbb{R}^n$ for some $n > 0$. Then for any $x_0 \in X$ has $\pi_1(X, x_0) = 1$. Indeed, use the convex hypothesis to shrink any path down to the constant path.

We can give $\pi_1(X, x_0)$ a product via composition.

Definition 2.7 (composition)**.** Let X be a topological space, and fix a basepoint $x_0 \in X$. Given paths $f, g\colon I \to X$ such that $f(1) = g(0)$, we define the path $(f \cdot g)\colon I \to X$ via

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ g(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Note that $f \cdot g$ is well-defined at $t = 1/2$ because $f(1) = g(0)$.

The point of the above definition is to "squish" a path to do both f and g in the interval I, but at twice the speed. One has the following checks.

- The class $[f \cdot g]$ does not depend on the choice of representatives f and g. Essentially, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then one can use these two homotopies to glue together to make a new homotopy $(f_1 \cdot g_1) \sim (f_2 \sim g_2)$.
- We have $[(f \cdot g) \cdot h] = [f \cdot (g \cdot h)]$, so composition associates. The point is that these are basically reparameterizations of each other.
- There is an identity path given by $e_{x_0}(t) := x_0$. The identity check is done again by some idea of reparameterization.
- For a given path $f\colon I \to X$, we can define $\overline{f}\colon I \to X$ by $\overline{f}(t) := f(1-t)$ and then check that

  $$f \cdot \overline{f} \sim e_{f(0)},$$

  so $\left[\overline{f}\right]$ provides the inverse path for $[f]$ in $\pi_1(X, x_0)$. The point is that $f \cdot \overline{f}$ is

  $$\left(f \cdot \overline{f}\right)(t) = \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ f(2-2t) & \text{if } 0 \leqslant t \leqslant 1/2. \end{cases}$$

  One can then provide a homotopy by

  $$h_s(t) := \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant s/2, \\ f(s) & \text{if } s/2 \leqslant t \leqslant 1-s/2, \\ f(2-2t) & \text{if } 1-s/2 \leqslant t \leqslant 1, \end{cases}$$

  so $h_0 = e_{f(0)}$ and $h_1 = f \cdot \overline{f}$.

For these checks, it is helpful to have lemmas establishing continuity of piecewise functions and establishing that reparameterization does not affect homotopy class.

Remark 2.8. Staring hard at our definition of composition, one sees that our reparameterization business is really just choosing various piecewise affine maps $I \to I$ with slopes in $2^{\mathbb{Z}}$ and breaks at the dyadic rationals $2^{\mathbb{Z}}\mathbb{Z} \subseteq \mathbb{Q}$. These maps form a group called the Thompson group.

Remark 2.9 (fundamental groupoid)**.** Fix a topological space X, and define a category where the objects are points $x \in X$, and the morphisms $x \to y$ are paths (up to homotopy fixing endpoints). The above checks now show that this is in fact a category, where each morphism has an inverse. This category is called the *fundamental groupoid*. Modding out by isomorphism, our objects are now path components in X, and choosing a particular component produces the fundamental group in its endomorphisms.

Remark 2.10. Verifying that $\pi_1(X, x_0)$ only required reparameterization. So as in Remark 2.9, we could also look at the category where paths are only considered up to reparameterization, and the above checks still go through. This is related to the notion of "thin homotopy."

Lemma 2.11. Fix a topological space X. Further, fix a path $p\colon I \to X$. Then $f \mapsto (\overline{p} \cdot f \cdot p)$ provides an isomorphism $\pi_1(X, p(1)) \to \pi_1(X, p(0))$.

Proof. This is well-defined because $f_1 \sim f_2$ implies $\overline{p} \cdot f_1 \sim \overline{p} \cdot f_2$ implies $\overline{p} \cdot f_1 \cdot p \sim \overline{p} \cdot f_2 \cdot p$. This is a group homomorphism because

$$\overline{p} \cdot f \cdot g \cdot p \sim \overline{p} \cdot f \cdot p \cdot \overline{p} \cdot g \cdot p.$$

Lastly, this is an isomorphism because $\overline{p}$ provides the inverse map. ■

Remark 2.12. The above result roughly says that we can indeed look at the fundamental groupoid only in terms of the path-connected components.

Thus, we see that $\pi_1(X, x_0)$ is well-defined up to base-point provided that X is path-connected. However, the isomorphism between base-points is only defined up to path between those basepoints! Roughly speaking, the problem is that elements of $\pi_1(X, x_0)$ should really only be thought of up to inner automorphism because we can pre- and post-compose by some loop at x_0.

Lemma 2.13. If X is homeomorphic to Y by $\varphi\colon X \to Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ for any $x_0 \in X$.

Proof. Use φ. ■

2.1.2 The Fundamental Group of S^1

Here is our result.

Theorem 2.14. Fix any $x \in S^1$. Then $\pi_1\left(S^1, x\right) \cong \mathbb{Z}$. In fact, there is an isomorphism $\Phi\colon \mathbb{Z} \to \pi_1\left(S^1, x\right)$ given by

$$n \mapsto [t \mapsto (\cos 2\pi nt, \sin 2\pi nt)].$$

Sketch without covering spaces. We show injectivity and surjectivity independently.

- Think of S^1 as embedded in $\mathbb{C}$ as $\{z : |z| = 1\}$ and take a smooth path $f\colon I \to S^1$, lift it to a map $\widetilde{f}\colon I \to \mathbb{R}$ via

$$\widetilde{f}(t) := \int_0^t d\theta,$$

where $d\theta$ is some differential form S^1 (say, $x\,dy - y\,dx$). Then $\widetilde{f}(1)$ is intuitively contained in $2\pi\mathbb{Z}$ and is homotopy invariant. Now, f is not smooth, then we can use some small homotopy to make f smooth and then use the above argument. This provides an inverse map to Φ and thus shows that Φ is injective.

- For surjectivity, one can use uniform continuity of any path $f\colon I \to S^1$ and the compactness of S^1 in order to divide up I into intervals on which f can be written as a composition of well-behaved paths, which eventually allows us to force f to make piecewise linear. Once f is piecewise linear, we go interval-by-interval and fix f to be constant speed. Eventually f becomes one of the $\Phi(n)$ for some n. ■

For the covering space approach, the point is that we understand the fundamental group of $\mathbb{R}$ well, and we have a fairly well-behaved "covering map" $p\colon \mathbb{R} \to S^1$ given by $p(\theta) := (\cos 2\pi\theta, \sin 2\pi\theta)$. The main claim, then, is that any path $\omega\colon I \to S^1$ has a unique lift $\widetilde{\omega}\colon I \to \mathbb{R}$ such that $\widetilde{\omega}(0) = \omega(0)$ and $p \circ \widetilde{\omega} = \omega$. The point is that once we lift, we can use a homotopy up in $\mathbb{R}$ (fixing the endpoints of $\widetilde{\omega}$), which will then go back down to a homotopy on S^1 if we are careful. Anyway, this lifting process can essentially be done as described in the surjectivity check above.

2.2 September 5

Today we actually prove $\pi_1^(S^1) \cong \mathbb{Z}$.

2.2.1 Eckmann–Hilton Argument

Because it is fun, we begin with some nonsense.

Proposition 2.15 (Eckmann–Hilton). Let X be a set equipped with the binary operations $\circ$ and $*$ such that the following hold.

(a) Identity: there are elements $1_\circ, 1_* \in X$ such that $1_\circ \circ a = a \circ 1_\circ = a$ and $1_* * a = a * 1_* = a$ for all $a \in X$.

(b) Distribution: we have $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$ for all $a, b, c, d \in X$.

Then $\circ$ and $*$ are the same operation and in fact are both commutative and associative.

Proof. This is purely formal. We proceed in steps.

1. We show that $1_\circ = 1_*$. Indeed, note
$$1_* = 1_* * 1_* = (1_* \circ 1_\circ) * (1_\circ \circ 1_*) = (1_* * 1_\circ) \circ (1_\circ * 1_*) = 1_\circ \circ 1_\circ = 1_\circ.$$
From now on, we use the symbol 1 to denote our identity $1_\circ = 1_*$.

2. We show that $a * b = a \circ b$. Indeed, note
$$a * b = (a \circ 1) * (1 \circ b) = (a * 1) \circ (1 * b) = a \circ b.$$
Thus, our operations are the same, and we will use the symbol $*$ to denote both of them now. Notably, our distribution law is $(a * b) * (c * d) = (a * c) * (b * d)$.

3. We show that $*$ is commutative. Indeed, for any $a, b \in X$, we see
$$a * b = (1 * a) * (b * 1) = (1 * b) * (a * 1) = b * a.$$

4. We show that $*$ is associative. Indeed,
$$(a * b) * c = (a * b) * (1 * c) = (a * 1) * (b * c) = a * (b * c),$$
for any $a, b, c \in X$. ■

As an application, we have the following result.

Corollary 2.16. Let G be a topological group with identity $e \in G$. Then $\pi_1(G,e)$ is abelian.

Proof. Let $\cdot$ denote the usual concatenation operation on $\pi_1(G,e)$. The point is to give another binary operation to $\pi_1(G,e)$ and then apply Proposition 2.15.

Well, let $*$ denote the group operation on G, and for paths $f,g\colon I \to G$ based at e, we define the path $(f*g)\colon I \to G$ by $(f*g)(t) := f(t)*g(t)$. Here are the necessary checks for our purposes.

- Note $f*g$ is a continuous map because it is the composite of the continuous maps

$$I \overset{(\mathrm{id}_I,\mathrm{id}_I)}{\to} I \times I \overset{(f,g)}{\to} G \times G \overset{*}{\to} G.$$

- We show $[f*g]$ does not depend on the choice of homotopy classes $[f]$ and $[g]$, so we may view $*$ as a binary operation on $\pi_1(G,e)$. Suppose $f \sim f'$ and $g \sim g'$ by the homotopies $F_\bullet$ and $G_\bullet$, respectively. We want to show that $f*g \sim f'*g'$. Well, define $H_\bullet\colon G \times I \to G$ by $H_t(x) := F_t(x)*G_t(x)$ for all $t \in I$ and $x \in G$. Then we see that $H_0 = F_0 * G_0 = f*g$ and $H_1 = F_1 * G_1 = f'*g'$, and $H_\bullet$ is continuous because it is the composite

$$G \times I \overset{(F_\bullet,G_\bullet)}{\to} G \times G \overset{*}{\to} G.$$

- Note that $*$ has an identity element given by the constant path $c(t) := e$ for all $t \in I$. Indeed, for any $[f] \in \pi_1(G,e)$, we see that $(f*c)(t) = f(t)*c(t) = f(t)$ for all $t \in I$, so $[f]*[c] = [f*c] = [f]$.

- Fix $[a],[b],[c],[d] \in \pi_1(G,e)$. We claim that

$$([a]\cdot[b])*([c]\cdot[d]) \overset{?}{=} ([a]*[c])\cdot([b]*[d]).$$

  Removing all the homotopy classes, it is enough to show that $(a\cdot b)*(c\cdot d) = (a*c)\cdot(b*d)$. Well, for any $t \in I$, we compute

$$((a\cdot b)*(c\cdot d))(t) = (a\cdot b)(t)*(c\cdot d)(t) = \begin{cases} a(t)*c(t) & \text{if } t \leqslant 1/2, \\ b(t)*d(t) & \text{if } t \geqslant 1/2, \end{cases}$$

  and

$$((a*c)\cdot(b*d))(t) = \begin{cases} (a*c)(t) & \text{if } t \leqslant 1/2, \\ (b*d)(t) & \text{if } t \geqslant 1/2, \end{cases}$$

  which is the same path.

Now, Proposition 2.15 shows that $*$ and $\cdot$ must be the same operation on $\pi_1(G,e)$ and that $\cdot$ is commutative, which is what we wanted. ■

2.2.2 Covering Spaces

Our computation is going to use the notion of a covering space.

Definition 2.17 (covering space). Fix a topological space X. Then a *covering space* is a topological space $\widetilde{X}$ together with a projection map $p\colon \widetilde{X} \to X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha\in\lambda} U_\alpha$ where U_α is open and $p\colon U_\alpha \to U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered.*

The fact we will require about covering spaces is the following "fibration property."

Proposition 2.18. Fix a topological space X and covering space $p\colon \widetilde{X} \to X$. Further, suppose we have maps $F\colon Y \times I \to X$ and $\widetilde{F}\colon Y \times \{0\} \to \widetilde{X}$ such that $p \circ \widetilde{F}|_{Y\times\{0\}} = F|_{Y\times\{0\}}$. Then there is a unique extension $\widetilde{F}\colon Y \times I \to \widetilde{X}$ such that $p \circ \widetilde{F} = F$.

Proof. We proceed in steps. Say that a subset $U \subseteq X$ is "evenly covered" if and only if $p^{-1}(U) = \bigsqcup_{\alpha\in\lambda} U_\alpha$ and $p\colon U_\alpha \to U$ is a homeomorphism. Note that making an evenly covered open subset smaller will retain it being evenly covered using the fact that the maps $p\colon U_\alpha \to U$ is a homeomorphism.

1. To set us up, given $y \in Y$, we claim that there we can find an open neighborhood V of y and a finite open cover $\mathcal{U}$ of I such that $F(V \times U)$ is contained in an evenly covered subset of X for any $U \in \mathcal{U}$. The point is to use compactness to shrink an evenly covered subset containing $F(V \times I)$ sufficiently. Well, for each $t \in I$, we may find and evenly covered subset $U_t \subseteq X$ containing $F(y,t)$ and then find $\varepsilon_t > 0$ and an open neighborhood V_t of y such that $V_t \times (t - \varepsilon_t, t + \varepsilon_t) \subseteq F^{-1}(U_{t_i})$.

   Now, by compactness, we may choose finitely many t labeled $\{t_1, \ldots, t_n\}$ and set $\varepsilon_i := \varepsilon_{t_i}$ and $V_i := V_{t_i}$ and $U_i := U_{t_i}$ such that the intervals $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ covers I and $F(V_i \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq F^{-1}(U_i)$. Now, set
   $$V := \bigcap_{i=1}^{n} V_i$$
   so any $t \in I$ lives in some $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ has $F(V \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq U_i$.

2. We prove uniqueness. It is enough to show this in the case where Y is a point. Namely, fix suppose we have two lifts $\widetilde{F}_1$ and $\widetilde{F}_2$ of F which agree with $\widetilde{F}$. Then, fixing some $y \in Y$, we see that $\widetilde{F}_1(y)$ and $\widetilde{F}_2(y)$ are maps $I \to \widetilde{X}$ lifting $F(y)\colon I \to X$ which equal $\widetilde{F}(y,0)$ at 0. In this setting, we want to show that $\widetilde{F}_1(y,t) = \widetilde{F}_2(y,t)$ for all $t \in I$. As such, we suppress the point $y \in Y$ in the argument which follows.

   The previous step promises us a finite open cover $\mathcal{U}$ of I such that $F(U)$ is contained in an evenly covered open subset of X for each $U \in \mathcal{U}$. Ordering the endpoints of $\mathcal{U}$, we produce a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that $F([t_i, t_{i+1}])$ is covered in an evenly covered subset of U_i for each i.

   We are now ready to show our uniqueness. We show that $\widetilde{F}_1(t) = \widetilde{F}_2(t)$ for each $t \in [0, t_i]$ by induction on i. At $i = 0$, there is nothing to say because $\widetilde{F}_1(0) = \widetilde{F}(0) = \widetilde{F}_2(0)$. Now, for the induction, we are given that $\widetilde{F}_1(t_i) = \widetilde{F}_2(t_i)$. The point is that $F([t_i, t_{i+1}])$ is contained in an evenly covered subset $U_i \subseteq X$, so $\widetilde{F}_1([t_i, t_{i+1}])$ lands in one of the disjoint copies of U_i of $p^{-1}(U_i)$, and it lands in exactly one because $[t_i, t_{i+1}]$ is connected; let $\widetilde{U}_i$ be the corresponding disjoint copy. The same statement holds for $\widetilde{F}_2$, and in fact $\widetilde{F}_2([t_i, t_{i+1}]) \subseteq \widetilde{U}_i$ because $\widetilde{F}_2([t_i, t_{i+1}])$ needs to land in the same copy of U_i containing $\widetilde{F}_1(t_i) = \widetilde{F}_2(t_i)$.

   We are now done. Note $p\colon \widetilde{U}_i \to U_i$ is injective, so
   $$p \circ \widetilde{F}_1 = p \circ \widetilde{F}_2$$
   for $t \in [t_i, t_{i+1}]$ forces equality after removing t.

3. Fix some $y \in Y$. We will extend locally: we construct some open neighborhood V of y and a lift $\widetilde{F}\colon V \times I \to \widetilde{X}$ of $F|_{V\times I}$. The point is to "spread out" from $\{y\} \times I$ using the previous step.

   As before, the first step promises us an open neighborhood V of y and a finite open cover $\mathcal{U}$ of I such that $F(V \times U)$ is contained in an evenly covered subset for each $U \in \mathcal{U}$. Arranging the endpoints of the open sets in $\mathcal{U}$, we may say that we have a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $F(V \times [t_i, t_{i+1}])$ is contained in an evenly covered open subset $U_i \subseteq X$ for each i.

   We now extend F to $\widetilde{F}$ on $[0, t_i]$ inductively. For $i = 0$, there is nothing to do because $\widetilde{F}|_{Y\times\{0\}}$ is already fixed. Now, suppose we have a definition of $\widetilde{F}$ on $V \times [0, t_i]$. Say $F(V \times [t_i, t_{i+1}]) \subseteq U_i$, and select the copy of U_i named $\widetilde{U}_i \subseteq p^{-1}(U_i)$ by requiring it to contain $\widetilde{F}(y, t_i)$. Now, shrink V so that $V \times \{t_i\}$

contains y still but now is contained in $\widetilde{U}_i$. Now, define $\widetilde{F}$ on $V \times [t_i, t_{i+1}]$ by pre-composing with the homeomorphism

$$p^{-1} \colon U_i \to \widetilde{U}_i,$$

and we produce a continuous map because we have agreed on the seam at $V \times \{t_i\}$.[1] This completes the lifting to a neighborhood V of y.

4. We can now glue the lifts $\widetilde{F}$ constructed in the previous step, and the gluing is well-defined because they must agree on intersections by the uniqueness of the second step. This completes the proof. ■

And now here is our result.

Theorem 2.19. For any $x \in S^1$, we have $\pi_1\left(S^1, x\right) \cong \mathbb{Z}$.

Proof. For brevity, embed S^1 into $\mathbb{C}$ as $S^1 = \mathbb{R}/\mathbb{Z}$, and let our basepoint be $0 \in S^1$. We now abbreviate our fundamental group to $\pi_1\left(S^1\right)$.

Now, we note that we have the continuous (in fact, holomorphic) path $\omega_n \colon [0,1] \to S^1$ given by $t \mapsto nt$. A reparameterization argument can show that $[\omega_n] \cdot [\omega_m] = [\omega_{m+n}]$ for any $m, n \in \mathbb{Z}$, so we have defined a homomorphism $\varphi \colon \mathbb{Z} \to \pi_1(S^1)$. We would like to show that this map is an isomorphism. We will use Proposition 2.18, for which we note that $p \colon \mathbb{R} \to S^1$ given by $p(t) := t$ is a covering space map. Indeed, for each $t \in S^1$, choose the neighborhood $(t - 0.1, t + 0.1)$ so that

$$p^{-1}((t-0.1, t+0.1)) = (t-0.1, t+0.1) + \mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} (t+n-0.1, t+n+0.1).$$

We now show that φ is an isomorphism.

- Surjective: let $f \colon I \to S^1$ be a loop, and we want to show that $f \sim \omega_n$ for some $n \in \mathbb{Z}$. By Proposition 2.18 applied with Y being a point, we get a path $\widetilde{f} \colon I \to \mathbb{R}$ such that $f = p \circ \widetilde{f}$. Now, set $n := \widetilde{f}(1)$, which is indeed an integer, and we claim $\widetilde{f} \sim \widetilde{\omega}_n$, where $\widetilde{\omega}_n(t) := nt$; this will finish after composing with the projection p as it shows that $f \sim \omega_n$ by Lemma 1.9.

  To see this, we define the map $h \colon I \times I \to \mathbb{R}$ by

  $$h_t(s) := (1-t)\widetilde{f}(s) + t\widetilde{\omega}_n(s).$$

  Then h is continuous because it is the composite

  $$I \overset{(\mathrm{id}, 1-\mathrm{id}, \widetilde{f}, \omega_n)}{\to} I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

  where the last map is taking a linear combination. Now, $h_0 = \widetilde{f}$ and $h_1 = \widetilde{\omega}_n$, so $\widetilde{f} \sim \widetilde{\omega}_n$ follows.

- Injective: suppose $\omega_n \sim \omega_0$, and we want to show that $n = 0$. Then we have a homotopy $h_\bullet \colon I \times I \to X$ such that $h_0 = \omega_n$ and $h_1 = \omega_0$. Then Proposition 2.18 produces a unique lift $\widetilde{h}_\bullet \colon I \times I \to \widetilde{X}$ of h such that $\widetilde{h}_t(0) = 0$ for each $t \in I$. Now, the map $t \mapsto \widetilde{h}_t(1)$ is continuous, and $h_t(1) = 0$ for each $t \in I$, so the map $t \mapsto \widetilde{h}_t(1)$ maps to the discrete space $\mathbb{Z}$. It follows that $\widetilde{h}_0(1) = \widetilde{h}_1(1)$, so $0 = n$ because of how ω_0 and ω_n lift to $\mathbb{R}$. ■

2.2.3 The Fundamental Group Functor

Let's do some nonsense checks, for fun.

[1] To avoid this annoyance at the seam, one can allow the partition to overlap a bit so that we only ever glue continuous maps along open sets, which is legal. I won't write this out.

Definition 2.20 (based topological space)**.** A *based topological space* (X, x_0) is a topological space X together with a basepoint $x_0 \in X$. A map of based topological spaces $\varphi\colon (X, x_0) \to (Y, y_0)$ is a continuous map $\varphi\colon X \to Y$ such that $\varphi(x_0) = y_0$. The category with these objects and morphisms is Top_*.

We won't bother to check that Top_* is a category. Here is the main point of this subsection.

Proposition 2.21. We have a functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$.

Proof. We already know that $\pi_1(X, x_0)$ is a group for each based topological space (X, x_0), so we really only have to check the functoriality properties.

Fix a map $\varphi\colon (X, x_0) \to (Y, y_0)$ of based topological spaces. We need to define a group homomorphism $\pi_1(\varphi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Well, given a loop $f : I \to X$ based at x_0, we note that $(\varphi \circ f)\colon I \to Y$ is a loop based at $y_0 = \varphi(x_0)$, so we hope that our desired map is $(\varphi \circ -)$. Here are our checks.

- Well-defined: if $f \sim f'$, we need to show that $\varphi \circ f \sim \varphi \circ f'$. This is simply Lemma 1.9.
- Group homomorphism: we need to show that $(\varphi \circ f) \cdot (\varphi \circ g) \sim \varphi \circ (f \cdot g)$ for loops $f, g\colon I \to X$ based at x_0. In fact, these paths are equal: for $t \in I$, we compute

$$((\varphi \circ f) \cdot (\varphi \circ g))(t) = \begin{cases} \varphi(f(2t)) & \text{if } t \leqslant 1/2, \\ \varphi(g(2t-1)) & \text{if } t \geqslant 1/2, \end{cases} = (\varphi \circ (f\dot{g}))(t).$$

We now prove functoriality of π_1.

- Identity: note that $\mathrm{id}_X : (X, x_0) \to (X, x_0)$ has $\mathrm{id}_X \circ f = f$ for any path $f : I \to X$, so $\pi_1(\mathrm{id}_X)([f]) = [f]$ for any $[f] \in \pi_1(X, x_0)$.
- Composition: given maps $\varphi\colon (X, x_0) \to (Y, y_0)$ and $\psi\colon (Y, y_0) \to (Z, z_0)$ and a loop $f : I \to X$ based at x_0, we see that

$$\pi_1(\psi \circ \varphi)([f]) = [\psi \circ \varphi \circ f] = \pi_1(\psi)([\varphi \circ f]) = (\pi_1(\psi) \circ \pi_1(\varphi))([f]),$$

  which finishes. ■

Of course, just being a functor is not terribly interesting. Here is a nice property.

Proposition 2.22. Fix based topological spaces (X, x_0) and (Y, y_0). Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $p_X : (X \times Y, (x_0, y_0)) \to (X, x_0)$ and $p_Y : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ denote the projections. Now, note that we have a map

$$(\pi_1(p_X), \pi_2(p_Y))\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

which we claim is an isomorphism. For brevity, let this morphism be φ. Of course, φ is a homomorphism because π_1 is a functor (see Proposition 2.21).

- Surjective: fix loops $f_X : I \to X$ and $f_Y : I \to Y$ based at x_0 and y_0 respectively. Then the map $f(t) := (f_X(t), f_Y(t))$ defines a loop $I \to X \times Y$ based at (x_0, y_0), and by construction $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$, so

$$\varphi(f) = ([p_X \circ f], [p_Y \circ f]) = ([f_X], [f_Y]).$$

- Injective: suppose $\varphi([f]) = \varphi([g])$, and we want to show that $[f] = [g]$. Well, we have homotopies $h_{X\bullet}\colon I \times I \to X$ and $h_{Y\bullet}\colon I \times I \to Y$ such that $h_{X0} = p_X \circ f$ and $h_{X1} = p_X \circ g$ and $h_{Y0} = p_Y \circ f$ and $h_{Y1} = p_Y \circ g$. Then we define $h_\bullet\colon I \times I \to X \times Y$ by

$$h_t(s) := (h_{Xt}(s), h_{Yt}(s)).$$

Note h_t is continuous because it is continuous in each coordinate. To finish, we see $h_0 = f$ and $h_1 = g$ by checking after applying the projections p_X and p_Y, so $f \sim g$ follows. ■

Remark 2.23. More precisely, the above proof has shown that π_1 preserves products.

Example 2.24. We have $\pi_1\left(S^1 \times S^1\right) \cong \mathbb{Z}^2$ by Proposition 2.22 and Theorem 2.19.

Example 2.25. We show that there is no retraction $r\colon D^2 \to S^1$. Let $i\colon S^1 \to D^2$ be the inclusion. If there is a retraction r, then we see that $r \circ i = \mathrm{id}_{S^1}$, so functoriality of π_1 means that the composite

$$\pi_1\left(S^1\right) \xrightarrow{i} \pi_1\left(D^2\right) \xrightarrow{r} \pi_1\left(S^1\right)$$

is an isomorphism. In particular, i is injective. However, $\pi_1\left(S^1\right) \cong \mathbb{Z}$ by Theorem 2.19, and $\pi_1\left(D^2\right) = 0$ because D^2 is convex and hence contractible.

Remark 2.26. One can use Example 2.25 to show Brouwer's fixed point theorem: we show that any continuous map $h\colon D^2 \to D^2$ has a fixed point. Well, suppose h has no fixed point. Then there is a continuous map sending $x \in D^2$ to the point on S^1 which intersects with the raw starting at $h(x)$ and then going through x. Then $h\colon D^2 \to S^1$ defines a retraction, contradicting Example 2.25.

2.3 September 7

Today we prove the van Kampen theorem.

2.3.1 Free Products of Groups

We will be somewhat brief on this because this is not an algebra class.

Definition 2.27 (free product)**.** Let $\{G_\alpha\}_{\alpha\in\lambda}$ be a collection of groups. Then we form the free product $*_{\alpha\in\lambda}G_\alpha$ as having underlying set given by strings of words whose letters are in the G_α, modded out by the relations $g_\alpha \cdot h_\alpha = g_\alpha h_\alpha$ whenever $g_\alpha, h_\alpha \in G_\alpha$ for some $\alpha \in \lambda$.

Perhaps one should check that this forms a group, so we will sketch what one should do.

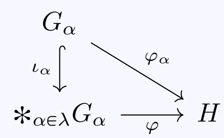
1. Let W be the set of finite strings (i.e., words) whose letters are g or g^{-1} where $g \in G_\alpha$ for some $\alpha \in \lambda$. Then we build $\overline{W}$ by allowing combining $g_\alpha \cdot h_\alpha$ into a single character $g_\alpha h_\alpha$ provided that g_α and h_α belong to the same group G_α. We will realize our desired group as a subgroup of $\operatorname{Aut}(W)$.

2. For each $g \in W$, define the function $L_g\colon W \to W$ by left concatenation. One should show that L_g is in fact a well-defined function, which depends on the equivalence relation defining W, but in short, one can show that having two words w and w' with $w \sim w'$ enforces $g \cdot w \sim g \cdot w'$ by using the same concatenation rules on both sides. A rigorous argument would need to use an induction, which we won't bother to write out.

3. Note that L_e (where e denotes the empty string) is the identity on W, and $L_{g^{-1}}$ is the inverse of L_g. Thus, the image of $L_\bullet$ in W is a subgroup of $\text{Aut}(W)$, and we call this subgroup $*_{\alpha\in\lambda}G_\alpha$. One realizes this group as the free product described above by identifying L_g with $L_g(e)$. The point of introducing $L_\bullet$ at all is to make the various group law checks easier.

One has the following universal property, which we will not prove, again because this is not an algebra class.

Proposition 2.28. Let $\{G_\alpha\}_{\alpha\in\lambda}$ be a collection of groups. Given homomorphisms $\varphi_\alpha\colon G_\alpha \to H$ to a target group H, there is a unique homomorphism $\varphi\colon *_{\alpha\in\lambda} G_\alpha \to H$ such that the following diagram commutes.



Here, $\iota_\alpha\colon G_\alpha \to *_{\alpha\in\lambda}G_\alpha$ is the inclusion.

Proof. Let's sketch the proof. We begin by showing uniqueness of φ. Given a word $g_{\alpha_1}\cdots g_{\alpha_n}$ in $*_{\alpha\in\lambda}$, we see that the commutativity of the diagram enforces

$$\begin{aligned}\varphi\left(g_{\alpha_1}\cdots g_{\alpha_n}\right) &= \varphi(g_{\alpha_1})\cdots\varphi(g_{\alpha_n})\\ &= \varphi(\iota_{\alpha_1}(g_{\alpha_1}))\cdots\varphi(\iota_{\alpha_n}(g_{\alpha_n}))\\ &= \varphi_{\alpha_1}(g_{\alpha_1})\cdots\varphi_{\alpha_n}(g_{\alpha_n}).\end{aligned}$$

Thus, φ is uniquely determined by the φ_α. It remains to show that the above formula in fact defines a group homomorphism, which follows roughly speaking by the minimal construction of $*_{\alpha\in\lambda}$. Namely, we have thus far defined a function $\varphi\colon W \to H$ where W is the set of all words, so one needs to check that we are still safe after modding out by the requisite equivalence relation on W. We will not do this, but in short, one can use induction on the various generators of the group presentation of $*_{\alpha\in\lambda}G_\alpha$. ■

In the discussion that follows, we will frequently use group presentations, which is an expression of the form

$$\langle a_1, a_2, \ldots, : w_1, w_2, \ldots\rangle,$$

where the $a_\bullet$ are generators for words giving the group and $w_\bullet$ are words intended to produce relations for the group, by default of the form $w_\bullet = 1$.

Example 2.29. The group $\langle a\rangle$ gives $\mathbb{Z}$. Namely, the group consists of the elements

$$\left\{\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots\right\}.$$

Example 2.30. The group $\langle a : a^2\rangle$ gives $\mathbb{Z}/2\mathbb{Z}$. Namely, our isomorphism is by sending $k \in \mathbb{Z}/2\mathbb{Z}$ to a^k. This is well-defined because $2 \mapsto a^2$, and a^2 is the identity of the group.

2.3.2 van Kampen's Theorem

In this subsection, we state and prove the van Kampen theorem. Let's explain the idea. Suppose we can decompose X into path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$. Then the inclusions $i_\alpha\colon A_\alpha \hookrightarrow X$ induce maps $\pi_1(A_\alpha) \to \pi_1(X)$, which by the nature of the free product induces a map

$$*_{\alpha\in\lambda}\pi_1(A_\alpha) \to \pi_1(X).$$

It is not too hard to see that this map is surjective.

Lemma 2.31. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$ each containing a basepoint $x_0 \in X$. For any loop $\gamma\colon I \to X$ based at x_0, there are loops $\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}$ based at x_0 such that

$$\gamma \sim \gamma_{\alpha_1} \cdot \cdots \cdot \gamma_{\alpha_n}$$

and γ_{α_n} is a path connected in A_{α_n} for each α_n.

Proof. For each $\alpha \in \lambda$, decompose $\gamma^{-1}(A_\alpha) \subseteq I$ into a collection of intervals $\mathcal{I}_\alpha$. Then

$$I = \gamma^{-1}(X) = \bigcup_{\alpha\in\lambda} \gamma^{-1}(A_\alpha) = \bigcup_{\alpha\in\lambda} \bigcup_{I'\in\mathcal{I}_\alpha} I'.$$

Now, I is compact, so this open cover can be turned into a finite subcover $\{(a_k, b_k)\}_{k=1}^n$ where $\gamma((a_k, b_k)) \subseteq A_{\alpha_k}$ for some $\alpha_k \in \lambda$. Ordering the (a_k, b_k), we produce a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma([t_k, t_{k+1}]) \subseteq A_{\alpha_k}$ for some perhaps different n and $\alpha_k \in \lambda$.

We are now ready to finish. For each $1 \leqslant k \leqslant n-1$, we recall that A_{α_k} is path-connected, so we can find a path η_k from $\gamma(t_k)$ to x_0. Then we see that

$$\begin{aligned}\gamma &\sim \gamma|_{[t_0,t_1]} \cdot \gamma|_{[t_1,t_2]} \cdot \cdots \cdot \gamma|_{[t_{n-2},t_{n-1}]} \cdot \gamma|_{[t_{n-1},t_n]} \\ &\sim \underbrace{\gamma|_{[t_0,t_1]} \cdot \eta_1}_{\gamma_0:=} \cdot \underbrace{\overline{\eta}_1 \cdot \gamma|_{[t_1,t_2]} \cdot \eta_2}_{\gamma_1:=} \cdot \overline{\eta}_2 \cdot \cdots \cdot \eta_{n-2} \cdot \underbrace{\overline{\eta}_{n-2} \cdot \gamma|_{[t_{n-2},t_{n-1}]} \cdot \eta_{n-1}}_{\gamma_{n-2}:=} \cdot \underbrace{\overline{\eta}_{n-1} \cdot \gamma|_{[t_{n-1},t_n]}}_{\gamma_{n-1}:=}.\end{aligned}$$

The above expression provides the desired factorization. ■

Corollary 2.32. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$ each containing a basepoint $x_0 \in X$. Then the map induced map

$$*_{\alpha\in\lambda}\pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

Proof. This is direct from Lemma 2.31. ■

We would now like to compute its kernel of our induced map. Well, if $A_\alpha \cap A_\beta$ is path-connected, then we let $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$ denote the inclusion, and we note that

$$\begin{array}{ccc} A_\alpha \cap A_\beta & \xrightarrow{i_{\alpha\beta}} & A_\alpha \\ {\scriptstyle i_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle i_\alpha} \\ A_\beta & \xrightarrow{i_\beta} & X \end{array}$$

commutes, so

$$\begin{array}{ccc} \pi_1(A_\alpha \cap A_\beta) & \xrightarrow{\pi_1(i_{\alpha\beta})} & \pi_1(A_\alpha) \\ {\scriptstyle \pi_1(i_{\beta\alpha})}\downarrow & & \downarrow{\scriptstyle \pi_1(i_\alpha)} \\ \pi_1(A_\beta) & \xrightarrow{\pi_1(i_\beta)} & \pi_1(X) \end{array}$$

also commutes. Thus, for any $\gamma \in \pi_1(A_\alpha \cap A_\beta)$, we see that $\pi_1(i_\alpha)(\pi_1(i_{\alpha\beta})([\gamma])) = \pi_1(i_\beta)(\pi_1(i_{\beta\alpha})([\gamma]))$, which produces a relation belonging to the kernel of our surjection $*_{\alpha\in\lambda}\pi_1(A_\alpha) \to \pi_1(X)$. Under favorable circumstances, van Kampen's theorem tells us that this is the entire kernel.

2.4 September 12

Let's wrap up some loose ends. People are doing pretty well on the homeworks, but please cite theorems and so on to be rigorous.

2.4.1 The Fundamental Group of a Torus Knot

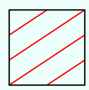
Let's give a few applications of van Kampen's theorem.

Example 2.33. Let $K \subseteq \mathbb{R}^n$ be a compact subset for $n \geqslant 3$. Embed $\mathbb{R}^n \subseteq S^n$ by stereographic projection, and van Kampen shows that

$$\pi_1\left(\mathbb{R}^n \backslash K\right) \cong \pi_1\left(S^n \backslash K\right).$$

More precisely, we have S^n sitting inside $S^{n-1} \times \mathbb{R}$ (place K inside a large ball, and we can continuously deform any loop in $\mathbb{R}^n \backslash K$ into this large ball), and the π_1 arising from this $\mathbb{R}$ cannot help you.

Example 2.34 (torus knots)**.** Fix positive integers $m, n \in \mathbb{Z}$ bigger than 1 with $\gcd(m,n) = 1$. Define the torus knot $K_{m,n} \subseteq T^2$ (where $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$) as the image of the line $my = nx$; alternatively, it is the image of the map $t \mapsto (mt, nt)$. For example, here is $K_{3,2}$ sitting inside the square $\mathbb{R}^2/\mathbb{Z}^2$.



We compute $\pi_1\left(\mathbb{R}^3 \backslash K_{m,n}\right)$.

Proof. Professor Agol seems to prefer the "Clifford torus" thought of as

$$T^2 = \left\{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\right\}.$$

This sits inside $S^3 = \left\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\right\}$. Anyway, we begin by giving us some breathing room. Define the "thickening" of K as

$$A := \left\{(z_1, z_2) : |z_1| < \frac{1}{\sqrt{2}} + \varepsilon\right\} \backslash \left\{\left(rz^m, \sqrt{1-r^2}z^n\right) : z \in S^1, \frac{1}{\sqrt{2}} \leqslant r \leqslant \frac{1}{\sqrt{2}} + \varepsilon\right\}$$

(namely, A is the torus thickened in such a way that it carries the subtraction of $K_{m,n}$) and in the other way as

$$B := \left\{(z_1, z_2) : |z_2| < \frac{1\sqrt{2}}{+}\varepsilon\right\} \backslash \left\{(\sqrt{1-r^2}z^m, rz^n) : \frac{1}{\sqrt{2}} \leqslant r \leqslant \frac{1}{\sqrt{2}} + \varepsilon\right\}.$$

Intersecting, we see that $A \cap B$ is $(S^1 \times S^1)\backslash K_{m,n}$ thickened by $(-\varepsilon, \varepsilon)$, which we note can be deformed to $(S^1 \times \mathbb{R}) \times (-\varepsilon, \varepsilon)$, which has fundamental group $\mathbb{Z}$. Notably, $\pi_1(A) \cong \mathbb{Z}$ and $\pi_1(B) \cong \mathbb{Z}$ by deforming them carefully to the circle S^1, so our fundamental group is going to be $(\mathbb{Z} * \mathbb{Z})/\mathbb{Z}$ by van Kampen.

However, we need to compute the image of $\pi_1(A \cap B)$ in $\pi_1(A) * \pi_1(B)$. In the retraction of A down to a circle says that the image in $\pi_1(A)$ is by multiplication by n, and similarly going to B is multiplication by m. We conclude that our fundamental group is

$$\langle a, b : a^n = b^m \rangle.$$

As an aside, we note that $S^3 \backslash K_{m,n}$ will have a deformation retract back to $K_{m,n}$ shifted upwards by some amount (for example, see the diagram and imagine a copy of $K_{m,n}$ shifted up by some small $\varepsilon > 0$). Anyway, for $m, n > 1$ we can see that the center of the above group is $\langle a^n \rangle$, so modding out by the center yields $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. In total, we are able to distinguish the torus knots $S^3 \backslash K_{m,n}$ from each other.[2] To deal with the signs of m and n, we need a notion of isotopy to distinguish a knot from its "mirror image." ■

[2] Alternatively, the abelianization $\pi_1(S^3 \backslash K_{m,n})$ is the free group with $(m-1)(n-1)$ generators, and the abelianization of $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ is mn, from which we can read off m and n.

2.4.2 The Fundamental Group of a CW Complex

Let's move on from knots and compute the fundamental group of some cell complexes.

Example 2.35. Let X be a connected graph (i.e., a 1-dimensional CW complex), then $\pi_1(X)$ is homotopy equivalent to a wedge of circles, which has fundamental group $\mathbb{Z}^{*r}$ for some r, which is the free group on r letters.

Now, if Y is a connected CW complex, then $\pi_1(Y^1)$ is a free group. Then $\pi_1(Y^2)$ might be complicated, but let's imagine computing $\pi_1(Y^3)$. The point is that we take some ball $e^3_\alpha \cong D^2$ and attach it via some $\varphi_\alpha \colon \partial D^3 \to Y^2$.

To compute the fundamental group of this, we cover $Y^2 \sqcup_{\varphi_\alpha} e^3_\alpha$ by $A := Y^2 \cup_{\varphi_\alpha} (e^3_\alpha \setminus \{x\})$ and $B = (e^\alpha_3)^\circ$ (Here, x is some point in the interior.) Notably, the intersection is simply $S^2 \times \mathbb{R}$, which is trivial, so we conclude that the attachment e^3_α did nothing to our fundamental group by van Kampen. Applying this argument inductively (perhaps transfinitely), we see that $\pi_1(Y^3) = \pi_1(Y^2)$. One can continue upwards to conclude that $\pi_1(Y) = \pi_1(Y^2)$.[3]

Now, let's say that we actually want to compute $\pi_1(Y^2)$. To do so, we note that we have a surjection $\pi_1(Y^1) \to \pi_1(Y^2)$ given by the inclusion (any loop can be deformed off the 2-skeleton to the 1-skeleton). Now, for each 2-cell e^2_α attached via $\varphi_\alpha \colon \partial e^2_\alpha \to Y^1$, we choose a path $\gamma_\alpha \colon I \to Y^1$ so that $\gamma_\alpha(0) = y$ and $\gamma_\alpha(1) = \varphi_\alpha(0)$ and then find that

$$\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$$

ought to be in the kernel of our projection. An argument shows that these elements will generate the needed kernel. One can show this by an analogous argument to the above: the point is that the attachment of e^2_α kills basically exactly the loop given above and nothing else, so we can use an inductive argument to conclude.

Remark 2.36. One can use this result to show that any group G arises as the fundamental group of a CW complex of dimension 2. Roughly speaking, the point is that any group is the quotient of a free group, and the above argument allows us to dictate relations, provided that we are sufficiently careful.

Example 2.37. Fix a positive integer g. Define S_g by starting with a $4g$-gon and attaching the edges. Namely, for $n < 3$, an $n \pmod 4$ edge is identified with the next over $n + 2 \pmod 4$ edge in the opposite direction. Roughly speaking, after some manipulation, one finds that S_g ought to be a g-hole torus. Using the above argument, one finds that $\pi_1(S_g)$ is generated by $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ modded out by the relations $a_i b_i a_i^{-1} b_i^{-1}$ for each i. In particular, the abelianization of $\pi_1(S_g)$ has all the commutators, so we get $\mathbb{Z}^{2g}$. Thus, π_1 distinguishes our surfaces.

Example 2.38 (projective space)**.** We note $\pi_1(\mathbb{RP}^\infty) \cong \pi_1(\mathbb{RP}^2)$ because the higher cells cannot help you in the fundamental group. Further, we see $\pi_1(\mathbb{RP}^2)$ is a disk with semicircles identified in the opposite direction, which we can see from the above argument is simply $\mathbb{Z}/2\mathbb{Z}$.

Example 2.39 (lens space)**.** Fix positive integers p and q with $\gcd(p, q) = 1$. Take S^2 and divide the equator into p circles, and we glue the top hemisphere to the bottom hemisphere by gluing after a $2\pi p/q$ rotation. The space has fundamental group $\mathbb{Z}/p\mathbb{Z}$. Indeed, our 1-skeleton is the equator, and the a^p comes from how we attached our disks together.

Next time we will talk about covering spaces.

[3] One can do this without transfinite induction by working with a single loop and arguing about homotopy equivalence. The point is that a single loop (and in fact a single homotopy) can is compact and therefore only cares about finitely many cells.

2.5 September 14

We're talking about covering spaces today.

2.5.1 Examples of Covering Spaces

Our goal is to generalize the method we used to compute $\pi_1\left(S^1\right)$. Let's recall our definition.

Definition 2.17 (covering space)**.** Fix a topological space X. Then a *covering space* is a topological space $\widetilde{X}$ together with a projection map $p\colon \widetilde{X} \to X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha\in\lambda} U_\alpha$ where U_α is open and $p\colon U_\alpha \to U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered*.

Example 2.40. The map $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi it}$ is a covering space map. Here, we are viewing S^1 as $\{z \in \mathbb{C} : |z| = 1\}$. The point is that, for any $e^{2\pi i\theta} \in S^1$, we have

$$p^{-1}\left(S^1 \setminus \left\{e^{2\pi i\theta}\right\}\right) = \bigsqcup_{n\in\mathbb{Z}} (\theta, \theta + 2\pi).$$

Non-Example 2.41. The map $p\colon (0,2) \to S^1$ given by $t \mapsto e^{2\pi it}$ is not a covering space map. For example, any open interval U around $1 \in S^1$ will have pre-image by p looking like $(0,\varepsilon) \sqcup (1-\varepsilon, 1+\varepsilon) \sqcup (2-\varepsilon, 2)$, and $(0,\varepsilon)$ is not mapped homeomorphically to our $U \subseteq S^1$.

Example 2.42. The map $f\colon \mathbb{C}^\times \to \mathbb{C}^\times$ given by $f\colon z \mapsto z^n$ for a positive integer n is a covering space map. Roughly speaking, for any ray ℓ through the origin in $\mathbb{C}$, one can define $\log\colon (\mathbb{C}\setminus\ell) \to \mathbb{C}$, which allows us to define an nth root $\sqrt[n]{w} := \exp\left(\frac{1}{n}\log w\right)$ on $\mathbb{C}\setminus\ell$; this makes $\mathbb{C}\setminus\ell$ into an evenly covered subset, so we are a covering space upon letting ℓ vary.

Example 2.43. Fix a topological space X and a discrete set E. Then of course $p\colon X \times E \to X$ is a covering space: indeed, X is an evenly covered subset. In fact, if $p\colon \widetilde{X} \to X$ is a covering space map where X is evenly covered, then the definition of p requires $\widetilde{X} \cong \bigsqcup_{\alpha\in\lambda} X_\alpha$

Example 2.44. Map $p\colon S^\infty \to \mathbb{RP}^\infty$ by sending $x \in S^\infty$ to the corresponding line in $\mathbb{RP}^\infty$. More precisely, embed some $S^n \subseteq S^\infty$ into $\mathbb{R}^{n+1}$ and then take lines down to $\mathbb{RP}^n$. Notably, $p(x) = p(-x)$ for each x (and conversely $p(x) = p(y)$ if and only if $\mathbb{R}x = \mathbb{R}y$ if and only if $x = \pm y$), so p is 2-to-1. One can check that p is a covering space map by looking on the level of cell complexes: the pre-image of the interior of the unique n-cell $(e^n)^\circ \subseteq \mathbb{RP}^\infty$ is the disjoint union of the interior of the two n-cells of S^∞. More precisely, the n-cell e_i^n inside $\mathbb{RP}^n$ given by

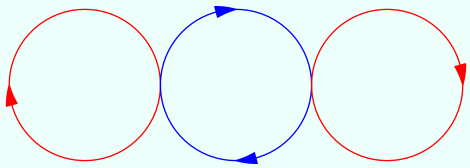
$$\{[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] : x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{R}\}$$

is evenly covered in the map $S^n \to \mathbb{RP}^n$. One can extend this idea up to $\mathbb{RP}^\infty$ to conclude: let e_i be the above subset except we do not terminate at x_n, and then e_i is covered by the open subsets $e_{i,\pm} \subseteq S^\infty$ defined as

$$e_{i,\pm} = \{(x_0, x_1, \ldots) \in S^\infty : x_i \text{ has sign } \pm\}.$$

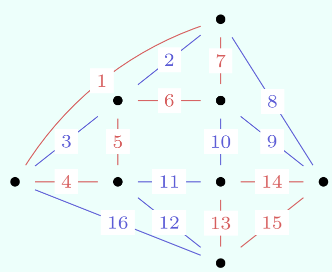
Let's do a few examples on $S^1 \vee S^1$.

Example 2.45. We examine 2-fold (i.e., 2-to-1) covers of $S^1 \vee S^1$. There is the trivial one with two copies of $S^1 \vee S^1$. As another example, note that $S^1 \vee S^1 \vee S^1$ loop around $S^1 \vee S^1$ twice: the first S^1 goes around the first S^1, then half of the second S^1 goes around the second S^1, then the third S^1 goes around the first S^1 around. Here is an image.

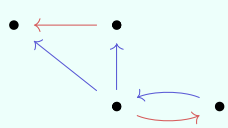


It turns out that, with one more, these are all the 2-to-1 covering maps, which can be seen by finding index-2 subgroups of $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, as we will soon see.

Example 2.46. Consider the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$. This is then a covering space of $S^1 \vee S^1$ by sending the $\mathbb{Z} \times \mathbb{R}$ to traverse one of the circles S^1 and the $\mathbb{R} \times \mathbb{Z}$ to traverse the other circle of S^1. More generally, it turns out that covering spaces are exactly graphs where every vertex has degree 4, which we can see by coloring the edges red and blue so that each vertex has exactly two red edges and two blue edges; then choosing an Euler cycle provides the needed covering space. The previous example is one way to do this. Here is another example of such a graph, with marked Euler cycle.



Example 2.47. We can take a subgroup of $\mathbb{Z} * \mathbb{Z}$ to produce a covering space of $S^1 \vee S^1$. As an example, take the subgroup generated by ab and $b^{-1}ab$. Reading off these generators produces a graph as follows.



In general, we basically fold edges together to make relations. For example, the multiple outgoing blue edges should be folded together.

Example 2.48. There is an infinite tree where each vertex has degree 4. A coloring of the edges produces a "Cayley graph" C_2, which will turn out to be the universal covering space once we define such a notion. It turns out to be maximal in the sense that it covers any path-connected cover of $S^1 \vee S^1$.

2.5.2 Lifting with Covering Spaces

We will want the following result.

Proposition 2.49. Covering spaces have the homotopy lifting property. In other words, given a covering space $p\colon \widetilde{X} \to X$, a "homotopy" $f_\bullet\colon Y \times I \to X$ with a given lift $\widetilde{f}_0\colon Y \to \widetilde{X}$ will lift uniquely to $\widetilde{f}_\bullet\colon Y \times I \to \widetilde{X}$ agreeing with X.

Proof. This is direct from Proposition 2.18. ■

Corollary 2.50. Fix a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Then $\pi_1(p)\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x)$ is injective.

Proof. Fix some loop $\widetilde{f}_0\colon I \to \widetilde{X}$ in the kernel of $\pi_1(p)$. Then there is a homotopy $f_\bullet\colon I \times I \to X$ from $\widetilde{f}_0$ to the constant path, which by Proposition 2.49 will lift uniquely to a homotopy $\widetilde{f}_\bullet\colon I \times I \to \widetilde{X}$ agreeing on $\widetilde{f}_0$. Now $p \circ \widetilde{f}_1$ is constant, so looking locally at $\widetilde{x}_0$, we conclude that $\widetilde{f}_1$ is constant, so $\widetilde{f}_0$ is homotopic to the constant map and hence vanishes in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. ■

2.6 September 19

Today we continue discussing covering spaces.

2.6.1 Using Path-Lifting

Last time we showed that covering space maps $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induce subgroups $\pi_1(\overline{X}, \overline{x}_0) \to \pi_1(X, x_0)$. Note this subgroup can then communicate information about the covering space.

Proposition 2.51. Fix a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of path-connected spaces. Then the number of sheets covering an evenly covered neighborhood of x_0 is the index

$$\left[\pi_1(X, x_0) : \pi_1(\widetilde{X}, \widetilde{x}_0)\right],$$

where we have implicitly embedded $\pi_1(\widetilde{X}, \widetilde{x}_0) \hookrightarrow \pi_1(X, x_0)$.

Remark 2.52. Because X is connected, the number of sheets of the covering space map is well-defined. Indeed, for any positive integer n, the set of $x \in X$ such that there is an n-sheeted evenly covered open neighborhood $U_x \subseteq X$ is open. So we produce a continuous map $X \to \mathbb{N}$ sending x to the number of sheets, so connectedness of X forces the number of sheets to be constant.

Proof. We roughly describe the idea. Let $\Omega(Y, y_1, y_2)$ denote the set of homotopy classes of paths from y_1 to y_2. The point is that $\Omega(X, x_0, x_0)$ is in bijection with

$$\bigsqcup_{\widetilde{x} \in p^{-1}(\{x_0\})} \Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$$

by lifting paths. Now, $\pi_1(\widetilde{X}, \widetilde{x}_0)$ acts on $\Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$ for each $\widetilde{x}$, and each orbit will correspond to a coset of our quotient. ■

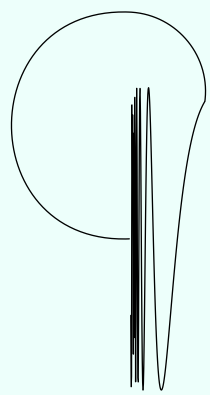
Remark 2.53. Proposition 2.51 can help us check that the covers of Example 2.45 are 2-to-1. For example, the subgroup corresponding to the shown covering space is $\langle a, b^2, bab\rangle$. Note that we have produced the free group with free generators as a subgroup of the free group with two generators.

We would like to go in the other direction, from subgroups back to covering space maps. This requires some technical hypotheses.

Definition 2.54 (locally path-connected)**.** A topological space X is *locally path-connected* if and only if each open neighborhood $U \subseteq X$ of a point $x \in X$ has some perhaps smaller open neighborhood $U' \subseteq U$ of $x \in X$ which is path-connected.

Example 2.55. CW complexes are locally path-connected.

Non-Example 2.56. The topologist's $\sin$ curve is not locally path-connected at the origin $(0, 0)$.



Being locally path-connected allows us to lift covering spaces.

Proposition 2.57. Fix a path-connected, locally path-connected topological space Y with basepoint $y_0 \in Y$. For a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and continuous map $f\colon (Y, y_0) \to (X, x_0)$, there is a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ making the following diagram commute if and only if $\pi_1(f)(\pi_1(Y, y_0)) \subseteq \pi_1(p)(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow{\widetilde{f}} & (\widetilde{X}, \widetilde{x}_0) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & (X, x_0) \end{array}$$

Proof. The backwards direction follows from functoriality of π_1 because we are asking for $\pi_1(f) = \pi_1(p) \circ \pi_1(\widetilde{f})$. For any $y \in Y$, composition with f defines a composite

$$\bigsqcup_{y \in Y} \Omega(Y, y_0, y) \to \bigsqcup_{x \in X} \Omega(X, x_0, x) \to \bigsqcup_{\widetilde{x} \in \widetilde{X}} \Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$$

where the last map is by path-lifting $\widetilde{\cdot}$. Then for any path $\gamma \in \Omega(Y, y_0, y)$, we simply define $\widetilde{f}(y) := \widetilde{f \circ \gamma}(1)$. To see that this is well-defined, the point is that choosing a different path $\gamma' \in \Omega(Y, y_0, y)$ produces is able to lift to basically a loop upstairs in $\widetilde{X}$, so the value of $\widetilde{f}(y)$ does not move.

For continuity, we will need to use that Y is locally path-connected. Fix $y \in Y$, and we will show that $\widetilde{f}$ is continuous at y. Set $x := f(y)$, and let U be an evenly covered neighborhood of x, and lift it to $\widetilde{U} \subseteq \widetilde{X}$ where $p\colon \widetilde{U} \to U$ is a homeomorphism, and $\widetilde{f}(y) \in \widetilde{U}$. We now may choose a path-connected open subset $V \subseteq f^{-1}(U)$ containing y and check continuity using $\widetilde{U}$, where $\widetilde{f}(y')$ for any $y' \in V$ can be somewhat easily defined because V is path-connected. ■

In fact, we have uniqueness of this lifting.

Proposition 2.58. Fix a connected topological space Y, and fix a covering space map $p\colon \widetilde{X} \to X$ and a map $f\colon Y \to X$. Given lifts $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ such that $p \circ \widetilde{f}_1 = f = \circ \widetilde{f}_2$ and $\widetilde{f}_1$ and $\widetilde{f}_2$ agree at a single point, we have $\widetilde{f}_1 = \widetilde{f}_2$.

Proof. Define the subsets

$$E := \left\{y \in Y : \widetilde{f}_1(y) = \widetilde{f}_2(y)\right\} \qquad \text{and} \qquad N := \left\{y \in Y : \widetilde{f}_1(y) \neq \widetilde{f}_2(y)\right\}.$$

One can use the covering space decomposition (by looking locally at $f(y)$ for some $y \in Y$) to show that both E and N are open, but they are disjoint with E nonempty, so connectedness of Y forces $Y = E$. ■

2.6.2 Classifying Covering Spaces

Our goal, roughly speaking, is to construct universal covers.

Definition 2.59 (univesal cover)**.** A covering space map $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a *universal cover* if and only if $\widetilde{X}$ is simply-connected (i.e., path-connected and $\pi_1(\widetilde{X}, \widetilde{x}_0) = 1$).

Remark 2.60. Proposition 2.57 tells us that a universal cover $\widetilde{X}$ will cover any covering space of X.

We will want yet another definition.

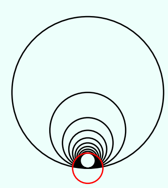
Definition 2.61 (semilocally simply-connected)**.** A space X is *semilocally simply-connected* if and only if each $x \in X$ has an open neighborhood U of x such that the induced inclusion $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map.

Remark 2.62. Let's explain this condition. Suppose $(\widetilde{X}, \widetilde{x}_0) \to (X, x)$ is a simply connected and path-connected covering space. Then any evenly covered subset $U \subseteq X$ with lift $\widetilde{U}$, then the inclusion $\pi_1(U) \to \pi_1(X)$ decomposes as

$$\pi_1(U) \to \pi_1(\widetilde{U}) \to \pi_1(\widetilde{X}) \to \pi_1(X),$$

which must be the trivial map because $\pi_1(\widetilde{X}) = 1$. In other words, we have checked that X is semilocally simply-connected.

Example 2.63. The earring space is not semilocally simply-connected at the origin because any neighborhood at the origin will have circles inside.



Being semilocally simply-connected is basically, then, the right hypothesis to have a universal cover.

Theorem 2.64. Let X be a topological space which is path-connected, locally path-connected, and semilocally simply-connected. Then X has a simply-connected covering space $\widetilde{X} \to X$ which is unique up to isomorphism of pointed topological spaces over X.

Proof. Uniqueness follows from Proposition 2.57 because the corresponding lifts we write down must be local homeomorphisms.

It remains to show existence. Fix a basepoint $x_0 \in X$. We simply define

$$\widetilde{X} := \{[\gamma] : \gamma \text{ is a path } I \to X \text{ with } \gamma(0) = x_0\}.$$

The point is that paths should lift uniquely up to $\widetilde{X}$ already, so we might as well define $\widetilde{X}$ in this way. We may define the function $p\colon \widetilde{X} \to X$ by sending $[\gamma] \mapsto \gamma(1)$. It remains to show that $\widetilde{X}$ is a simply-connected topological space and that p is a covering space map.

Let's produce a topology on $\widetilde{X}$. Using our hypotheses on X, each $x \in X$ has a path-connected open neighborhood $V \subseteq X$ such that $\pi_1(V) \to \pi_1(X)$ is trivial. We then use V to define a subset around $[\gamma]$ with $\gamma(0) = x_0$ and $\gamma(1) = x$ by

$$\widetilde{V} := \{[\gamma \cdot \gamma'] : \gamma' \text{ is a path } I \to V \text{ such that } \gamma'(0) = x_0 \text{ and } \gamma'(1) = y\}.$$

Now, $\widetilde{V}$ is in bijection with V by p, so we make the restricted map $p\colon \widetilde{V} \to V$ a homeomorphism. One can check that the topology is well-defined and that p becomes a covering space map from this. ■

One can now use the universal cover to produce any covering space.

Theorem 2.65. Let X be a pointed topological space which is path-connected, locally path-connected, and semilocally simply-connected, and let $x_0 \in X$ be a basepoint. Then there is a bijection between pointed path-connected covering spaces $(Y, y_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$. Unpointed covering space maps correspond to conjugacy classes of subgroups.

To produce the desired covering space given a subgroup, one repeats the proof of Theorem 2.64 by taking a quotient of the produced $\widetilde{X}$. Then one shows that this is a bijection with some work.

Remark 2.66. One can also use permutations of the pre-image of a basepoint in order to describe our covering spaces. Namely, if $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, then any loop $[\alpha] \in \pi_1(X, x_0)$ will lift to a permutation of $p^{-1}(\{x_0\})$. Conversely, such automorphisms are able to produce an automorphism of the universal covering space $\widetilde{X} \to \widetilde{X}$. (On the level of paths, we send $[\gamma] \in \widetilde{X}$ to $[\gamma \cdot \alpha]$. One can check that this is continuous with continuous inverse and thus a homeomorphism.)

2.7 September 21

We continue to cover spaces.

2.7.1 Deck Transformations

Let X be a path-connected, locally path-connected, and semilocally simply-connected space with universal cover $\widetilde{X} \to X$. We would like to use the universal cover to produce intermediate covering maps.

Definition 2.67 (deck transformation)**.** Let X be a path-connected, locally path-connected, and semilocally simply-connected space with cover $p\colon \widetilde{X} \to X$. A homeomorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $p = p \circ f$ is called a *deck transformation*.

In our set-up, let G be the group of deck transformations of the universal cover $\widetilde{X} \to X$. Then $G \cong \pi_1(X)$. Let's explain why. Fix a basepoint $\widetilde{x}_0 \in \widetilde{X}$ lying over $x_0 \in X_0$. The point is that a deck transformation is uniquely determined by where it sends $\widetilde{x}_0$ by how path-lifting works. So a deck transformation $f\colon \widetilde{X} \to \widetilde{X}$ produces a path from $\widetilde{x}_0$ to $f(\widetilde{x}_0)$ (which is unique up to homotopy class because $\widetilde{X}$ is simply-connected), and then mapping this down to p produces an element of $\pi_1(G, x_0)$. And conversely a loop in $\pi_1(X, x_0)$ lifts to a path up in $\widetilde{X} \to \widetilde{X}$ sending $\widetilde{x}_0 \mapsto f(\widetilde{x}_0)$, and there is a unique automorphism $f\colon \widetilde{X} \to \widetilde{X}$ sending $\widetilde{x}_0$ to the right place.[4]

Remark 2.68. More generally, if $(Y, y_0) \to (X, x_0)$ is any covering space, one has a bijection between $\pi_1(X, x_0)/\pi_1(Y, y_0)$ and points in the fiber of x_0.

Extending the above discussion, we have the following result.

Theorem 2.69. Fix a path-connected covering space $p\colon (Y, yx_0) \to (X, x_0)$, and let G be the group of deck transformations. Then X is homeomorphic to Y/G in the natural way if and only if $\pi_1(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$. In this case, $G \cong \pi_1(X)/\pi_1(Y)$.

Proof. Let $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal cover. Then the universal property allows us to factor as follows.

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & \xrightarrow{r} & (Y, y_0) \\ & {}_{q}\searrow & \downarrow p \\ & & (X, x_0) \end{array}$$

Now, for all $y \in p^{-1}(\{x_0\})$, we see that $r^{-1}(\{y\})$ will correspond to a coset of $\pi_1(\widetilde{X})$ in $\pi_1(X)$ via the discussion with the universal cover; looping over y, we produce a bijection with points in the fiber of $q^{-1}(\{x_0\})$. Normality of the subgroup then follows because the action of G here is trying to act on cosets. ■

What?

A less careful version of this discussion lets us work with more general subgroups.

Proposition 2.70. Let X be a path-connected, locally path-connected, and semilocally simply connected space with universal cover $p\colon \widetilde{X} \to X$. For any subgroup $H \subseteq \pi_1(X, x_0)$, the quotient space $\widetilde{X}/H$ is a covering space of X and has fundamental group H.

Proof. Track through the above discussion without focusing on the group being normal. ■

2.7.2 Attempts for Universal Covers

We are interested in the universal covering space construction having the lifting property. For our purposes, we will assume that our topological space (X, x_0) which is locally path-connected, and we can still just try to define $\widetilde{X}$ as the set of homotopy classes of paths starting at x_0. Then the topology is defined by building a sub-base as follows: for open path-connected subsets $V \subseteq X$, one defines an open set around $[\gamma]$ with $\gamma(0) = x_0$ and $\gamma(1) = x$ by

$$\widetilde{V} := \{[\gamma \cdot \gamma'] : \gamma' \text{ is a path } I \to V \text{ such that } \gamma'(0) = x_0 \text{ and } \gamma'(1) = y\}.$$

Let's see some examples.

Example 2.71. Let's apply this to the earring E. One can show that this construction produces an open map $p\colon \widetilde{E} \to E$, but it is not a covering space. Nonetheless, $\widetilde{E}$ is path-connected, locally path-connected, and simply-connected, and it has the unique path-lifting property. Indeed, for any locally path-connected map $f\colon Y \to E$ where $f_*\pi_1(Y) \subseteq \pi_1(X)$ is trivial, the map f factors through p uniquely.

One might want to try to draw $\widetilde{E}$, but this is hard: for example, with $e \in E$ the vertex of the earring, one has $p^{-1}(\{e\})$ uncountable, and E is an $\mathbb{R}$-tree, meaning any two points has a unique path connecting them.

[4] At this point, it is perhaps clearer to use the direct construction of $\widetilde{X}$ as homotopy classes of paths starting at x_0.

Example 2.72. Let $X := \prod_{i\in\mathbb{N}} S^1$. By how the product topology works, this remains path-connected (as the product of path-connected spaces) but is not semilocally simply-connected because any open set contains at least one S^1, which fails to be simply-connected. Nonetheless, the map $\mathbb{R} \to S^1$ remains continuous, so there is a map

$$\prod_{i\in\mathbb{N}} \mathbb{R} \to \prod_{i\in\mathbb{N}} S^1$$

which behaves like a covering space.

Example 2.73. CW-complexes X are locally contractible and hence locally path-connected and locally simply-connected. Thus, our construction provides a universal covers for connected CW complexes. For simplicity, we work with the 2-skeleton $X^{(2)}$, which encodes all π_1-information anyway, and we will focus on constructing $\widetilde{X}$. One can show that the covering space of a CW-complex remains a CW-complex because one can lift sufficiently small evenly covered cells to produce a CW-structure on the covering space. Looking at how $X^{(2)}$ is constructed by adding 2-cells to produce quotients, we see that $\widetilde{X}^{(1)}$ corresponds to the kernel of $\pi_1(X^{(1)}) \to \pi_1(X)$, which by van Kampen is the normal subgroup generated by 2-cells as $\pi_1(\partial e^2_\bullet)$ for the various $e^2_\bullet$.

Example 2.74. Fix coprime positive integers p and q, and construct the lens space $L(p,q)$ by taking the quotient of D^3 by dividing an equator S^1 into q pieces and then gluing the top and bottom hemisphere after rotating by $2\pi q/p$. Equivalently, one can view this as $S^3/(\mathbb{Z}/p\mathbb{Z})$, where the action is given by $k \cdot (z_1, z_2) := \left(\zeta_p^k z_1, \zeta_{qk} z_2\right)$. One sees that $L(p,q)$ is a CW-complex with 1-skeleton given by S^1 and two-skeleton by attaching D^2 and identifying z with $\zeta_q z$ for each z.

- $\widetilde{L(p,q)}^{(1)}$ is S^1 again, but it is viewed as the p-fold cover of S^1.
- $\widetilde{L(p,q)}^{(2)}$ is p disks glued at their boundaries.
- $\widetilde{L(p,q)}$ fills in these disks with 3-balls.

Example 2.75. Suppose $X^{(1)} = \bigvee_S S^1$, then $\pi_1\left(X^{(1)}\right)$ is the free group on S as letters. Each attached 2-cell to $X^{(1)}$ gives a relation for $G := \pi_1\left(X^{(2)}\right)$. Now, $\widetilde{X^{(2)}}^{(1)}$ turns out to be Cayley graph of G, and its 0-skeleton is in bijection with G, where edges are given by group elements in the natural way.

Let's be more explicit: for any generating set $S \subseteq G$, let N be the kernel of the surjection $F(S) \twoheadrightarrow G$, and then we can view our Cayley graph via some covering space quotient.

For the next few examples, we have the following definition.

Definition 2.76. We say that a CW-complex X is $K(G,1)$ if and only if it has fundamental group G and has contractible $\widetilde{X}$.

It turns out that $K(G,1)$ is unique up to homotopy equivalence, so it allows us to talk more canonically about the group G via topology. Here are some examples.

Example 2.77. Note $K(\mathbb{Z},1) = S^1$ because $\widetilde{S^1} = \mathbb{R}$ is contractible.

Example 2.78. Note $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty$ because $S^\infty = \widetilde{\mathbb{RP}^\infty}$ is contractible.

Example 2.79. We see $S^1 \times S^1 = K(\mathbb{Z}^2, 1)$ because the universal cover of $S^1 \times S^1$ is the contractible space $\mathbb{R}^2$. Of course, we can take arbitrary powers and products like this.

2.8 September 26

Today we discuss free groups and graphs.

2.8.1 Spanning Tree

For technical reasons, it will be helpful to rigorize give graphs a CW topology.

Definition 2.80 (graph)**.** A *graph* is a 1-dimensional CW complex X built as follows: the vertices are X^0, and the edges are built by taking two vertices $v_1, v_2 \in X^0$ and connecting them by an edge e_α with $\partial e_\alpha = \{v_1, v_2\}$.

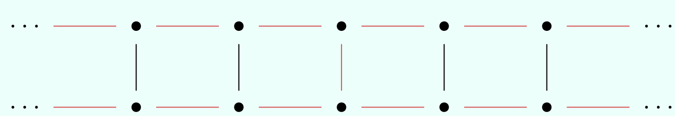
Remark 2.81. A graph X with a vertex $v \in X$ of infinite degree fails to be locally compact. Indeed, any open neighborhood of v will intersect infinitely many edges, which is not contained in any compact set because one can build an open cover with an open set from each of the individual edges, from which no finite subcover is possible to construct.

Definition 2.82 (subgraph)**.** A *subgraph* is a closed CW subcomplex of a graph.

Trees are the simplest graphs.

Definition 2.83 (tree)**.** A *tree* is a contractible graph. A subtree T of a graph X is *maximal* or *spanning* if and only if $T^0 = X^0$.

Example 2.84. The highlighted edges make a maximal subtree of the following graph.



We have the following result on trees.

Proposition 2.85. Any connected graph X contains a maximal tree. In fact, any subtree can be extended to a maximal tree.

Proof. We begin by fixing some subtree $X_0 \subseteq X$. Then to construct X_{n+1} from X_n, we look at the set of vertices adjacent to a vertex in X_n, and we add exactly one edge to X_{n+1} to add in all these vertices. Each added edge maintains being contractible, and adding them all in at once will continue to be contractible; explicitly, X_{n+1} has a deformation retract back to X_n and will therefore be contractible by induction.

Eventually the union T of $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ will hit every vertex: note X is connected and locally path-connected hence path-connected, so it follows that any two vertices can be connected by a path, which may only hit finitely many vertices and edges along its path by compactness of the interval.[5] Thus, T is the desired subtree. ■

[5] Hitting infinitely many vertices or edges implies that the image of $[0, 1]$ has an infinite discrete closed subset (choose a single point from each vertex and from each hit edge), violating compactness.

Remark 2.86. One needs some form of the axiom of choice to achieve the above result because we may be making infinitely many choices in the construction of X_{n+1} from X_n.

2.8.2 Fundamental Groups of Graphs

Having spanning trees allows us to compute fundamental groups. Fix a spanning tree $T \subseteq X$. Fix a basepoint $x_0 \in T$. Then each edge e_α of $X \setminus T$ produces a loop based at x_0: if e_α connects v_1 and v_2, then we have a loop going from x_0 to v_1 (through T) to v_2 (through e_α) and back to x_0 (through T again). These loops generate the fundamental group.

Proposition 2.87. Fix a connected graph X with spanning tree T. Then $\pi_1(X)$ is a free group with basis $[e_\alpha]$ where e_α is an edge of $X \setminus T$.

Proof. The quotient map $X \twoheadrightarrow X/T$ is a homotopy equivalence because T is contractible (it's a tree). However, X/T now only has a single vertex x_0, and we see that each edge e_α of $X \setminus T$ then goes down to a loop at x_0. Thus, X/T is S^1 wedged with itself once for each edge in $X \setminus T$, so the result follows. ■

Our work allows us the following application.

Lemma 2.88. Every covering space of a graph X is itself a graph whose vertices and edges as pre-images.

Proof. Let $p\colon \widetilde{X} \to X$ be a covering space. Set vertices of $\widetilde{X}$ to be $p^{-1}\left(X^0\right)$, and our edges are similarly given by pre-images because p is locally a homeomorphism, we see that $\widetilde{X}$ has the desired topology. ■

Theorem 2.89. Any subgroup of a free group is free.

Proof. A free group F generated by κ generators is the fundamental group of the graph $X := \left(S^1\right)^\kappa$. Then any subgroup $F' \subseteq F$ arises from the fundamental group of the covering space $p\colon \widetilde{X} \to X$, and the lemma tells us that $\widetilde{X}$ is a graph, so its fundamental group is in fact also free by Proposition 2.87. ■

The above result is quite nice: it is quite non-obvious that this result should be true purely from the algebra, but the topology makes it easier to attack.

Remark 2.90. There is an algorithm (due to Reidemeister–Schreier) to find a generating set for finite-index subgroups of a free group.

2.8.3 $K(G,1)$s

We have the following definition.

Definition 2.91 ($K(G,1)$)**.** Fix a group G. A path-connected topological space X is a $K(G,1)$ if and only if $\pi_1(X) \cong G$, and X has a contractible universal cover.

It turns out that $K(G,1)$ is unique up to homotopy equivalence. Here are some examples.

Example 2.92. The space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$. The fundamental group can be computed by seeing that the universal cover is $S^\infty \twoheadrightarrow \mathbb{RP}^\infty$. Let's see that S^∞ is in fact contractible: the map $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$ defines an embedding $i\colon S^\infty \to S^\infty$. However, i has a linear homotopy to id given by

$$f_t(x_1, x_2, \ldots) := \frac{(1-t)(x_1, x_2, \ldots) + t(0, x_1, \ldots)}{\|(1-t)(x_1, x_2, \ldots) + t(0, x_1, \ldots)\|},$$

and then i has a linear homotopy to a constant map by

$$g_t(x_1, x_2, \ldots) := \frac{(1-t)(1, 0, \ldots) + t(0, x_1, \ldots)}{\|(1-t)(1, 0, \ldots) + t(0, x_1, \ldots)\|}.$$

(Note we needed the inclusion i because the linear combination $(1-t)(1, 0, \ldots) + t(x_1, x_2, \ldots)$ goes through the origin if we use the point $(x_1, x_2, \ldots) = (-1, 0, \ldots)$.)

Example 2.93. The space $S^\infty/(\mathbb{Z}/m\mathbb{Z})$ is a $K(\mathbb{Z}/m\mathbb{Z}, 1)$. Here, $\mathbb{Z}/m\mathbb{Z}$ acts on S^∞ by having $1 \in \mathbb{Z}/m\mathbb{Z}$ be pointwise multiplication by $e^{2\pi i/m}$. The covering space is still S^∞, which is contractible by the previous example.

Example 2.94. Fix a closed, connected subspace $K \subseteq S^3$ (thought of as a knot). If $G := \pi_1\left(S^3 \backslash K\right)$, then $S^3 \backslash K$ is a $K(G, 1)$; this is a result to Papakyriakopoulos (yes, this name is hard to spell). Note that having S^3 is important; otherwise, if K is bounded, we could just place a large box around $K \subseteq \mathbb{R}^3$, and it is not possible to contract this box in $\mathbb{R}^3 \backslash K$. Instead, we want to contract it in S^3 by passing to the point at infinity.

Example 2.95. Let X_G be a $K(G, 1)$, and let X_H be a $K(H, 1)$, and we assume that both are CW complexes. Then $X_G \times X_H$ (given the product topology!) becomes a $K(G \times H, 1)$ because the universal cover of $X_G \times X_H$ is the product of the universal covers, which will then remain contractible.

Taking a product of $K(\mathbb{Z}/m\mathbb{Z}, 1)$s, we see that there is a $K(G, 1)$ for a finitely generated abelian group G. One can in fact give a $K(G, 1)$ for any group G, though this trickier. Let's see this. The following notions will be helpful.

Definition 2.96 (simplex)**.** An n-*simplex* is constructed by taking affinely linearly independent vectors $v_0, \ldots, v_n \in \mathbb{R}^m$ (i.e., the set $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent—note that this condition is independent of rearranging the $v_\bullet$) and setting

$$[v_0, v_1, \ldots, v_n] := \left\{ \sum_{i=0}^n t_i v_i : 0 \leqslant t_i \text{ for each } i \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

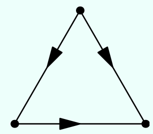
Namely, $[v_0, v_1, \ldots, v_n]$ is the convex hull of the $v_\bullet$; a *face* of this n-simplex is an $(n-1)$-simplex of the form $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$ attained by deleting one of the vertices v_i. Then the boundary of the n-simplex is

$$\partial[v_0, \ldots, v_n] := \bigsqcup_{i=1}^n [v_0, \ldots \widehat{v}_i, \ldots, v_n],$$

and the interior is defined in the obvious way.

Definition 2.97 (Δ-complex)**.** A Δ-*complex* is a CW complex X such that the cells e^n_α are homeomorphic to $(\Delta^n)^\circ$, where we require that the attaching maps $\varphi_\alpha\colon \partial\Delta^n \to X^{n-1}$ restricts to a face $\varphi_\alpha|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]}$ is an attaching map $\varphi_\beta\colon \Delta^{n-1} \to X^{n-1}$ for some β.

Example 2.98 (dunce cap)**.** Glue the following 2-simplex to a 1-simplex following the arrows.



This is weird, but we allow it.

We now describe $K(G,1)$ for a general group G. We begin by constructing the universal cover EG, which will be a Δ-complex. The vertices of EG are elements of G. Then the n-simplices of EG (for $n \geqslant 1$) are simply $[g_0,\ldots,g_n]$ attached to the $(n-1)$-simplices $[g_0,\ldots,\widehat{g}_i,\ldots,g_n]$ in the obvious way.

Example 2.99. Take G to be the trivial group. Then we have a single n-simplex $[e,\ldots,e]$ for each n. For example, the two-simplex $[e,e]$ is attaching at its ends to a single vertex. Then the $[e,e,e]$ is attaching its edges to the loops as in Example 2.98.

Example 2.100. Take G to be $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$. Then we have 2^{n+1} total n-simplices.

Note that G acts freely on EG by multiplication of the vertices, so we produce a covering space $EG \to BG$, where $BG := EG/G$. We claim that EG is contractible, which will complete our construction with BG as our $K(G,1)$. Indeed, inside any n-simplex $[g_0,\ldots,g_n]$, we embed it into $[e,g_0,\ldots,g_n]$ and then use the linear homotopy to the identity e. This will be well-defined with respect to our gluing, so we have indeed produced contraction.

2.9 September 28

Today we talk about graphs of groups.

Remark 2.101. Problem 1.B.9 on the homework needs to assume that the edge maps are injective.

2.9.1 Using Classifying Spaces

Given a group G, last time we constructed a contractible Δ-complex EG, and from there we built $BG := EG/G$, and we argued that BG is a $K(G,1)$ because the action of G on EG was free, making $\pi_1(BG) = \pi_1(EG/G) = G$. Though huge, the EG and BG construction are nice because they are functorial: a homomorphism $\varphi\colon G \to H$ of groups produces a continuous map $E\varphi\colon EG \to EH$ by moving the vertices (which continuously will send simplices to simplices), and this commutes with the group actions on both spaces, so we produce a map $BG\colon BG \to BH$. Explicitly, $B\varphi([g]) = [\varphi(g)]$, so

$$B\varphi([g_1,\ldots,g_n]) = [\varphi(g_1),\ldots,\varphi(g_n)],$$

and this map is preserved by the group actions because

$$B\varphi(g\cdot[g']) = B\varphi([gg']) = [\varphi(gg')] = \varphi(g)\cdot[\varphi(g')] = \varphi(g)\cdot B\varphi([g']),$$

so there is a quotient down to a map $E\varphi\colon EG \to EH$.

One might now hope that we can produce a map $K(\varphi,1)\colon K(G,1) \to K(H,1)$, but for this to make sense, we need to know that $K(G,1)$ is well-defined in some sense.

Theorem 2.102. The homotopy type of a CW-complex $K(G,1)$ is uniquely determined by G.

The main input to the theorem is the following functoriality result.

Proposition 2.103. Fix a connected CW-complex X, and let Y be a $K(G,1)$. Then any homomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is induced by a map $\Phi\colon (X,x_0) \to (Y,y_0)$ which is unique up to homotopy (relative to basepoints).

Proof. We construct $X \to Y$ inductively. Map X^0 to y_0. As in our discussion of graphs, choose a spanning tree T of X^1, and we see that each edge e of $X^1 \backslash T$ determines a generator $[e]$ of $\pi_1(X^1)$, and we map these down to the corresponding generator in $\pi_1(Y,y_0)$ as required by φ.

By way of example, we can take $X = S^1 \times S^1$ to be the torus, mapping the two generators of π_1 to $1 \in \mathbb{Z}$. Then may extend Φ on the vertices to X^2 linearly via this triangulation (check up in the covering space to be told how to do this), viewing things as simplices. One can then keep going up to higher X^n by continuing to go linearly, noting that the effect on the fundamental group is now not doing anything.

For the uniqueness, suppose we have two maps $\Phi, \Psi\colon (X,x_0) \to (Y,y_0)$. This will essentially follow from the homotopy extension property. If they induce the same map $\pi_1(\Phi) = \pi_1(\Psi)$, then we move them up to the universal cover, and the convex combinations as described in the previous paragraph are forced and homotopic (linearly), where we are essentially using contractability of our universal cover. One needs to do this by induction on the skeletons: there is a homotopy on the 0-skeleton by moving, there is a homotopy on the 1-skeleton because they have the same π_1, there is a homotopy on the 2-skeleton because the relations are the same, and from here one inducts upwards. ■

We can now prove Theorem 2.102.

Proof of Theorem 2.102. One has identities relating fundamental groups on two $K(G,1)$s, so one produces maps in both directions by the proposition, and then the composition of these maps (in both directions) are homotopy equivalent to identity maps by uniqueness of these maps up to homotopy. ■

Classifying spaces allow one to classify principal bundles with fiber given by a particular group. For example, the annulus A provides a double-cover of the Möbius strip M, so we see that this double-cover corresponds to $2\mathbb{Z} \subseteq \mathbb{Z}$. (Note the Möbius strip has a deformation retraction to S^1, so the fundamental groups are the same.) Now, each fiber has a $(\mathbb{Z}/2\mathbb{Z})$-action, and mapping $M \to \mathbb{RP}^\infty$ (given by the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ and using the $K(\mathbb{Z}/2\mathbb{Z},1)$ universal property), we see that the composite $A \to M \to \mathbb{RP}^\infty$ is now trivial on π_1, so we induce a map making the following diagram commute.

$$\begin{array}{ccc} A & \dashrightarrow & S^\infty \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathbb{RP}^\infty \end{array}$$

Namely, this map is given by tracking fibers through on the map $M \to \mathbb{RP}^\infty$.

More generally, if we have a covering space $\widetilde{X} \to X$, where G acts freely and transitively (as deck transformations), then $G = \pi_1(X)/\operatorname{im}\pi_1(p)$, so maps $\pi_1(X) \twoheadrightarrow G$ will be given by maps $X \to K(G,1)$ via the above construction. So $K(G,1)$ in some sense allows us to classify these covering spaces $\widetilde{X} \to X$, which is of interest. Indeed, one can go the other direction: given a map $\varphi\colon X \to K(G,1)$, we pull back the bundle $p\colon EG \to K(G,1)$ to X to produce the necessary covering space. Namely, set

$$\widetilde{X} := \{(x,y) \in X \times EG : \varphi(x) = p(y)\} \subseteq X \times EG,$$

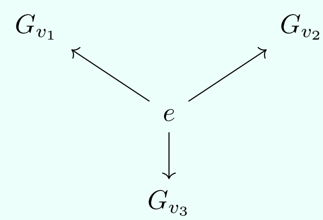
and one can check that the induced map $\widetilde{X} \to EG$ is continuous, and the map $\widetilde{X} \to X$ is a covering space map where G is acting on the fibers via EG.

2.9.2 Graphs of Groups

Fix a connected directed graph Γ, and for each vertex $v \in \Gamma^0$, we place a group G_v, and for each edge $e \in \Gamma^1$ connecting v to w, we place a homomorphism $\varphi_e\colon G_v \to G_w$. This will be our set-up for this subsection.

We are going to build a classifying space $B\Gamma$ for this graph by putting a classifying space BG_v (which is a CW-complex) at each vertex and attaching these along vertices with the mapping cylinders $MB\varphi_e$ for each $B\varphi_e \colon BG_v \to BG_w$. Notably, $B\varphi_e$ can always be constructed by Proposition 2.103. We will be interested in $\pi_1(B\Gamma)$. Note that $\pi_1(B\Gamma)$ does not depend on the choices of $BG - v$ and $B\varphi_e$ because these things are all well-defined up to homotopy.

Example 2.104. Consider the following graph.



Now, $K(e,1)$ is just a point, so the corresponding $B\Gamma$ is just a wedge product, so van Kampen tells us that this is $G_{v_1} * G_{v_2} * G_{v_3}$.

Example 2.105. Consider the following graph.

$$\mathbb{Z} \xleftarrow{\;q\;} \mathbb{Z} \xrightarrow{\;p\;} \mathbb{Z}$$

Applying van Kampen to the resulting $B\Gamma$, we get a group presentation of $\langle a, b : a^p = b^q \rangle$. If $p = q = 2$, one can squint very hard and see a Klein bottle as we are in some sense attaching two Möbius strips.

Example 2.106. Consider the following graph.

$$C \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} A$$

This looks like $\pi_1(B\Gamma) = \langle A, t : t\varphi_2(c)t^{-1} = \varphi_1(c) \text{ for } c \in C \rangle$, again by some van Kampen argument.

Anyway, here is our main theorem.

Theorem 2.107. Fix everything as above, and further assume that the φ_e maps are injective. Then $B\Gamma$ is a $K(G,1)$ where $G := \pi_1(B\Gamma)$, and the maps $\pi_1(BG_v) \to \pi_1(B\Gamma)$ are injective.

Proof. Start with a specific edge $B\varphi_e \colon BG_v \to BG_w$. Then the $MB\varphi_e$ connecting the two will lift to connect EG_v and EG_w by checking each "end" of this cylinder. We now build upwards via a tree to slowly encompass the entire graph. Being path-connected implies that this inductive process will union out to give us a legitimate "tree of spaces" connecting all the groups. Now, each vertex group G_v successfully acts on EG_v and then goes on to act on the mapping cylinders adjacent, so we have the right fundamental group. And we can see by reversing the inductive constructive process that we can deformation retract our mapping cylinders away to show that our covering space is contractible. ∎

THEME 3
HOMOLOGY

I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial.

—Andre Weil, [Wei59]

3.1 October 3

The homeworks will now get a little longer.

3.1.1 Homology for Δ-Complexes

Let's recall our construction of Δ-complexes.

Definition 3.1 (simplex)**.** We define the n-simplex

$$\Delta^n := \left\{ (t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : \sum_{k=0}^{n} t_k = 1 \right\}.$$

The ith face $\Delta_i^{n-1} \subseteq \Delta^n$ consists of the points with $t_i = 0$. An *orientation* of the simplex consists of an ordering of the vertices modulo the action of A_{n+1} on the vertices $\{0, 1, \ldots, n\}$.

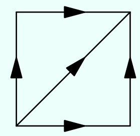
The orientation basically indicates which vertices are "small" and which are "large."

Definition 3.2 (Δ-complex)**.** A Δ-*complex* is a CW-complex X with maps $\sigma_\alpha \colon \Delta^n \to X$ satisfying the following properties.

- Interiors: the map σ_α is injective on the interior of Δ^n.
- Faces: the map σ_α restricted to the face Δ_i^{n-1} is simply another map $\sigma_\beta \colon \Delta^{n-1} \to X$.
- Continuity: if $A \subseteq X$ is open, then $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

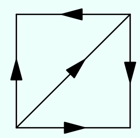
Given a Δ-complex X, orientations will tend to extend uniquely to X.

Example 3.3. We provide an orientation on the torus T^2.



Note that the diagonal arrow cannot go the other way to have an orientation because this would create a loop!

Example 3.4. We provide an orientation on the projective plane $\mathbb{P}^2$.



We would like to define homology. For this, we have a notion of a chain.

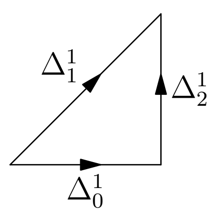
Definition 3.5 (chain)**.** Fix a Δ-complex X with maps $\sigma_\alpha \colon \Delta^n \to X$. Then we define *chains* $\Delta_n(X)$ to be the formal sums

$$\Delta_n(X) := \left\{ \sum_\alpha n_\alpha \sigma_\alpha : n_\alpha \in \mathbb{Z} \right\},$$

and then we define the *chain map* $\partial_n \colon \Delta_n(X) \to \Delta_{n-1}(X)$ given by

$$\partial_n(\sigma_\alpha) := \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{\Delta_i^{n-1}}.$$

The point of the signs in the definition of ∂_n is to have the correct orientation. For example, suppose we want to go "around" Δ^2 as in this diagram.



One now has the following check.

Proposition 3.6. Fix a Δ-complex X. For any positive integer n, we have $\partial_{n-1} \circ \partial_n = 0$.

Proof. Direct computation. It suffices to show this for Δ^n because $\Delta_n(X)$ is freely generated by images of this Δ^n. And for Δ^n, the point is that our signs are going to cancel:

$$\begin{aligned}(\partial_{n-1} \circ \partial_n)(\Delta^n) &= \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \Delta_i^{n-1}\right) \\ &= \sum_{i=0}^{n} (-1)^i \partial_{n-1}\left(\Delta_i^{n-1}\right).\end{aligned}$$

Now, for some notation, writing out the vertices Δ^n as $\{0, 1, \ldots, n\}$, we write $\Delta^n = [0, 1, \ldots, n]$ so that $\Delta_i^{n-1} = [0, \ldots, \widehat{i}, \ldots, n]$, so w are looking at

$$\begin{aligned}
(\partial_{n-1} \circ \partial_n)(\Delta^n) &= \sum_{i=0}^{n} (-1)^i \partial_{n-1}([0, \ldots, \widehat{i}, \ldots, n]) \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{i-1} (-1)^i (-1)^j [0, \ldots, \widehat{j}, \ldots, \widehat{i}, \ldots, n] + \sum_{j=i+1}^{n} (-1)^i (-1)^{j+1} [0, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n] \right) \\
&= \sum_{j<i} (-1)^{i+j} [0, \ldots, \widehat{j}, \ldots, \widehat{i}, \ldots, n] - \sum_{i<j} (-1)^{i+j} [0, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n] \\
&= 0,
\end{aligned}$$

as desired. ■

We are now ready to define homology.

Definition 3.7 (simplicial homology)**.** Fix a Δ-complex X. Then we define $\Delta(X)$ to be the graded module $\bigoplus_{n=0}^{\infty} \Delta_n(X)$, and we define the n*th homology group* as

$$H_n^{\Delta}(X) = H_n(\Delta(X)) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

For notation, we set $Z_n(X) := \ker \partial_n$ to be n-*cycles* and $B_n(X) := \operatorname{im} \partial n + 1$ to be n-*boundaries*. Then $H_n(\Delta(X)) = Z_n(X)/B_n(X)$, so we are measuring cycles which are not boundaries, which approximately is finding holes.

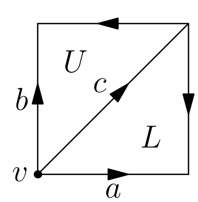
Note that we have not shown that $H_\bullet$ does not depend on the choice of Δ-structure, which is why we are marking our H_n^{Δ} by Δ, but we will do this in due time.

Example 3.8. Give S^1 a Δ-complex structure by attaching both endpoints of Δ^1 together at some vertex v as an edge e.

- We see $H_0^{\Delta}(S^1)$ is $\ker \partial_0 / \operatorname{im} \partial_1$, but $\operatorname{im} \partial_1 = 0$ because we are looking at $\partial_1(e) = v - v = 0$. However, $\ker \partial_0$ is simply all $\mathbb{Z}v$, so we have $\mathbb{Z}$.
- We see $H_1^{\Delta}(S^1)$ is $\ker \partial_1 / \operatorname{im} \partial_2$, and then $\partial_1 = \mathbb{Z}e$ as shown in the previous point, but $\operatorname{im} \partial_2 = 0$ because there is nothing to map, so we have $\mathbb{Z}$.

We note that all the higher homology groups vanish because there is nothing to compute.

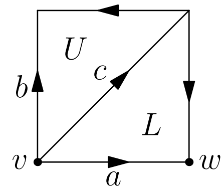
Example 3.9. Give T^2 the Δ-complex as described earlier. We expect to have a two-dimensional hole and two one-dimensional holes. We compute some homology.



Proof. Now, $\partial_2(U) = b - c + a$ and $\partial_2(L) = a - c + b$, which is the same, so $\ker \partial_2$ is generated by $U - L$. Now, $\operatorname{im} \partial_3 = 0$ (there is nothing to compute), so $H_2^{\Delta}(T^2) \cong \mathbb{Z}$. As for H_1^{Δ}, we note that ∂_1 is identically zero because there is only a single vertex, so $\ker \partial_1 = \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$, so $H_1^{\Delta}(T^2) = \ker \partial_1 / \operatorname{im} \partial_2 \cong \mathbb{Z}a \oplus \mathbb{Z}b$. ■

Example 3.10. Give $\mathbb{P}^2$ the Δ-complex as described earlier. We compute some homology.

Proof. Here is our structure.



Here are our computations.

- We see $H_0^\Delta(\mathbb{P}^2) = \mathbb{Z}v \oplus \mathbb{Z}w/(\mathbb{Z}(v-w)) \cong \mathbb{Z}$, where the point is that $\partial_1(c) = 0$ and $\partial_1(a) = w - v$ and $\partial_1(b) = v - w$.
- Next up, we compute $\partial_2(U) = b-a+c$ and $\partial_2(L) = a-b+c$, so ∂_2 is injective, so $H_2^\Delta(\mathbb{P}^2) = 0$. Further, we note $\ker \partial_1 = \mathbb{Z}c \oplus \mathbb{Z}(a-b)$, and we have $\partial_2(U+L) = 2c$ and $\partial_2(U-L) = 2a-2b$, so we have

$$H_1^\Delta(\mathbb{P}^2) = \frac{\mathbb{Z}c \oplus \mathbb{Z}(a-b)}{\mathbb{Z}(2c) \oplus \mathbb{Z}(2a-2b) \oplus \mathbb{Z}(a-b+c)} \cong \frac{\mathbb{Z}}{2\mathbb{Z}},$$

finishing. ■

Example 3.11. We note that $\partial\Delta^{n+1} \cong S^n = \Delta^n$, so we can give S^n a natural Δ-complex structure. Then we can compute that $H_n^\Delta(\partial\Delta^{n+1}) \cong \mathbb{Z}$, where the point is that $\partial_{n+1}(\Delta^{n+1})$ does provide a cycle, and all cycles are generated in this way.

3.1.2 Singular Homology

Let's define singular homology now.

Definition 3.12 (singular simplex)**.** Fix a topological space X. A *singular n-simplex* is simply a map $\sigma\colon \Delta^n \to X$ to a topological space, with no other requirements. We define our n-chains $C_n(X)$ to be the $\mathbb{Z}$-linear formal sums of such σs, and we define our chain maps $\partial_n\colon C_n(X) \to C_{n-1}(X)$ given in the usual way by

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|_{\Delta_i^{n-1}}.$$

As before, one can do the exact same proof to show that $\partial_n \circ \partial_{n+1} = 0$, and so we may define homology.

Definition 3.13 (singular homology)**.** Fix a topological space X. Then we define $S(X)$ to be the Δ-complex with exactly one $n$simplex Δ_σ^n for each singular n-simplex $\sigma\colon \Delta^n \to X$, attached via faces. Then we define $H_n(X)$ to be the nth homology on $S(X)$ of the chain

$$\cdots \to C_{n+1}(X) \to C_n(X) \to C_{n-1}(X) \to \cdots.$$

Dealing with $S(X)$ is a little annoying. By allowing for repetitions, we may assume that all our $\mathbb{Z}$-coefficients are actually 1. For $n = 1$, one can realize these as oriented loops, and for $n = 2$, we can think of these as maps of oriented surfaces.

3.2 October 5

We continue our discussion of homology.

3.2.1 Basic Homology Facts

Let's continue working with our singular homology because it is a little more canonical. To begin, it suffices to look at path-connected spaces.

Proposition 3.14. Fix a topological space X with path-connected components X_α for $\alpha \in \pi_0(X)$. Then

$$H_n(X) \cong \bigoplus_{\alpha\in\pi_0(X)} H_n(X_\alpha)$$

Proof. Note that

$$C_\bullet(X) \cong \bigoplus_{\alpha\in\pi_0(X)} C_\bullet(X_\alpha)$$

because any map $\Delta^n \to X$ must land in a single path-connected component. We can see that this provides an isomorphism of chain complexes, so the isomorphism in homology follows. ■

Proposition 3.15. Fix a nonempty path-connected topological space X. Then $H_0(X) \cong \mathbb{Z}$.

Proof. Let $\varepsilon\colon C_0(X) \to \mathbb{Z}$ be the map given by sending

$$\sum_\sigma \alpha_\sigma \sigma \mapsto \sum_\sigma \alpha_\sigma.$$

Intuitively, some $\sigma\colon \Delta^0 \to X$ is just marking a point of X. Now, when X is path-connected, we see that $\operatorname{im}\partial_1 = \ker\varepsilon$. Note that $\ker\varepsilon$ is generated by differences $p-q$ for points $p, q \in X$. So to get these differences, note that for any two points $p, q \in X$, we have a path $f\colon \Delta^1 \to X$ with $f(0) = q$ and $f(1) = p$, so $\partial_1(f) = p-q$, as needed. So we see that

$$H^0(X) \cong \frac{C_0(X)}{\operatorname{im}\partial_1} = \frac{C_0(X)}{\ker\varepsilon} \cong \mathbb{Z},$$

as needed. ■

Remark 3.16. The above points we are checking go under the "Eilenberg–Steenrod axioms."

Proposition 3.17. If X is a point, then $H_n(X) = 0$ for $n > 0$.

Proof. We do this computation by hand. Notably, for each n, there is a unique n-simplex $\sigma_n\colon \Delta^n \to X$ sending everyone to the point. Then we note

$$\partial\sigma_n = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{is odd}, \\ \sigma_{n-1} & \text{if } n \text{ is even}. \end{cases}$$

Thus, our chain complex looks like

$$\cdots \cong \underbrace{C_3(X)}_{\mathbb{Z}\sigma_3} \xrightarrow{0} \underbrace{C_2(X)}_{\sigma_2} \cong \underbrace{C_1(X)}_{\mathbb{Z}\sigma_1} \xrightarrow{0} \underbrace{C_0(X)}_{\mathbb{Z}\sigma_0} \to 0.$$

At odd degrees $2n+1$, we have $\ker\partial_{2n+1} = C_{2n+1}(X) = \operatorname{im}\partial_{2n+2}$, so homology vanishes; at even degrees $\ker\partial_{2n} = 0 = \operatorname{im}\partial_{2n+1}$, so homology still vanishes. ■

The following technical definition will be helpful, mostly for functoriality reasons.

Definition 3.18 (reduced homology)**.** Fix a topological space X, and let $\varepsilon\colon C_0(X) \to \mathbb{Z}$ be the augmentation map. Then we define

$$\widetilde{H}_0(X) = \frac{\ker \varepsilon}{\operatorname{im} \partial_1},$$

and $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$. In particular, $\widetilde{H}_0(\{p\}) = 0$.

3.2.2 Functoriality of Homology

Note that H_n is in fact a functor.

Proposition 3.19. Fix a continuous map $f\colon X \to Y$. Then there is an induced map $H_\bullet(f)\colon H_\bullet(X) \to H_\bullet(Y)$.

Proof. Post-composition will send some $\sigma\colon \Delta^n \to X$ to some $(f \circ \sigma)\colon \Delta^n \to Y$. This extends to a map of chain complexes

$$C_\bullet(f)\colon C_\bullet(X) \to C_\bullet(Y),$$

so we induce a map on homology. Rigorously, one notes that $(f \circ -)$ commutes with ∂: one checks that

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial_n^X} & C_{n-1}(X) \\ {\scriptstyle (f\circ -)}\downarrow & & \downarrow{\scriptstyle (f\circ -)} \\ C_n(Y) & \xrightarrow{\partial_n^Y} & C_{n-1}(Y) \end{array} \qquad \begin{array}{ccc} \sigma & \longmapsto & \sum_{i=0}^{n} \sigma|_{\Delta_{n-1}^i} \\ \downarrow & & \downarrow \\ (f\circ\sigma) & \longmapsto & \sum_{i=0}^{n} (f\circ\sigma)|_{\Delta_{n-1}^i} \end{array}$$

commutes, and this is enough to induce a map on the homology upon checking what lives in what kernels and images. Let's explain this: to begin, we note that $C_n(f)$ maps $\ker \partial_n^X \to \ker \partial_n^Y$ because $\partial_n^Y(C_n(f)(\alpha)) = C_n(f)(\partial_n^X(\alpha)) = 0$. Similarly, we note that $C_n(f)$ maps $\operatorname{im} \partial_{n+1}^X \to \operatorname{im} \partial_{n+1}^Y$ because $C_n(f)(\partial_{n+1}^X(\alpha)) = \partial_{n+1}^Y(C_n(f)(\alpha))$. Thus, we get to produce a map

$$H_n(f)\colon \underbrace{\frac{\ker \partial_n^X}{\operatorname{im} \partial_{n+1}^Y}}_{H_n(X)} \to \underbrace{\frac{\ker \partial_n^Y}{\operatorname{im} \partial_{n+1}^Y}}_{H_n(Y)},$$

as needed. ■

Remark 3.20. As usual, one can check the usual functoriality checks such as that $H_\bullet(f \circ g) = H_\bullet(f) \circ H_\bullet(g)$ and $H_\bullet(\mathrm{id}_X) = \mathrm{id}_{H_\bullet(X)}$. These facts follow directly from the definition of $H_\bullet$.

More generally, the above proof establishes the following result.

Proposition 3.21. Fix a homomorphism $f\colon (C, \partial^C) \to (D, \partial^D)$ of chain complexes such that $\partial^C \circ f = f \circ \partial^D$. Then f induces a natural map on homology.

Proof. This is the last half of the proof of the above proposition. ■

We are now ready to show homotopy invariance. This will follow from the following result.

Theorem 3.22. Fix homotopic maps $f, g\colon X \to Y$ of topological spaces. Then $H_n(f) = H_n(g)$.

Proof. The point is to construct a "chain homotopy" between the maps $H_n(f)$ and $H_n(g)$. Let $F_\bullet\colon X \times I \to Y$ be the needed homotopy from f to g with $F_0 = f$ and $F_1 = g$. Then any singular simplex $\sigma\colon \Delta^n \to X$ will induce a map $(F_\bullet \circ \sigma)\colon \Delta^n \times I \to Y$ with $(F_0 \circ \sigma) = (f \circ \sigma)$ and $(F_1 \circ \sigma) = (g \circ \sigma)$. Technically, $F \circ \sigma$ is not a singular chain, but it is somewhat close.

The goal is as follows: for any chain $[c] \in C_n(X)$, we would like to produce a chain $[d] \in C_{n+1}(X)$ such that $[\partial d] = [f(c)] - [g(c)]$, and this will show that $H_n(f) = H_n(g)$. For this, we would like to make $\Delta^n \times I$ more like a simplex, so we triangulate it in a way which will be compatible with restricting to faces (and hence compatible with ∂).

As a warm-up, let's explain how to triangulate $I^{n+1} = [0,1]^{n+1}$. This is a cube with vertices of the form $(x_0, \ldots, x_n)$ where $x_\bullet\{0,1\}$ for each $x_\bullet$. Now, for each $\sigma \in S_{n+1}$, we choose the $(n+1)$-simplex given by

$$\Delta_\sigma := \{(x_0, \ldots, x_n) : x_{\sigma(0)} \leqslant x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

Notably, every face will be homeomorphic to I^n, and we roughly respect rearranging the coordinates (it just moves simplices around), though reflections will reverse the orientation of the simplex; also, there are $(n+1)!$ total simplices. Summing, we see that I^n is triangulated as

$$\sum_{\sigma \in S_{n+1}} (-1)^{\operatorname{sgn}\sigma} \Delta_\sigma.$$

Now, each simplex contains $(0, \ldots, 0)$ to $(1, \ldots, 1)$, and one can read off σ by noting the simplex has a unique monotonic path along the vertices of the cube from $(0, \ldots, 0)$ to $(1, \ldots, 1)$.

We now return to note that $\Delta^n \times I = \Delta^n \times \Delta^1$ embeds into $(\Delta^1)^n = I^n$, so we may triangulate $\Delta^n \times I$ as a Δ-subcomplex. Explicitly, we see that we are essentially choosing our monotonic path as having its first $i+1$ vertices in $\Delta^n \times \{0\}$ and its last $n - i + 1$ vertices in $\Delta^n \times \{1\}$. Anyway, for this chosen Δ-complex structure on $\Delta^n \times I$, there is a "prism operator," we get something

$$\rho_n := \sum_i (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_n],$$

where the vertices of $\Delta^n \times \{0\}$ are given by $v_0, \ldots, v_n$, and the vertices of $\Delta^n \times \{1\}$ are given by $w_0, \ldots, w_n$. Taking faces, we see that

$$\partial \rho_n = [v_0, \ldots, v_n] - [w_0, \ldots, w_n] + \sum_i (-1)^i F_i \circ \rho_{n-1},$$

where F_i corresponds to the ith face. But by construction of $\rho_\bullet$ and our Δ-complex structure, it follows that this summation is merely $\rho_{n-1} \circ \partial$, so we get the inductive equation

$$\partial \rho_n = [v_0, \ldots, v_n] - [w_0, \ldots, w_n] + \rho_{n-1}\partial.$$

Applying F, we get the needed chain homotopy: given a singular simplex $\sigma\colon \Delta^n \to X$, we define

$$P(\sigma) := (F \circ \sigma)(\rho_n),$$

which is a map $P\colon C_n(X) \to C_{n+1}(Y)$, and the relation tells us that

$$\partial \circ P = C_\bullet(g) - C_\bullet(f) - P \circ \partial,$$

so upon going down to homology, we are done. ■

Remark 3.23. Here is an intuitive argument, using the notation of the first paragraph of the above proof. As a reduction step, we let $i_0\colon X \to X \times I$ and $i_1\colon X \to X \times I$ be the embeddings so that $i_t(a) := (a, t)$. Now, $f = F \circ i_0$ and $g = F \circ i_1$, so by functoriality, it is enough to check that $H_n(i_0) = H_n(i_1)$. Thus, we may as well assume that Y is $X \times I$ and that f and g are i_0 and i_1 respectively. At this point, the result is somewhat intuitive because one should be able to continuously deform $i_0 \circ \sigma$ to $i_1 \circ \sigma$ for any $\sigma\colon \Delta^n \to X$. However, it is mildly difficult to make this argument precise.

Corollary 3.24. Fix a homotopy equivalence $f\colon X \to Y$. Then $H_n(f)\colon H_n(X) \to H_n(Y)$ is an isomorphism.

Proof. This follows from functoriality. Let $g\colon Y \to X$ be the inverse homotopy equivalence for f. Then

$$H_n(f) \circ H_n(g) = H_n(f \circ g) \stackrel{*}{=} H_n(\mathrm{id}_Y) = \mathrm{id}_{H_n(Y)},$$

where $\stackrel{*}{=}$ follows from Theorem 3.22. A symmetric argument shows that $H_n(g) \circ H_n(f) = \mathrm{id}_{H_n(X)}$, so $H_n(f)$ is an isomorphism with inverse given by $H_n(g)$. ■

3.3 October 10

We would like to compute homology groups. The main tool for π_1 was van Kampen's theorem, which essentially allowed us to compute $\pi_1(A \cup B)$ from $\pi_1(A)$ and $\pi_1(B)$. Our goal is to build a similar computation for homology. To do this, we will require a little more homological algebra.

3.3.1 The Mayer–Vietoris Sequence

Let's discuss chain complexes on their own terms.

Definition 3.25 (chain complex)**.** Fix a ring R, and fix a sequence of maps of R-modules

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n_1} \to \cdots.$$

This is a *chain complex* if and only if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ for each n; it is *exact* or *acyclic* if equality holds. We may write this chain complex as $(A_\bullet, \alpha_\bullet)$. A *morphism* of chain complexes $(\varphi_\bullet)\colon (A_\bullet, \alpha_\bullet) \to (B_\bullet, \beta_\bullet)$ is a sequence of maps $\varphi_\bullet\colon A_\bullet \to B_\bullet$ commuting with the boundaries.

Definition 3.26 (homology group)**.** Given a chain complex $(A_\bullet, \alpha_\bullet)$ of R-modules, we define the n*th homology group* to be

$$H_n(A_\bullet) := \frac{\ker \alpha_n}{\operatorname{im} \alpha_{n+1}}.$$

Example 3.27. Given a topological space X, we have shown that

$$\cdots \to C_{n+1}(X) \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \to C_0(X) \to 0$$

is a chain complex.

Example 3.28. The sequence $0 \to A \to B$ is exact if and only if $A \to B$ is injective.

Example 3.29. The sequence $A \to B \to 0$ is exact if and only if $A \to B$ is surjective.

Example 3.30. The sequence $0 \to A \to B \to 0$ if and only if $A \to B$ is an isomorphism.

Example 3.31. The sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is a short exact sequence.

To compute our homology groups, it will help to have the following terminology.

Definition 3.32. A *good pair* of spaces (X, A) is a topological space X along with a closed subspace $A \subseteq X$ such that A is a deformation retract of some open subset $U \subseteq X$ containing A.

Example 3.33. If A is a CW-subcomplex of a CW-complex X, then (X, A) is a good pair by very slightly expanding the CW cells around $A \subseteq X$.

And now here is our result.

Theorem 3.34 (Mayer–Vietoris)**.** Fix a good pair (X, A). Then there is a long exact sequence as follows.

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \longrightarrow \widetilde{H}_{n-1}(X) \longrightarrow \widetilde{H}_{n-1}(X/A) \longrightarrow \cdots$$

Here, the maps $\widetilde{H}_n(A) \to \widetilde{H}_n(X)$ are given by inclusion $A \subseteq X$, and the maps $\widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$ are given by the quotient map $X \twoheadrightarrow X/A$. Note that we have not currently defined the boundary map ∂.

It will take us a while to prove Theorem 3.34. Here is an application.

Example 3.35. We show that

$$\widetilde{H}_i\left(S^n\right) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

Proof. Note that $S^{n-1} \subseteq D^n$ makes a good pair, and D^n is contractible, so $\widetilde{H}^\bullet\left(D^n\right) = 0$ always. Thus, for each i, we find

$$\underbrace{\widetilde{H}_i\left(D^n\right)}_{0} \to \widetilde{H}_i\left(S^n\right) \to \widetilde{H}_{i-1}\left(S^{n-1}\right) \to \underbrace{\widetilde{H}_i\left(D^n\right)}_{0},$$

so the result follows by induction, where the base case is given by $\widetilde{H}_0\left(S^0\right) \cong \mathbb{Z}$ and $\widetilde{H}_i\left(S^0\right) \cong 0$ for $i > 0$, which can be checked directly because S^0 is just two points. ■

3.3.2 Building Long Exact Sequences

The proof of Theorem 3.34 will make use of "relative homology groups."

Definition 3.36 (relative homology)**.** Fix a subspace $A \subseteq X$. We define the *relative chains* by $C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A)$. Then the boundary maps $\partial^X : C_\bullet(X) \to C_\bullet(X)$ and $\partial^A : C_\bullet(A) \to C_\bullet(A)$ induce a boundary map $\partial : C_\bullet(X, A) \to C_\bullet(X, A)$, granting us a chain complex

$$\cdots \to C_{n+1}(X, A) \to C_n(X, A) \to C_{n-1}(X, A) \to \cdots .$$

From here, the *relative homology groups* are the homology groups of the above chain complex.

In particular, we see that some $[\alpha] \in H_n(X, A)$ has $\alpha \in C_n(X)$, where $[\alpha]$ will vanish only when $\alpha = \partial\beta + \gamma$ where $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$. Namely, $H_n(X, A)$ is $\subseteq \partial_n^X \subseteq C_n(X)$ upon taking a quotient by $\operatorname{im} \partial_{n+1}^X$ and by $C_n(A)$.

We are now equipped to show a long exact sequence close to Theorem 3.34.

Proposition 3.37. Fix a subspace $A \subseteq X$. Then there is a long exact sequence as follows.

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \longrightarrow \widetilde{H}_{n-1}(X) \longrightarrow \widetilde{H}_{n-1}(X/A) \longrightarrow \cdots$$

Proof. By construction, we have a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0.$$

Explicitly, for each $n \geqslant 1$, the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ 0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0 \end{array}$$

As such, the result follows directly from the following proposition. ■

Proposition 3.38. Fix a short exact sequence

$$0 \to (A_\bullet, \alpha_\bullet) \xrightarrow{\varphi_\bullet} (B_\bullet, \beta_\bullet) \xrightarrow{\psi_\bullet} (C_\bullet, \gamma_\bullet) \to 0$$

of chain complexes of R-modules; i.e., this is a short exact sequence at each fixed index. Then there is a long exact sequence in homology as follows.

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C) \longrightarrow \cdots$$

Proof. Let's describe the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, which is really the only interesting thing. Well, given $[z] \in H_n(C)$ with $z \in \ker \gamma_n$, we can lift it up to some $y \in B_n$ such that $\varphi_n(y) = z$. Then take $\beta_n(y)$, which we see lives in the kernel of φ_n, so exactness finds some $x \in A_{n-1}$ such that $\psi_{n-1}(x) = \beta_n(y)$. We can check that $\alpha_{n-1}(x) = 0$ by construction, so it follows that x represents some class in $H_{n-1}(A)$, which is the desired class.

For completeness, we describe why this is well-defined. The content is in explaining why the choice of lift y does not affect our element in $H_{n-1}(A)$. Well, choosing a separate element y' in B_n will have $y - y'$ in the image of A_n by exactness, say equal to $\alpha_n(x_0)$. Then choosing $x, x' \in A_{n-1}$ such that $\varphi_{n-1}(x) = \beta_n(y)$ and $\varphi_{n-1}(x') = \beta_n(y')$, we claim that $x - x' = \alpha_n(x_0)$. For this, it is enough to check after applying the injective map φ_{n-1}, which is true by construction of x_0.

Let's quickly sketch some exactness arguments.

- Exact at $H_n(A)$: on one hand, we note that any $[z] \in H_{n+1}(C)$ will have $\varphi_n(\partial([z])) = 0$ by construction of the boundary map. Explicitly, $\varphi_n(\partial(z))$ (suitably defined) will live in the image of β_{n+1}, which is what vanishing means.

  On the other hand, given $[x] \in H_n(A)$ which vanishes under φ_n, meaning that $\varphi_n(x) = \beta_{n+1}(y')$ for some y', allowing us to set $z' := \psi_{n+1}(y')$. The construction of the boundary maps shows $\partial([z']) = [x]$, as needed.

- Exact at $H_n(B)$: on one hand, we note that any $[x] \in H_n(A)$ has $\psi_n(\varphi_n([x])) = 0$ because $\psi_n \circ \varphi_n = 0$.

  On the other hand, given $[y] \in H_n(A)$ which vanishes under ψ_n, we see that $\psi_n(y)$ must be in $\operatorname{im} \gamma_{n+1}$, so write $\psi_n(y) = \gamma_{n+1}(z')$, but then ψ_{n+1} is surjective, so $\psi_n(y) = \gamma_{n+1}(\psi_{n+1}(y')) = \psi_n(\beta_{n+1}(y'))$, so replacing y with $y - \beta_{n+1}(y')$ (which is in the same class) provides $\psi_n(y) = 0$. Thus, exactness grants $y \in \operatorname{im} \varphi_n$, as needed.

- Exact at $H_n(C)$: on one hand, we note that any $[y] \in H_n(B)$ has $\partial(\psi_n([y])) = 0$ by construction of the boundary map: $\psi_n([y])$ has a lift in B_n given by y itself, which by definition of $H_n(B)$ will vanish upon applying β_n.

  On the other hand, given $[z] \in H_n(C)$, going down to 0 in $H_{n-1}(A)$ implies that means that there is a lift $y \in C_n(B)$ of z such that $\beta_n(y) = 0$. But then $[y]$ is a class in $H_n(B)$ mapping to $[z]$, exhibiting our exactness.

That's enough for me. ■

Remark 3.39. One can define the boundary map $H_n(X, A) \to H_{n-1}(A)$ more explicitly by taking some class $[z] \in H_n(X, A)$ and then viewing z as a class of objects in $C_n(X)$, we can literally take its boundary as a chain in X and note that ∂z must then vanish in $C_{n-1}(X)/C_{n-1}(A)$ by construction of the reduced homology, so we produce a chain in $C_{n-1}(A)$. This is essentially the above construction where we have described our objects topologically.

More generally, the above arguments are able to prove the following result.

Lemma 3.40 (Snake). Fix a "snake" (commutative) diagram as follows.

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

The following are true.

(a) There is an exact sequence

$$\ker a \xrightarrow{f} \ker b \xrightarrow{g} \ker c \xrightarrow{\delta} \operatorname{coker} a \xrightarrow{f'} \operatorname{coker} b \xrightarrow{g'} \operatorname{coker} c,$$

where $\ker x \xrightarrow{h} \ker y$ is restriction, δ is the connecting morphism, and $\operatorname{coker} x \xrightarrow{h'} \operatorname{coker} y$ is induced by h' by modding out.

(b) If f is injective, then $\ker a \xrightarrow{f} \ker b$ is injective.

(c) If g' is surjective, then $\operatorname{coker} b \xrightarrow{g'} \operatorname{coker} c$ is surjective.

Proof. Analogous to the last half of the proof of Proposition 3.38. Namely, the construction of the boundary map δ is exactly what we constructed: pull back along g, push through b, and then pull back along f'. ■

Anyway, let's see an example.

Example 3.41. Analogous to Example 3.35, we see that Proposition 3.37 produces in the long exact sequence the exact sequence

$$\underbrace{\widetilde{H}_i\left(D^n\right)}_{0} \to \widetilde{H}_i\left(D^n, S^{n-1}\right) \to \widetilde{H}_{i-1}\left(S^{n-1}\right) \to \underbrace{\widetilde{H}_{i-1}\left(D^n\right)}_{0}.$$

Thus, the middle map is an isomorphism.

3.4 October 17

I am stressed. We're talking about excision today.

3.4.1 Excision

We close class by stating excision, which is a primary tool to compute homology groups.

Theorem 3.42 (excision). Fix subspaces $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A$. Then the inclusion $(X \backslash Z, A \backslash Z) \subseteq (X, A)$ induces isomorphisms $H_n(X \backslash Z, A \backslash Z) \to H_n(X, A)$.

Of course, we see that there is a map $C_n(X \backslash Z, A \backslash Z) \to C_n(X, A)$ given by the inclusions $C_n(X \backslash Z) \subseteq C_n(X)$ and $C_n(A \backslash Z) \subseteq C_n(A)$. The main content, then, is in going the other way. Approximately speaking, the idea is to take some $\alpha \in C_n(X, A)$ and then attempt to throw out the parts of α that live in Z. But for this to make sense, we must subdivide $X \backslash Z$ in order to make sure that we are going to get a chain at the end of this process.

Let's restate this result into something without differences.

Theorem 3.43 (excision)**.** Fix a topological space $X = A \cup B$ where $A, B \subseteq X$ are open. Then the map of open pairs $(B, A \cap B) \to (X, A)$ induces a family of isomorphisms on relative cohomology $H_n(B, A \cap B) \to H_n(X, A)$.

The following tool will be useful.

Definition 3.44. Fix a topological space X, and let $\mathcal{U}$ be an open cover of X. We then let $C_n^{\mathcal{U}}(X)$ denote the subgroup of $C_n(X)$ consisting of chains which output to some open set in $\mathcal{U}$. Notably, $\partial\colon C(X) \to C(X)$ restricts to $\partial\colon C^{\mathcal{U}}(X) \to C^{\mathcal{U}}(X)$.

The main technical result is the following.

Proposition 3.45. Fix a topological space X with open cover $\mathcal{U}$. Then the inclusion of chain complex $C^{\mathcal{U}}(X) \to C(X)$ is an isomorphism on homology.

Remark 3.46. It turns out that there is an inverse map so that composites are chain homotopic to identities, but we will not show this.

Let's see how Theorem 3.43 follows from Proposition 3.45.

Proof of Theorem 3.43. Let $\mathcal{U}$ be the open cover $\{A, B\}$. Then Proposition 3.45 grants $\rho\colon C^{\mathcal{U}}(X) \to C(X)$ which is a section of the inclusion i and a chain homotopy $D\colon C_n(X) \to C_{n+1}(X)$ so that $\partial D + D\partial = \mathrm{id} - i\rho$. It will be a property of the construction that ρ sends $C_n(A) \to C_n(A)$ and D sends $C_n(A)$ to $C_{n+1}(A)$, so the quotient maps

$$\frac{C_n^{\mathcal{U}}(X)}{C_n(A)} \to \frac{C_n(X)}{C_n(A)}$$

is an isomorphism on homology. Continuing, we note that

$$\frac{C_n(B)}{C_n(A \cap B)} \to \frac{C_n^{\mathcal{U}}(X)}{C_n(A)}$$

is an isomorphism because these are both free groups whose generators are given by chains landing in B but not in A. So we have a composite map

$$\frac{C_n(B)}{C_n(A \cap B)} \to \frac{C_n(X)}{C_n(A)},$$

which is an isomorphism on homology, so we are done. ■

So we now turn to the proof of Proposition 3.45. The main point is to use barycentric subdivision to replace a chain with smaller chains which will hopefully land in $\mathcal{U}$. We proceed in stages.

1. For a simplex $[v_0, \ldots, v_n]$, the barycenter is the average of all the coordinates; we denote this point by $\widehat{[v_0, \ldots, v_n]}$.

   Now, for $\Delta^n = [v_0, \ldots, v_n]$, we mark all the barycenters of all the various simplices arising as substrings. Now, given a permutation τ of $\{0, \ldots, n\}$, we have the simplex

$$\Delta(\tau) := \left[v_{\tau(0)}, \widehat{[v_{\tau(0)}, v_{\sigma(1)}]}, \ldots, \widehat{[v_{\tau(0)}, \ldots, v_{\tau(n)}]}\right]$$

   One can see that these turn Δ^n into a Δ-complex, and we are able to define

$$S(\Delta^n) = \sum_{\tau \in \mathrm{Sym}(\{0,\ldots,n\})} (-1)^{\operatorname{sgn}\tau} \Delta(\tau),$$

This can then be extended to chains: we send a chain $\sigma\colon \Delta^n \to X$ to the chain given by passing the terms of $S(\Delta^n)$ through σ.

As an aside, note that each $\Delta(\sigma)$ has the diameter go down by a factor of $\frac{n}{n+1}$ by the nature of how we chose our simplices, so this subdivision will exponentially decrease our diameters. As such, for any chain $\sigma\colon \Delta^n \to X$, we can find i such that $S^i\sigma \in C_n^{\mathcal{U}}(X)$. The point is that we can pull back the open cover $\mathcal{U}$ to Δ^n, reduce to a finite subcover, and then we note that any point in Δ^n has an open neighborhood fully contained in one of the $\mathcal{U}$, so we can merely keep shrinking our diameters via S until we full live in $\mathcal{U}$.

2. Next up, we remark that $S\colon C_n(X) \to C_n(X)$ is chain homotopic to the identity, and the chain homotopy restricts to a map $C_n^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)$. The idea is to work with $\Delta^n \times I$ imagining Δ^n on one end and $S(\Delta^n)$ on the other end. In particular, choose an increasing subsequence $i_0 < i_1 < \cdots < i_n$ of vertices of Δ^n, and we can produce an $(n+1)$-simplex

$$\left[v_{i_0}, \ldots, v_{i_k}, [\widehat{v_{i_0}, \ldots, v_{i_k}}], \ldots, [\widehat{v_{i_0}, \ldots, v_{i_k}}]\right].$$

This will subdivide $\Delta^n \times I$, and we can sum over all these simplices to produce the desired element of $C_{n+1}(\Delta^n)$, and then this becomes a map on $C_n(X)$ by the usual pushing around. Then one can check that $\partial D + D\partial = S - \mathrm{id}$ be a direct computation.

We now argue that we have an isomorphism on homology even though we needed a little stronger for our proof of Theorem 3.43. The point is that we can take any chain $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(U)$ and find j so that $S^j\alpha \in C_n^{\mathcal{U}}(X)$. Because S is chain homotopic to the identity, so S^j, so $[S^j\alpha] = [\alpha]$ in $H_n(X, U)$. Then one needs to argue that this is a bijection.

3.4.2 Fixing Relative Homology

We have the following coherence check.

Proposition 3.47. Fix a good pair (X, A). Then the quotient map $q\colon (X, A) \to (X/A, A/A)$ induces an isomorphism on homology $H_n(X, A) \to \widetilde{H}_n(X/A, A/A)$.

Proof. Being a good pair promises us some open neighborhood V of A with a deformation retract to A. Now, (A, V) and (V, X) are also good pairs, so the usual argument is able to produce a long exact sequence

$$H_n(V, A) \to H_n(X, A) \to H_n(X, V) \to H_{n-1}(V, A),$$

but the end terms vanish, so we see $H_n(X, A) = H_n(X, V)$. Similarly, we get isomorphisms $H_n(X/A, A/A) \cong H_n(X/A, V/A)$, so we put everything together into the following picture.

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\sim} & H_n(X, V) & \xleftarrow{\sim} & H_n(X \setminus A, V \setminus A) \\ {\scriptstyle q}\downarrow & & \downarrow & & {\scriptstyle q}\downarrow \\ H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, V/A) & \xleftarrow{\sim} & H_n((X/A)\setminus(A/A), V/A\setminus(A/A)) \end{array}$$

We have argued that the horizontal arrows are isomorphisms, and we note that $(X \setminus A, V \setminus A)$ is homeomorphic to $((X/A)\setminus(A/A), (V/A)\setminus(A/A))$, so the right arrow is an isomorphism, so we conclude that the left arrow is also an isomorphism. ∎

Remark 3.48. Fix an arbitrary pair (X, A). Then we claim that $H_n(X, A) \cong \widetilde{H}_n(X \cup_A CA)$, where CA is a cone over A (effectively contracting it to a point). Because CA is contractible, we note that the long exact sequence of the pair $(X \cup_A CA, CA)$ produces isomorphisms

$$\widetilde{H}_n(X \cup_A CA) \cong H_n(X \cup_A CA, CA).$$

Now, we apply excision, puncturing CA at the point of the cone, and then $CA\backslash\{0\}$ has a deformation retract to A, so we get an isomorphism

$$H_n(X \cup_A CA, CA) \cong H_n(X, A).$$

This sort of remark turns into an "exact sequence of spaces" where the point is that the composite $A \hookrightarrow X \hookrightarrow X \cup_A CA$ trivializes A, and A is somehow exactly what gets trivialized.

3.5 October 19

Here we go.

3.5.1 Excision for Fun and Profit

Let's use excision to compute homology of some spaces.

Proposition 3.49. Fix pointed topological spaces (X_α, x_α) for $\alpha \in \lambda$, and let X denote the wedge sum of these spaces. Then the induced map

$$\bigoplus_{\alpha\in\lambda} \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(X)$$

is an isomorphism.

Proof. Apply Proposition 3.47 to the good pair given by the disjoint union of the X_αs and the disjoint union of the x_αs. ■

Proposition 3.50. Fix nonempty open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ which are homeomorphic. Then $m = n$.

Proof. Fix some $x \in U$. Then find an open ball $B(x, r) \subseteq U$, so excision tells us that

$$\widetilde{H}_\bullet(U, U\backslash\{x\}) = \widetilde{H}_\bullet(B(x, r), B(x, r)\backslash\{x\}).$$

This is then isomorphic to $\widetilde{H}_\bullet(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\})$ by using an isomorphism $B(x, r) \cong \mathbb{R}^m$.

Now, we claim that $\widetilde{H}_\bullet(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\})$ is $\mathbb{Z}$ if $k = m$ and 0 otherwise, which will complete the proof because it allows us to read off m from U. This follows from the long exact sequence

$$\underbrace{\widetilde{H}_k(\mathbb{R}^m)}_{0} \to \widetilde{H}_k(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \to \widetilde{H}_{k-1}(\mathbb{R}^m\backslash\{0\}) \to \underbrace{\widetilde{H}_{k-1}(\mathbb{R}^n)}_{0}.$$

Now, $\widetilde{H}_{k-1}(\mathbb{R}^m\backslash\{0\})$ was computed in Example 3.35, so the result follows. ■

3.5.2 Functoriality of Long Exact Sequences

Let's prove a few things.

Proposition 3.51. Fix a map of pairs $f\colon (X,A) \to (Y,B)$. Then this induces a morphism of long exact sequences as follows.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

Proof. Commutativity of all squares not involving the boundary map is automatic because H_n is a functor. Anyway, the point is that we actually have a homomorphism of short exact sequences of chain complexes as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X)/C_\bullet(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_\bullet(B) & \longrightarrow & C_\bullet(Y) & \longrightarrow & C_\bullet(Y)/C_\bullet(B) & \longrightarrow & 0 \end{array}$$

One sees that this diagram commutes for any given n because the left square commutes by functoriality of $C_\bullet$, and the right morphism is simply taking the cokernel. So the result will now follow from the following piece of homological algebra. ■

Proposition 3.52. Fix a morphism of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}'_\bullet & \longrightarrow & \mathcal{A}_\bullet & \longrightarrow & \mathcal{A}''_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{B}'_\bullet & \longrightarrow & \mathcal{B}_\bullet & \longrightarrow & \mathcal{B}''_\bullet & \longrightarrow & 0 \end{array}$$

Then there is a morphism of induced long exact sequences as follows.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(\mathcal{A}'_\bullet) & \longrightarrow & H_n(\mathcal{A}_\bullet) & \longrightarrow & H_n(\mathcal{A}''_\bullet) & \longrightarrow & H_{n-1}(\mathcal{A}'_\bullet) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(\mathcal{B}'_\bullet) & \longrightarrow & H_n(\mathcal{B}_\bullet) & \longrightarrow & H_n(\mathcal{B}''_\bullet) & \longrightarrow & H_{n-1}(\mathcal{B}'_\bullet) & \longrightarrow & \cdots \end{array}$$

Proof. Again, the commutativity of any square not involving the boundary is automatic. So it remains to check commutativity of the boundary square

$$\begin{array}{ccc} H_n(\mathcal{A}''_\bullet) & \longrightarrow & H_{n-1}(\mathcal{A}'_\bullet) \\ \downarrow & & \downarrow \\ H_n(\mathcal{B}''_\bullet) & \longrightarrow & H_{n-1}(\mathcal{B}'_\bullet) \end{array}$$

which can be done directly. Well, choose $[\alpha''] \in H_n(\mathcal{A}''_\bullet)$ where $\alpha \in \mathcal{A}''_n$, and we track it through the diagram.

- Along the top, we pull α'' back to some $\alpha \in \mathcal{A}_n$, take boundary down to $\partial\alpha \in \mathcal{A}_{n-1}$, and then we find $\alpha' \in \mathcal{A}'_{n-1}$ such that $\alpha' \mapsto \partial\alpha$. This is then passed through the map $\mathcal{A}'_{n-1} \to \mathcal{B}'_{n-1}$.

- Along the bottom, we push α'' to some $\beta'' \in \mathcal{B}''_n$. Now we compute the boundary. We need to pull β'' back to some $\beta \in \mathcal{B}_n$, but we might as well use the image of $\alpha \in \mathcal{A}_n$. Then we take boundary down to $\partial\beta$, which we might as well take as the image of $\partial\alpha$. Then we find $\beta' \in \mathcal{A}'_{n-1}$ such that $\beta' \mapsto \partial\beta$, but again, we may as well take the image of α'.

The above computation completes the proof. ■

This sort of naturality allows us to derive an equivalence between simplicial and singular homology; as a corollary, this will imply that the simplicial homology is invariant under the chosen Δ-complex structure. We will purely formally use the axioms we have built.

Proposition 3.53. Fix a Δ-complex X a subcomplex $A \subseteq X$. Then (X, A) is a good pair, and there is an isomorphism

$$H_n^\Delta(X, A) \to H_n(X, A).$$

Proof. Checking that (X, A) is a good pair follows from the case of a CW-complex, which can be checked by manually finding the needed open neighborhood of all the cells. We now proceed in many steps.

1. To begin, note that there is at least an embedding $C_\bullet^\Delta(X, A) \to C_\bullet(X, A)$ always. Our goal is to show that the induced map on homology is an isomorphism.
2. Take $A = \varnothing$ and X is a point. Then we manually computed both sides are isomorphic to $\mathbb{Z}$ at degree 0 and no nonzero higher homology.
3. Take $A = \varnothing$ and X is some set of points. Then we take disjoint unions (which cohere for both of our homology theories) to conclude.
4. Take $A = \varnothing$ and X a finite-dimensional Δ-complex. Let's say X is k-dimensional so that $X = X^{(k)}$. Then we use the previous piece of homological algebra to produce a morphism of long exact sequences as follows.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_n^\Delta\left(X^{(k-1)}\right) & \longrightarrow & H_n^\Delta\left(X^{(k)}\right) & \longrightarrow & H_n^\Delta\left(X^{(k)}, X^{(k-1)}\right) & \longrightarrow & H_{n-1}^\Delta\left(X^{(k-1)}\right) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_n\left(X^{(k)}, X^{(k-1)}\right) & \longrightarrow & H_n\left(X^{(k)}\right) & \longrightarrow & H_n\left(X^{(k)}, X^{(k-1)}\right) & \longrightarrow & H_{n-1}\left(X^{(k-1)}\right) & \longrightarrow \cdots \end{array}$$

   By induction, the leftmost and rightmost arrows are isomorphisms. Now, we show that the right middle morphism is an isomorphism by hand, which forces the remaining map to be an isomorphism by the Five lemma (see Proposition 3.54 below). Well, note that $\Delta_n(X^{(k)}, X^{(k-1)})$ is zero for $n \neq k$ and free abelian group with basis given by the k-simplices for $k = n$. (For $n < k$, everything is in $X^{(k-1)}$, and for $n > k$, there is nothing there to begin with.) As such, the same will be true for $H_n^\Delta(X^{(k)}, X^{(k-1)})$. On the other hand, consider the maps

$$\frac{\bigsqcup_\alpha \Delta_\alpha^k}{\bigsqcup_\alpha \partial\Delta_\alpha^k} \to \frac{X^{(k)}}{X^{(k-1)}}$$

   which is a homeomorphism and thus an isomorphism on singular homology. So our singular homology is again we are again zero for $n \neq k$ and when $n = k$ we have the same presentation as before via a computation of $H_n(\Delta^k, \partial\Delta^k)$ done in Example 3.41.
5. Let A be empty and X be an infinite-dimensional complex. Then we note $H_n^\Delta(X^{(n+1)}) = H_n^\Delta(X)$ because all the relevant Δ-complexes for $H_n^\Delta(X)$ will come from $X^{(n+1)}$. So by the previous step, this is $H_n(X^{(n+1)})$. For the other side,

$$\varinjlim H_n(X^{(k)}) = H_n(X)$$

   because the computation of $H_n(X)$ can only ever use finitely many simplices from X^k. (The map is also injective because anything $[\alpha] \in H_n(X^{(k)})$ landing in the trivial class of $H_n(X)$ will be the boundary of some chain, but then this chain can be witnesses again by some $X^{(\ell)}$ for perhaps different but still finite ℓ.) This colimit completes our argument because $H_n(X^{(k)})$ has been dealt with in the finite case.

6. For A nonempty, we simply use the induced morphism of long exact sequences given as follows.

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X,A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \cdots \end{array}$$

Everything but the middle morphism is an isomorphism by the previous steps, so we complete by the Five lemma again (see Proposition 3.54).

■

Professor Agol then proceeded to prove the five lemma. I have copy-pasted a proof using the Snake lemma from a previous homework below.

Proposition 3.54. Consider a commutative diagram of R-modules and homomorphisms such that each row is exact.

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

(a) If f_1 is surjective and f_2, f_4 are monomorphisms, then f_3 is a monomorphism.

(b) If f_5 is a monomorphism and f_2, f_4 are surjective, then f_3 is surjective.

Proof. Label the diagram as follows.

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{a_1} & M_2 & \xrightarrow{a_2} & M_3 & \xrightarrow{a_3} & M_4 & \xrightarrow{a_4} & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \xrightarrow[b_1]{} & N_2 & \xrightarrow[b_2]{} & N_3 & \xrightarrow[b_3]{} & N_4 & \xrightarrow[b_4]{} & N_5 \end{array}$$

Very quickly, we claim that we can induce the following diagram with exact rows.

$$\begin{array}{ccccccccc} & & M_2 & \xrightarrow{a_2} & M_3 & \xrightarrow{a_3} & a_3M_3 & \longrightarrow & 0 \\ & & {\scriptstyle \overline{f_2}}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle \overline{f_4}}\downarrow & & \\ 0 & \longrightarrow & N_2/b_1N_1 & \xrightarrow[b_2]{} & N_3 & \xrightarrow[b_3]{} & N_4 & & \end{array}$$

Here, $\overline{f_2}$ is induced as the composite of $M_2 \xrightarrow{f_2} N_2 \twoheadrightarrow N_2/b_1N_1$; and $\overline{f_2}$ is induced as the restriction of $M_4 \xrightarrow{f_4} N_4$ to a_3M_3. We also note that $a_3 : M_3 \to a_3M_3$ is well-defined because a_3 outputs into its image; $b_2 : N_2/b_1N_1 \to N_3$ is well-defined because $b_1N_1 = \operatorname{im} b_1 \subseteq \ker b_2$ by the exactness of the original diagram.

We now check the exactness of the rows.

- Exact at M_3: we still have $\operatorname{im} a_2 = \ker a_3$ by exactness of the original diagram.
- Exact at a_3M_3: we note that $a_3 : M_3 \to a_3M_3$ is surjective by definition of a_3M_3.
- Exact at N_2/b_1N_1: we note that $\ker b_2 = \operatorname{im} b_1$ by exactness of the original diagram, so $b_2 : N_2/\operatorname{im} b_1 \to N_3$ has trivial kernel.
- Exact at N_3: we still have $\operatorname{im} b_2 = \ker b_3$ by exactness of the original diagram.

We now attack the parts of the problem individually.

(a) The trick is to claim that we have the following commutative diagram with exact rows, where $\widetilde{f}_2$ and $\overline{f}_4$ are monic.

$$\begin{array}{ccccccc} & & M_2/a_1M_1 & \xrightarrow{a_2} & M_3 & \xrightarrow{a_3} & a_3M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{f_2}} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \overline{f_4}} & & \\ 0 & \longrightarrow & N_2/b_1N_1 & \xrightarrow[b_2]{} & N_3 & \xrightarrow[b_3]{} & N_4 & & \end{array}$$

We start by showing that the map $\widetilde{f}_2 : M_2/a_1M_1 \to N_2/b_1N_1$ is actually well-defined with trivial kernel. It suffices to show that the composite $M_2 \xrightarrow{f_2} N_2 \twoheadrightarrow N_2/b_1N_1$ has kernel a_1M_1.

Well, α lives in the kernel of the composite if and only if $f_2\alpha \in b_1N_1$ if and only if $f_2\alpha \in b_1(f_1M_1)$ (because f_1 is surjective) if and only if $f_2\alpha \in \operatorname{im}(b_1 \circ f_1)$ if and only if $f_2\alpha \in \operatorname{im}(f_2 \circ a_1)$ (by commutativity) if and only if $f_2\alpha \in f_2(\operatorname{im} a_1)$ if and only if $\alpha \in \operatorname{im} a_1$ (because f_2 is monic and hence injective). So indeed,

$$\ker(M_2 \to N_2 \twoheadrightarrow N_2/b_1N_1) = \operatorname{im}(M_1 \to M_2),$$

which is what we needed to show that $M_2/a_1M_1 \hookrightarrow N_2/b_1N_1$ is well-defined and monic.

We now note that the rows of the diagram are exact. The only modified point here is exactness at M_3, which now must accommodate for $M_2/a_1M_1 \to M_3$. This map is well-defined because $\operatorname{im} a_1 \subseteq \ker a_2$ by exactness of the original diagram, and we are exact at M_3 because

$$\ker(M_3 \to a_3M_3) = \operatorname{im}(M_2 \to M_3) = \operatorname{im}(M_2/a_1M_1 \to M_3)$$

because modding in the domain does not alter the image.

To finish, we note that $\widetilde{f_2}$ and $\overline{f_4}$ being monic imply that f_3 is monic by Lang III.14 part (a).

(b) Similarly, the trick is to claim that we have the following commutative diagram with exact rows, where $\overline{f_2}$ and $\widetilde{f}_4$ are surjective.

$$\begin{array}{ccccccc} & & M_2 & \xrightarrow{a_2} & M_3 & \xrightarrow{a_3} & a_3M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{f_2}} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \widetilde{f_4}} & & \\ 0 & \longrightarrow & N_2/b_1N_1 & \xrightarrow[b_2]{} & N_3 & \xrightarrow[b_3]{} & b_3N_3 & & \end{array}$$

We quickly note that the map $N_3 \to b_3N_3$ is well-defined because b_3 always outputs to b_3N_3 by definition. We also note that the top row is exact as checked earlier, and the only perturbation to the bottom row is exactness at N_3, which holds because the kernel of b_3 has not changed and will still be $\operatorname{im} b_2$.

Next we show that $\widetilde{f}_4$ is well-defined. For this, we need to show that the image of $f_4 : M_4 \to N_4$ under the restriction to $a_3M_3 \to N_4$ will always output to N_4. Well, we see

$$f_4(\operatorname{im} a_3) = \operatorname{im}(f_4 \circ a_3) = \operatorname{im}(b_3 \circ f_3) \subseteq \operatorname{im} b_3,$$

so we are indeed safe.

We now note that $\overline{f}_2 : M_2 \to N_2/b_1N_1$ is surjective because it is the composite of the surjective maps $f_2 : M_2 \to N_2$ and $N_2 \twoheadrightarrow N_2/b_1N_1$. (Any element of N_2/b_1N_1 can be pulled back to a representative in N_2, which can then be pulled back along f_2 to a representative in M_2.)

Further, we claim that $\widetilde{f}_4$ is surjective. Well, find any $\beta \in \operatorname{im} b_3$ that we want to hit. Because f_4 is surjective, there exists $\alpha \in M_4$ such that $f_4\alpha = \beta$, and we will show that $\alpha \in a_3M_3$, which will be enough.

Indeed, $f_4\alpha \in \operatorname{im} b_3$ if and only if $f_4\alpha \in \ker b_4$ (by exactness) if and only if $\alpha \in \ker(b_4 \circ f_4)$ if and only if $\alpha \in \ker(f_5 \circ a_4)$ (by commutativity) if and only if $a_4\alpha \in \ker f_5$ if and only if $a_4\alpha = 0$ (f_5 is monic) if and only if $\alpha \in \ker a_4$ if and only if $\alpha \in \operatorname{im} a_3$ (by exactness).

In total, the fact that $\overline{f_2}$ and $\widetilde{f}_4$ are surjective implies that f_3 is surjective by Lang III.14 part (b). ■

3.5.3 Degrees

As an application, let's talk a bit about degrees. For example, any map $f\colon S^n \to S^n$ induces a map on homology $H_n(S^n) \to H_n(S^n)$. This is a map $\mathbb{Z} \to \mathbb{Z}$, so it will have to be multiplication by some integer d (independent of the choice of isomorphism $H_n(S^n) \cong \mathbb{Z}$), which is called the degree of f.

Example 3.55. The degree of id_{S^n} is 1.

Example 3.56. Suppose f is not surjective. Then $\deg f = 0$. The point is that f lands in S^n minus a point, which contracts to a point, so the image of f factors through $H_n(S^n \setminus \{*\}) = 0$.

Remark 3.57. Fix $f, g\colon S^n \to S^n$. If $f \sim g$, then $\deg f = \deg g$ because homotopic maps produce the same map on homology.

Remark 3.58. Fix $f, g\colon S^n \to S^n$. We have $\deg(f \circ g) = (\deg f)(\deg g)$ by tracking through the composite maps as $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$.

Example 3.59. If $f\colon S^n \to S^n$ is a continuous bijection, then it is a homeomorphism and so has an inverse map, so $\deg f$ must be a unit in $\mathbb{Z}$, so $\deg f \in \{\pm 1\}$.

Example 3.60. The degree of a reflection $f\colon S^n \to S^n$ is -1. Namely, let Δ_1^n denote the top hemisphere and Δ_2^n denote the bottom hemisphere, and we see that f flips Δ_1^n and Δ_2^n. Noting that $H_n(S^n)$ is generated by $\Delta_1^n - \Delta_2^n$ (one can track through the boundary maps to show this or see it directly on simplicial homology), the fact that $\deg f = -1$ follows.

Example 3.61. The degree of the antipodal map $x \mapsto -x$ is $(-1)^{n+1}$ because it is a composite of $(n+1)$ reflections.

Example 3.62. Suppose $f\colon S^n \to S^n$ has no fixed points. Then one can find a homotopy from f to the antipodal map because the "straight-line" path from $f(x)$ to $-x$ fails to go through the origin. So $\deg f = (-1)^{n+1}$.

3.6 October 24

We continue with some applications of homology.

3.6.1 Applications of Degree

Let's give a few fun applications of the degree.

Proposition 3.63. Fix an integer n. Then S^n has a continuous vector field nonzero everywhere if and only if n is odd.

Proof. Quickly, recall that a vector field is a function assigning a tangent vector to each point. Namely, for each $x \in S^n$, there is a tangent plane $T_x S^n \subseteq \mathbb{R}^{n+1}$ consisting of the vectors $y \in \mathbb{R}^{n+1}$ such that $(y - x) \cdot x =$

0. Shifting down by x, we may as well as say that T_xS^n intersects the origin, and so we are asking for a continuous map $f\colon S^n \to \mathbb{R}^{n+1}$ such that $f(x)\cdot x = 0$ for each $x \in S^n$.

For example, there is a continuous vector field nonzero everywhere on S^1 given by $(x,y) \mapsto (y,-x)$. More generally, if n is odd, then $S^n \subseteq \mathbb{R}^{n+1}$ can have its coordinates enumerated by $(x_1, y_1, \ldots, x_n, y_n)$, and we have a continuous vector field given by

$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (-y_1, x_1, \ldots, -y_n, x_n).$$

Thus, if n is odd, we have a nonzero continuous vector field.

For the other direction, suppose we have a nonzero continuous vector field $f\colon S^n \to \mathbb{R}^{n+1}$. Applying a deformation retraction, we may assume that f actually maps $S^n \to S^n$. But then f maps a vector to a perpendicular vector, so it has no fixed points, so we have a homotopy to the antipodal map, so $\deg f = (-1)^{n+1}$. On the other hand, f is homotopic to the identity by simply following the vector field backwards to the original point. So $\deg f = 1$ also, so n must be odd. ■

Remark 3.64. Colloquially, this is the hairy ball theorem: there is no way to comb the hair of a ball $S^2 \subseteq \mathbb{R}^3$.

Remark 3.65. A more interesting question one can ask is for which n do there exist n pointwise orthogonal vector fields which vanish nowhere. This is equivalent to saying that the tangent bundle TS^n is trivializable. We discussed how to do this for S^1, and there is a similar process for S^3 (viewing $\mathbb{R}^4$ as the underlying vector space for a quaternion algebra) as well as S^7 (using the octonions). It turns out that these are the only such n.

Proposition 3.66. Let n be an even integer. Then $\mathbb{Z}/2\mathbb{Z}$ is the only group which can act freely on S^n.

Proof. Suppose G is a group acting freely on S^n. Then we show that G has an injection into $\mathbb{Z}/2\mathbb{Z}$. Note that each $g \in G$ must act by a homeomorphism on S^n because it has inverse given by g^{-1}, so the action of g must be surjective, so we see that $\deg g \in \{\pm 1\}$. Because $\deg$ is multiplicative, this is actually a homomorphism $\deg\colon G \to \{\pm 1\}$. We argue that this map is injective, which will complete the proof.

Well, suppose $g \neq e$ for some $g \in G$, and we show that $\deg g = -1$. To see this, note that having a free action implies that g has no fixed points, so as usual g is homotopic to the antipodal map, so $\deg g = (-1)^{n+1} = -1$. ■

Remark 3.67. Of course $\mathbb{Z}/2\mathbb{Z}$ acts on any S^n because the antipodal map $x \mapsto -x$ has order 2.

Remark 3.68. For odd spheres, the story is more complicated. We have classified all the groups which act linearly on spheres, but we don't know all the actions explicitly.

3.6.2 Local Degree

Take $n > 0$. Let's discuss a way to compute degree via a "signed point count." Given a map $f\colon S^n \to S^n$, we can try to look locally at some point $y \in \operatorname{im} f$ and attempt to count the number of points in the pre-image of f. Signed appropriately, this will turn into the degree. For example, if we are looking at (say) differentiable maps $f\colon S^1 \to S^1$, counting signed by direction turns into the winding number.

Explicitly, fix $y \in \operatorname{im} f$ such that the fiber $f^{-1}(\{y\})$ is finite, whose points we number off as $\{x_1, \ldots, x_n\}$. By choosing a radius less than half of the smallest distance between any two $x_\bullet$s, we may fix disjoint open neighborhoods U_i around each x_i. We now draw the following rather large diagram.

$$\begin{array}{ccccccc} & H_n\left(S^n, S^n\backslash\{x\}\right) & \xrightarrow{1} & H_n\left(U_i, U_i\backslash\{x\}\right) & \xrightarrow{f} & H_n(S^n, S^n\backslash\{y\}) & \\ & \downarrow 2 & \nwarrow f & \downarrow & f \nearrow & \downarrow 2 & \\ \mathbb{Z} = & H_n\left(S^n\right) & \longrightarrow & H_n\left(S^n, S^n\backslash f^{-1}(\{y\})\right) & & H_n\left(S^n\right) & = \mathbb{Z} \end{array}$$

To begin, we note excision by $S^n\backslash U_i$ implies that the 1 arrow is an isomorphism. Because $S^n\backslash\{*\}$ is contractible for any point $*$, we see that the 2 arrows are isomorphisms. We are now equipped to make the following definition.

Definition 3.69 (local degree)**.** Fix everything as above. Then the *local degree* $\deg f|_{x_i}$ is the degree of the induced map $H_n(S^n) \to H_n(S^n)$ as above.

Proposition 3.70. Fix everything as above. Then

$$\deg f = \sum_{i=1}^{n} \deg f|_{x_i}.$$

Proof. We basically take direct sums of our large diagram, as follows.

$$\begin{array}{ccccc} H^n\left(S^n, S^n\backslash f^{-1}(\{y\})\right) & = & \bigoplus_{i=1}^{n} H_n(U_i, U_i\backslash\{x_i\}) & \xrightarrow{\oplus f_i} & H_n\left(S^n, S^n\backslash\{y\}\right) \\ & & \downarrow & & \| \\ & & \bigoplus_{i=1}^{n} H_n(S^n, S^n\backslash\{x_i\}) & & \| \\ & & \uparrow & & \| \\ & & H^n(S^n) & \xrightarrow{f} & H^n(S^n) \end{array}$$

By excision to delete everything outside the $U_\bullet$s, we see that the top-left arrow is an isomorphism. Then the vertical rectangle commutes by tracking through how $H_n(S^n) \cong \mathbb{Z}$ goes around (this is really the diagram we drew above the definition), so we are done because the vertical maps are all isomorphisms. ■

Remark 3.71. Any map is homotopic to a map with finite fibers somewhere, so this local degree check can usually be carried through. Explicitly, cover S^n by convex balls, such as the hemispheres

$$H_i^\pm := \{(x_0, \ldots, x_n) : \pm x_i > 0\}.$$

Now, for $f\colon S^n \to S^n$, do a barycentric subdivision repeatedly until the diameter is smaller than the Lebesgue number of the cover $f^{-1}(H_i^\pm)$: i.e., we want a cover of S^n such that each point in one of the covering sets lands inside some hemisphere. Then we can "straighten" the map f inside one of the convex hemispheres to make the map f piecewise affine. So the size of our fibers is bounded by the number of simplices of f.

Remark 3.72. In fact, one can show that two maps $f, g\colon S^n \to S^n$ are homotopic if and only if $\deg f = \deg g$, which allows us to strengthen the above result.

Let's use this to show that any degree is achievable.

Example 3.73. For $n = 1$, the map $S^1 \to S^1$ given by $z \mapsto z^k$ has degree k.

We now go up from $n = 1$.

Proposition 3.74. Fix a map $f\colon S^n \to S^n$. Then the suspension map $Sf\colon S^{n+1} \to S^{n+1}$ has $\deg Sf = \deg f$.

Proof. The main concern is that we must go up in the dimension of our homology groups, for which we want to use the long exact sequence. Note that we have a map $Cf\colon (CS^n, S^n \times \{0\}) \to (CS^n, S^n \times \{0\})$, so the quotient space is S^n. Naturality of our long exact sequences now produces the following commutative diagram.

$$\begin{array}{ccccccccc} H_{n+1}(CS^n) & \longrightarrow & H_{n+1}(CS^n, S^n) & = & H_{n+1}(S^{n+1}) & \xrightarrow{\partial} & H_n(S^n) & \longrightarrow & H_n(CS^n) \\ \downarrow{\scriptstyle Cf} & & \downarrow{\scriptstyle Cf} & & \downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Cf} \\ H_{n+1}(CS^n) & \longrightarrow & H_{n+1}(CS^n, S^n) & = & H_{n+1}(S^{n+1}) & \xrightarrow{\partial} & H_n(S^n) & \longrightarrow & H_n(CS^n) \end{array}$$

Here, S^n has been embedded into CS^n via the copy in the code, and the point is that the quotient CS^n/S^n is simply $SS^n = S^{n+1}$. All terms on the ends vanish because CS^n is contractible, so ∂ is an isomorphism, so the proof is complete. ■

Remark 3.75. If a map $f\colon S^n \to S^n$ is differentiable at a point x, then an exercise we did on the homework allows us to compute $\deg f|_x$ as $\det Df_x$. Indeed, f is locally linear at x, so we choose the corresponding neighborhood where f is homotopic to a linear map, and the degree of linear maps was computed on the homework.

3.7 October 26

Today we discuss cellular homology.

3.7.1 Cellular Homology

Let's attempt to compute the homology of a CW-complex.

Lemma 3.76. Fix a CW-complex X and indices k and n.

(a) $\widetilde{H}_k\left(X^n/X^{n-1}\right) = 0$ if $k \neq n$.

(b) $H_n\left(X^n/X^{n-1}\right)$ is free abelian if $k = n$, with a basis given by the n-cells.

(c) $H_k\left(X^n\right) = 0$ if $k > n$.

(d) The inclusion $i\colon X^n \to X$ induces an isomorphism $H_n(i)\colon H_k\left(X^n\right) \to H_k(X)$ if $k < n$ and is a surjection if $k = n$.

Proof. This is similar to what we saw with Δ-complexes.

(a) We see
$$H_k\left(X^n, X^{n-1}\right) \cong \widetilde{H}_k\left(X^n/X^{n-1}\right) = \widetilde{H}_k\left(\bigvee S^n\right),$$
and we know the homology of S^n already.

(b) This follows by induction on n and using (a). The base case is that $H_k\left(X^0\right) = 0$ for $k > 0$. The long exact sequence provides

$$H_k\left(X^{n-1}\right) \to H_k\left(X^n\right) \to H_{k-1}\left(X^n, X^{n-1}\right).$$

The left term vanishes by the inductive hypothesis, and the right term vanishes by (a), so the middle term will also vanish.

(c) A similar exact sequence as in (b) shows that $H_k(X^n) \to H_k\left(X^{n+1}\right)$ is an isomorphism if $k < n$ and surjective when $k = n$. Indeed, we simply write down

$$H_{k+1}\left(X^{n+1}, X^n\right) \to H_k(X^n) \to H_k\left(X^{n+1}\right) \to H_k\left(X^{n+1}, X^n\right)$$

to achieve the result. If X is finite-dimensional, we are done because $X = X^n$ for some n large enough. In the infinite-dimensional case, we use the fact that

$$H_k(X) = \varinjlim H_k(X^n)$$

because any cycle or boundary lives in some fixed chain. So we get this result purely algebraically. ■

We now build a complex from X using its skeletons. For each n, we acknowledge that we have maps $H_{n+1}\left(X^{n+1}, X^n\right) \to H_n(X^n)$ and $H_n(X^n) \to H_n\left(X^n, X^{n-1}\right)$ induced by some long exact sequences, so we get a map $H_{n+1}\left(X^{n+1}, X^n\right) \to H_n\left(X^n, X^{n-1}\right)$ via composition. So we have a sequence

$$\cdots \to H_{n+1}\left(X^{n+1}, X^n\right) \to H_n\left(X^n, X^{n-1}\right) \to H_{n-1}\left(X^{n-1}, X^{n-2}\right) \to \cdots.$$

Quickly, we claim that this is a chain complex. Indeed, the main point is that the composition of two consecutive maps amounts to a long composition

$$H_{n+1}\left(X^{n+1}, X^n\right) \to H_n(X^n) \to H_n\left(X^n, X^{n-1}\right) \to H_{n-1}\left(X^{n-1}\right) \to H_{n-1}\left(X^{n-1}, X^{n-2}\right).$$

However, the composite of the middle three maps must vanish by the relevant long exact sequence. So we are allowed to make the following definition.

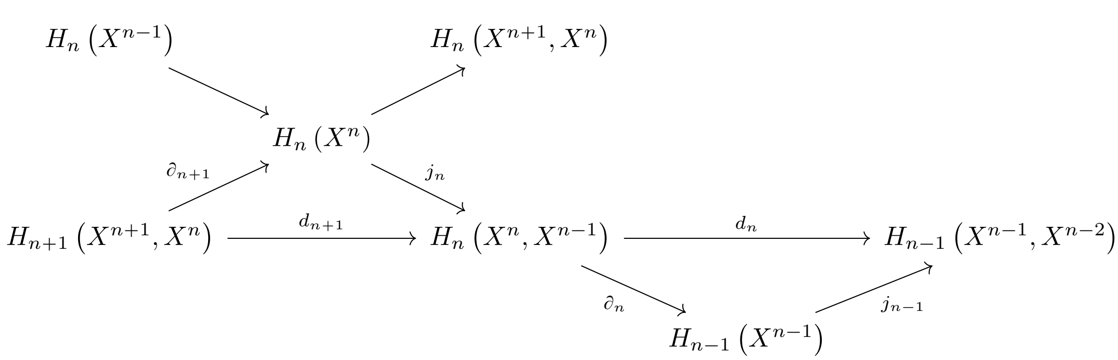
Definition 3.77 (cellular homology)**.** Fix a CW-complex X. Then the *cellular homology groups* $H_n^{CW}(X)$ is the homology of the chain complex

$$\cdots \to H_{n+1}\left(X^{n+1}, X^n\right) \to H_n\left(X^n, X^{n-1}\right) \to H_{n-1}\left(X^{n-1}, X^{n-2}\right) \to \cdots.$$

Of course, we would like to see that this is independent of the chosen CW-structure. In fact, we have the following result.

Proposition 3.78. Fix a CW-complex X. For all n, we have $H_n^{CW}(X) = H_n(X)$.

Proof. Draw the following very large diagram.



Now, by the lemma, we see that $H_n\left(X^{n+1}\right) = H_n(X)$, and we know this is isomorphic to $H_n\left(X^n\right)/\operatorname{im}\partial_{n+1}$. The diagram above has $H_n(X^{n-1}) = 0$, so j_n is injective, and similarly, j_{n-1} is injective. As such, we see $H_n\left(X^n\right)/\operatorname{im}\partial_{n+1}$ is isomorphic to

$$\frac{\operatorname{im} j_n}{\operatorname{im}(j_n \circ \partial_{n+1})}.$$

Again, because j_{n-1} is injective, it follows that $\ker d_n = \ker \partial_n$, which we know by the long exact sequence is the image of j_n, so the numerator is $\ker d_n$. Similarly, we know that the denominator is $\operatorname{im}\, d_{n+1}$ by definition, so we are done. ■

Example 3.79. If X has some r number of n-cells, then $H_n(X)$ is a subgroup of the free abelian group $H_n(X^n, X^{n-1})$ on r generators, so $\ker d_n$ is a free abelian group of at most r generators, so the quotient $H_n^{CW}(X)$ is an abelian group on at most r generators as well.

Example 3.80. Take $X = \mathbb{CP}^n$. This has exactly one cell in each even dimension. So Lemma 3.76 tells us that the cellular homology sequence has every other term equal to $\mathbb{Z}$ up to $2n$, so

$$H_i\left(\mathbb{CP}^n\right) = \begin{cases} \mathbb{Z} & \text{if } i \in \{0, 2, \ldots, 2n\}, \\ 0 & \text{else.} \end{cases}$$

We would like to use $H_\bullet^{CW}$ to actually compute some homology groups, but for this we need to be able to compute the boundary maps $d_\bullet$.

Proposition 3.81. Fix a CW-complex X. The boundary map $d_n\colon H_n\left(X^n, X^{n-1}\right) \to H_{n-1}\left(X^{n-1}, X^{n-2}\right)$ sends some n-cell e_n^α representing a class in $H_n\left(X^n, X^{n-1}\right)$ to

$$\sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of the composite $\Delta_{\alpha\beta}$

$$\underbrace{S_\alpha^{n-1}}_{\partial e_\alpha^n} \to X^{n-1}/X^{n-2} \to \underbrace{S_\beta^{n-1}}_{e_\beta^{n-1}}.$$

Here, the second map is induced via the retraction q_β of X^{n-1}/X^{n-2} onto S_β^{n-1}, squishing $X^{n-1}\backslash e_\beta^{n-1}$ to a point.

Proof. Let $\Phi_\alpha\colon D_\alpha^n \to X^n$ denote the embedding of this n-cell, and $\varphi_\alpha\colon \partial D_\alpha^n \to X^{n-1}$ denote the attaching map. We now draw the following very large diagram.

$$\begin{array}{ccccc} H_n(D_\alpha^n, \partial D_\alpha^n) & \xrightarrow{\partial} & H_{n-1}(\partial D_\alpha^n) & \xrightarrow{\Delta_{\alpha\beta}} & H_{n-1}\left(e_\beta^{n-1}/\partial e_\beta^{n-1}, *\right) \\ \downarrow{\scriptstyle \Phi_\alpha} & & \downarrow{\scriptstyle \varphi_\alpha} & & \uparrow{\scriptstyle q_\beta} \\ H_n\left(X^n, X^{n-1}\right) & \xrightarrow{d_n} & H_{n-1}\left(X^{n-1}, X^{n-2}\right) & \xrightarrow{q} & H_{n-1}(X^{n-1}/X^{n-2}, *) \end{array}$$

Here, q and q_β are the relevant quotient maps. Then one tracks around the relevant diagram and sums over all β to achieve the result. In particular, q_β detects the coordinate of e_β^{n-1} in $d_n(e_n^\alpha)$, and e_n^α is the image of a generator of $H_n(D_\alpha^n, \partial D_\alpha^n)$ passed through Φ_α. So the top composite tells us what the coordinate of e_β^{n-1} in $d_n(e_\alpha^n)$ should look like, which we see is the degree of $\Delta_{\alpha\beta}$, as needed. (Note that ∂ above is an isomorphism because the relevant long exact sequence has the terms before and after the homology of a disk, which vanishes because disks are contractible.) ■

Example 3.82. Consider the surface Σ_2 produced by identifying opposite ends of an octagon. This has one vertex, four edges, and one face, so our cellular homology chain complex is

$$0 \to \mathbb{Z} \to \mathbb{Z}^4 \to \mathbb{Z} \to 0.$$

Using the above formula, we see that each edge in $\mathbb{Z}^4$ goes to 0 (the main point is that we are taking the edge and doing a signed sum of its boundary, but the boundary points have been identified), so we verify that $H_0(\Sigma_2) = \mathbb{Z}$. Next, for the face e^2 generating the left $\mathbb{Z}$, one checks that the identified edges are in such a way that the differential again vanishes, so $H_1(\Sigma_2) = \mathbb{Z}^4$ and $H_2(\Sigma_2) = \mathbb{Z}$. All higher homology vanishes.

Example 3.83. Consider the surface X produced by identifying adjacent edges of an octagon. There is still one vertex, four edges, and one face, so our cellular homology chain complex is

$$0 \to \mathbb{Z} \to \mathbb{Z}^4 \to \mathbb{Z} \to 0.$$

For the same reason as in the previous example, one sees that $\mathbb{Z}^4 \to \mathbb{Z}$ is the zero map, verifying $H_0(\Sigma_2) = \mathbb{Z}$. Computing using the boundary formula, we see that $d_2 \colon \mathbb{Z} \to \mathbb{Z}^4$ is the diagonal map multiplied by 2. So $H_2(\Sigma_2) = 0$ because d_2 is injective, and $H_1(\Sigma_2) = \mathbb{Z}^4/(2,2,2,2)\mathbb{Z}$. One can see this group is $\mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})$, where the point is that we have given $\mathbb{Z}^4$ a new basis given by $(1,0,0,0)$ and $(0,1,0,0)$ and $(0,0,1,0)$ and $(1,1,1,1)$.

3.7.2 Euler Characteristic

Fix a finite CW-complex X.

Definition 3.84 (Euler characteristic)**.** Fix a finite CW-complex X. Then the *Euler characteristic* $\chi(X)$ is the alternating sum

$$\sum_{n \geqslant 0} (-1)^n c_n,$$

where c_n is the number of n-cells of X.

A priori, $\chi(X)$ depends on the CW-structure of X, but we can remove this dependency.

Proposition 3.85. Fix a finite CW-complex X. Then

$$\chi(X) = \sum_{n \geqslant 0} (-1)^n \operatorname{rank} H_n(X).$$

Here, $\operatorname{rank} H_n(X)$ is the number of $\mathbb{Z}$-summands in the finitely generated abelian group $H_n(X)$

Alternatively, the rank is $\dim_{\mathbb{Q}}(H_n(X) \otimes_{\mathbb{Z}} \mathbb{Q})$, where the point is that tensoring by $\mathbb{Q}$ deletes the torsion. We will want the following result.

Lemma 3.86. Fix a short exact sequence

$$0 \to A \to B \to C \to 0$$

of finitely generated abelian groups. Then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof. Tensor with $\mathbb{Q}$ and then use the corresponding fact for dimensions of $\mathbb{Q}$-vector spaces, which is proven directly by counting bases. ■

We are now ready to prove Proposition 3.85.

Proof of Proposition 3.85. More generally, suppose we have a finite chain complex

$$0 \to C_k \to C_{k-1} \to \cdots \to C_1 \to C_0 \to 0$$

with boundary maps $d_n\colon C_n \to C_{n-1}$, and we let $\chi(C_\bullet)$ denote the sum

$$\chi(C_\bullet) := \sum_{n\geqslant 0} \operatorname{rank} H_n(C_\bullet).$$

Note that we have the short exact sequence

$$0 \to \ker d_k \to C_k \to \operatorname{im} d_k \to 0,$$

so we find that $\operatorname{rank} C_k = \operatorname{rank}\ker d_k + \operatorname{rank}\operatorname{im} d_k = \operatorname{rank} H_k(C_\bullet) + \operatorname{rank}\operatorname{im} d_k$. In particular, we find that, trying to reduce ourselves down to

$$0 \to \frac{C_{k-1}}{\operatorname{im} d_k} \to C_{k-2} \to \cdots \to C_1 \to C_0 \to 0,$$

we have

$$\sum_{n\geqslant 0} (-1)^n \operatorname{rank} C_n = \sum_{n\geqslant 0} (-1)^n \operatorname{rank} H_n(C_\bullet).$$

Anyway, for our application, we take $C_k = H_k\left(X^k, X^{k-1}\right)$ to be our cellular homology chain complex, so it follows $H_n(C_\bullet) = H_n(X)$ and $\operatorname{rank} C_n$ is the number of n-cells. This completes the proof. ■

Remark 3.87. As a nice corollary, we see that $\chi(X)$ is homotopy invariant, which is not so obvious. Notably, this allows us to define $\chi(X)$ whenever X is homotopy equivalent to a CW-complex.

3.7.3 Homology with Coefficients

If A is any abelian group, we can define simplicial homology with coefficients in A simply using by replacing simplicial chains $C_n(X)$ with $C_n(X;A) := C_n(X) \otimes_{\mathbb{Z}} A$, which can be intuitively thought of as the free A-module with basis given by singular complexes $\sigma_\bullet\colon \Delta^n \to X$. Notably, if A is in fact a ring, then these are R-modules, so if A is a field, these are F-vector spaces!

Example 3.88. Let R be a ring, and consider the surface Σ_2 from earlier. Then the same computation as in Example 3.82 reveals

$$H_i(\Sigma_2;R) = \begin{cases} R & \text{if } i \in \{0,2\}, \\ R^4 & \text{if } i = 1, \\ 0 & \text{else.} \end{cases}$$

Example 3.89. Consider the space X constructed in Example 3.83 and work with coefficients in $\mathbb{F}_2$. Then the same computation as in the example tells us that the relevant cellular homology sequence

$$0 \to \mathbb{F}_2 \to \mathbb{F}_2^4 \to \mathbb{F}_2 \to 0$$

has differentials equal to 0! So the homology changes.

3.8 October 31

Here we go. Today we'll do more examples with CW-complexes.

3.8.1 Cellular Homology Examples

Here are some examples.

Example 3.90. Let X be a dodecahedron where opposite sides have been identified via a 180° rotation. We compute the homology of X.

Proof. There is some very large diagram which I cannot be bothered to draw. There are thirty edges on the dodecahedron, and each are identified with three edges total, so X has ten 1-cells. Continuing, there are twenty vertices on the dodecahedron, and each is identified with four edges total, so X has five 0-cells. There are twelve faces to start, so X has six 2-cells. Lastly, X has one 3-cell. In total, our chain complex is

$$0 \to \mathbb{Z} \to \mathbb{Z}^6 \to \mathbb{Z}^{10} \to \mathbb{Z}^5 \to 0.$$

One can draw everything out and note that any pair of vertices has exactly one edge connecting them, so X^1 is the complete graph of 5 vertices. From here, one can compute $d_1 \colon \mathbb{Z}^{10} \to \mathbb{Z}^5$ as mapping to $e_i - e_j$ for distinct $i, j \in \{1,2,3,4,5\}$. One can also see that the map $d_3 \colon \mathbb{Z} \to \mathbb{Z}^6$ is the zero map because each face has one copy plus another copy with a reflection afterwards, which sums to zero. It remains to compute $d_2 \colon \mathbb{Z}^6 \to \mathbb{Z}^{10}$. One can track the cellular boundary formula to see that we are outputting any path of length 3.

This then allows us to see that all homology vanishes except $H_3(X) = \mathbb{Z}$. The main point is that X has the same homology as S^3 but is not homeomorphic to it; for example, one can compute that $\pi_1(X)$ is a $(\mathbb{Z}/2\mathbb{Z})$-extension of A_5. ■

Example 3.91 (Moore spaces)**.** Fix an abelian group G and index $n \geqslant 1$. Then there is a space $X = M(G,n)$ with $H_n(X) = G$ while $\widetilde{H}_i(X) = 0$ for $i \neq n$. We write down some X.

Proof. If G is $\mathbb{Z}/m\mathbb{Z}$ for $m \geqslant 1$, we can take X to be S^n with a single e^{n+1} attached of degree m. Then the cellular boundary formula is able to compute the needed homology. From here, wedges are able to take products of these groups to achieve any finitely generated abelian group. (Note the single point can do $G = \mathbb{Z}$.)

The general case requires some thinking. Find a free abelian group F surjecting onto G via $\pi \colon F \twoheadrightarrow G$; say $F = \bigoplus_{\alpha\in\kappa} \mathbb{Z}$. Then begin with the space $X^1 = \bigvee_{\alpha\in\kappa} S^1$. Now, $\ker\varphi$ is a free subgroup of F, so write $\ker\varphi = \bigoplus_{\beta\in\lambda} \mathbb{Z}y_\beta$ where $y_\beta \in F$. For each β, let $f_\beta \colon S^1 \to X^1$ be the corresponding attaching map with $f_\beta(1) = y_\beta$, so we attach a two-cell to fill in this boundary as f_β. From here, one finds that our cellular homology chain complex is just going to exactly be

$$0 \to \bigoplus_{\beta\in\lambda} \mathbb{Z}y_\beta \to \bigoplus_{\alpha\in\kappa} \mathbb{Z} \to 0$$

whose quotient is precisely the needed G. This achieves the correct H_1; from here, one can use suspension n times to get general $M(G,n)$, which works by Example 3.97 (as we will see later from Mayer–Vietoris). Alternatively, we can achieve the same by simply replacing S^1 in the construction above with S^n and directly using the cellular boundary formula in the same way. ■

3.8.2 Group Homology

Let's talk about lens spaces as a way into group homology.

Example 3.92 (lens space)**.** Recall that the lens space $L_m(\ell_1, \ldots, \ell_n)$ is defined by taking $S^{n-1} \subseteq \mathbb{C}^n$ and modding out by a $\mathbb{Z}/m\mathbb{Z}$-action given by

$$\rho(z_1, \ldots, z_m) = \left(\zeta_m^{\ell_1} z_1, \ldots, \zeta_m^{\ell_n} z_n\right),$$

where ρ is a generator of $\mathbb{Z}/m\mathbb{Z}$. Here, the integers $\ell_1, \ldots, \ell_n$ are all coprime to n, so the action of $\mathbb{Z}/m\mathbb{Z}$ is free. We compute the homology of these spaces.

Proof. The main point is to figure out how to put a reasonable CW-structure on the lens space. View S^{2n-1} as n-iterate of the join $S^1 * \cdots * S^1$: send some $(t_1 z_1, \ldots, t_n z_n)$ where $\sum_{i=1}^n t_n = 0$ and $x_1, \ldots, x_n \in S^1$ to the point $(\sqrt{t_1} z_1, \ldots, \sqrt{t_n} z_n) \in S^{2n-1}$.

We will produce a CW-structure with one cell in each dimension; by induction, we may assume that this exists for $L_{m-1}(\ell_1, \ldots, \ell_{n-1})$. Now, the action on the last coordinate S^1 has fundamental domain given by the arc

$$I_m := \left\{e^{2\pi i t/m} : 0 \leqslant t \leqslant 1\right\} \subseteq S^1.$$

Now, $I_m * S^{2n-1}$ attaches to S^{2n-3} as a covering map, and our map is degree m. What happens is that we produce two new 2-cells given by $1 * S^{2n-3} = CS^{2n-3} \cong B^{2n-2}$ and $I_m * S^{2m-3} \cong B^{2n-1}$. The boundary of $I_m * S^{2n-3}$ then attaches with degree 0. Totaling everything, we produce a cellular homology chain complex

$$\cdots \stackrel{m}{\to} \mathbb{Z} \stackrel{0}{\to} \mathbb{Z} \stackrel{m}{\to} \mathbb{Z} \to 0,$$

so our homology is $\mathbb{Z}$ in degrees 0 and $2n-1$, it's $\mathbb{Z}/m\mathbb{Z}$ if k is odd and between 0 and $2n-1$, and it's zero everywhere else. Notably, looking at our homology, we have produced an essentially minimal cell structure: we have a nontrivial torsion group in every other position, so the cell complex structure must have at least one cell in each entry to produce this kind of behavior. ■

Remark 3.93. It is known that $L_q(1,p) \cong L_{q'}(1,p')$ if and only if $q = q'$ and $p \equiv \pm p^{\pm 1} \pmod q$. This is rather hard to show. Notably, some of these spaces are not even homotopic (e.g., $L_5(1,1)$ is not homotopic to $L_5(1,2)$) or can be homotopic but not homeomorphic (e.g., $L_7(1,1)$ and $L_7(1,2)$).

Remark 3.94. Even though we have $\mathbb{RP}^n$ for every n, we can only have these lens spaces in the odd dimensions $2n-1$. The reason is that the only group acting on spheres S^{2n} of even dimension is $\mathbb{Z}/2\mathbb{Z}$.

Remark 3.95. One can write down the cohomology groups $H_\bullet(G; A)$ as $H_\bullet(K(G,1); A)$, but in practice these $K(G,1)$s might be hard to write down. One can use the "infinite lens space" $S^\infty/(\mathbb{Z}/m\mathbb{Z})$ as a $K(\mathbb{Z}/m\mathbb{Z}, 1)$, but this is hard to work with in practice. As another difficult example, we note that any finite-dimensional CW-complex X which is a $K(G,1)$ must have $\pi_1(X)$ torsion-free. Indeed, suppose $a \in \pi_1(X)$ has order $m > 1$. Now, use the subgroup $\langle a \rangle \subseteq \pi_1(X)$ to produce a covering space $p \colon X' \to X$, meaning that X' is homotopy equivalent to $K(\mathbb{Z}/m\mathbb{Z}, 1)$, which is not possible by cellular homology arguments because $K(\mathbb{Z}/m\mathbb{Z}, 1)$ has homology in arbitrarily large coefficients! (Note X' must also be a finite CW-complex because it is a finite cover of a finite complex.)

3.8.3 Mayer–Vietoris

Let's discuss a more convenient version of excision.

Theorem 3.96 (Mayer–Vietoris)**.** Let X be a topological space which is the union of the interiors of two subspaces $A, B \subseteq X$. Then we have a long exact sequence of homology groups

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots.$$

The point here is that $C_n(A) + C_n(B) \subseteq C_n(X)$, and we can then write down the following diagram which turns out to be a chain homotopy.

$$\begin{array}{ccc} C_n(A)+C_n(B) & \longrightarrow & C_n(X) \\ \downarrow \partial & & \downarrow \partial \\ C_{n-1}(A)+C_{n-1}(B) & \longrightarrow & C_{n-1}(X) \end{array}$$

One can then try to take this into the needed long exact sequence. Somehow the main point is to try to use barycentric subdivision to view $C_n(A \cap B)$ as the kernel of the map $C_n(A) \oplus C_n(B) \to C_n(X)$.

3.9 November 2

Today let's discuss the axioms for homology.

3.9.1 More on Mayer–Vietoris

We continue our discussion of Mayer–Vietoris.

Theorem 3.96 (Mayer–Vietoris)**.** Let X be a topological space which is the union of the interiors of two subspaces $A, B \subseteq X$. Then we have a long exact sequence of homology groups

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots .$$

Proof. Note that we have the short exact sequence of simplices

$$0 \to C_\bullet(A \cap B) \to C_\bullet(A) \oplus C_n(B) \to C_\bullet(A) + C_\bullet(B) \to 0,$$

where the left map is $x \mapsto (x, -x)$ and the right map is $(x, y) \mapsto x + y$. Notably, this is exact because the kernel of the map $C_n(A) \oplus C_n(B) \to C_n(A) + C_n(B)$ is simply $C_n(A) \cap C_n(B)$, but the only way to have an n-chain land in both A and in B is for it to land in $A \cap B$, so $C_n(A \cap B) = C_n(A) \cap C_n(B)$ follows. Further, the inclusion $C_n(A) + C_n(B) \subseteq C_n(X)$ is a chain homotopy equivalence by Proposition 3.45 because X is covered by $\{A, B\}$. So we have a long exact sequence in homology, which is the desired one upon noting that

$$H_n(C_\bullet(A) \oplus C_\bullet(B)) = H_n(A) \oplus H_n(B) \qquad \text{and} \qquad H_n(C_\bullet(A) + C_\bullet(B)) = H_n(C_\bullet(X)) = H_n(X),$$

where the left equality is because H_n is additive, and the right equality is by the chain homotopy equivalence as just discussed. ■

Example 3.97. We compute the homology of the suspension $SX = CX \cup_X CX$. Well, let A be some open neighborhood around the left CX, and let B be some open neighborhood around the left CX. Rigorously, if SX is $X \times [-1, 1]$ where we collapse $X \times \{-1\}$ and $X \times \{1\}$, then $A := X \times [-1, 0.1)$ and $B := X \times (-0.1, 1]$ will do. Then $A \cap B$ is homotopic to X, but A and B are both contractible to CX and thus to a point, so Theorem 3.96 tells us that

$$0 \to H_n(SX) \to \underbrace{\widetilde{H}_{n-1}(A \cap B)}_{H_n(X)} \to 0$$

is exact, so $H_n(SX) = H_{n-1}(X)$ follows. Approximately speaking, the geometric content here is that we can turn an $(n-1)$-cycle (made out of some simplices) and bring it up to an n-cycle by taking its cone.

Example 3.98. Consider the torus knot $K_{n,m}$ of Example 2.34, and set $X := S^3 \backslash K_{n,m}$. Now one can choose A to be the space outside the torus and B to be the space inside the torus so that X is covered by $A \cap B$, and both A and B include the boundary. However, A and B can both be contracted to S^1, as can their intersection, so

$$\widetilde{H}_n(A) = \widetilde{H}_n(B) = \widetilde{H}_n(A \cap B) = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

However, $H_1(C) \to H_1(A)$ is multiplication by p by because C winds around p times around the torus in one direction that way by construction, and similarly $H_1(C) \to H_1(B)$ is multiplication by q. The point is that Theorem 3.96 yields

$$\widetilde{H}_n(C) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(X) \to 0,$$

so (for example) $\widetilde{H}_1(X) = \mathbb{Z}/pq\mathbb{Z}$, and the other homology will vanish.

Example 3.99. Suppose $X = A \cup B$ where X is a finite CW-complex, and A and B and $A \cap B$ are homotopic to finite CW-complexes. Then we claim $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$. Indeed, Theorem 3.96 tells us that we have an exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots.$$

Taking alternating sum of ranks, the total sum must vanish, so we conclude that

$$\sum_{n=0}^{\infty} (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(X) + \sum_{n=0}^{\infty} (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(A \cap B) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(A) + \sum_{n=0}^{\infty} (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(B),$$

which is what we wanted.

3.9.2 More on Homology with Coefficients

As usual, G is an abelian group, and X is a space, and we recall $C_\bullet(X;G) := C_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}[G]$. The arguments we made for $G = \mathbb{Z}$ generalize immediately; for example, if X is a point,

$$H_n(X, G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

We are able to define simplicial, singular, and cellular homology theories all in the same way, but we now allow coefficients in $\mathbb{Z}[G]$ instead of merely $\mathbb{Z}$. One complication is in computing the boundary map for the cellular chain complex, for which we need to understand how to compute the degree of a map. So we have the following result.

Lemma 3.100. If $f \colon S^n \to S^n$ is a map of degree m, then the map $H_n(f) \colon H_n(S^n;G) \to H_n(S^n;G)$ is multiplication by m.

We will prove this as a result of naturality in G.

Remark 3.101. Notably, we are allowing for multiplication by m to be zero here!

Anyway, here is our notion of naturality.

Lemma 3.102. Fix a pair (X, A). A homomorphism of groups $\varphi\colon G \to H$ induces a homomorphism of chain complexes

$$C_n(\varphi)\colon C_n(X, A; G) \to C_n(X, A; H)$$

which is functorial.

Proof. The map is simply given by passing the coefficients in $C_n(X, A; G)$ through φ. It will commute with the boundary morphism of $C_n(X, A; G)$ and of $C_n(X, A; G)$ by a direct check. ■

Having a map of chain complexes will thus induce a map on homology, allowing us to prove Lemma 3.100.

Proof of Lemma 3.100. Fix any $g \in G$ representing a class in $H_n(S^n; G)$. Then consider the map $\varphi\colon \mathbb{Z} \to G$ sending $\varphi(1) := g$, and the above naturality tells us that the following diagram commutes.

$$\begin{array}{ccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{H_n(f)} & H_n(S^n;\mathbb{Z}) \\ \varphi\downarrow & & \downarrow\varphi \\ H_n(S^n;G) & \xrightarrow{H_n(f)} & H_n(S^n;G) \end{array} \qquad \begin{array}{ccc} 1 & \longmapsto & m \\ \downarrow & & \downarrow \\ g & \longmapsto & mg \end{array}$$

This is exactly what we wanted to prove. ■

So we can compute our cellular chain complex boundary maps in the usual way.

Example 3.103. Fix a field F, and we will compute the homology on $\mathbb{RP}^n$. Our discussion with lens spaces in Example 3.92 produces a chain complex

$$0 \to F \to F \to F \to \cdots \to F \to F \to 0$$

where the maps alternate being doubling or zero. So if $\operatorname{char} F = 2$, then all these maps are the zero map, so we get $H_k(\mathbb{RP}^n F; F) \cong F$ for $0 \leqslant k \leqslant n$. And if $\operatorname{char} F \neq 2$, then multiplication by 2 is an isomorphism, so we get $H_k(\mathbb{RP}^n; F) = F$ at only $k \in \{0, n\}$ where n is odd. One can check that the Euler characteristic is zero in odd dimensions and one in even dimensions. Of course, a similar computation will work for more arbitrary lens spaces $L_m(\ell_1, \ldots, \ell_n)$, where the point is that multiplication by m as a map $F \to F$ is zero if $\operatorname{char} F \mid m$ and is an isomorphism otherwise.

3.9.3 Axioms for Homology

To give some perspective, let's provide a version of the Eilenberg–Steenrod axioms for reduced homology theories for CW-complexes. Namely, for each integer $n \in \mathbb{Z}$, we want a functor $\widetilde{h}_n$ from the category of CW-complexes to AbGrp. We now add in the following extra conditions.

1. If f is homotopic to g, then $\widetilde{h}_\bullet(f) = \widetilde{g}_\bullet(g)$.
2. For each CW-pair (X, A), we have a long exact sequence

$$\cdots \to \widetilde{h}_n(A) \to \widetilde{h}_n(X) \to \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A) \to \cdots.$$

   This long exact sequence is functorial in the pair (X, A).
3. If $X = \bigwedge_\alpha X_\alpha$ with inclusions $i_\alpha\colon X_\alpha \to X$, then the induced map on homology

$$\bigoplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$$

   is an isomorphism.

4. Take X to be a point. Then we have $\widetilde{h}_n(X) = 0$ for any integer n.

One can show that the above axioms are sufficient to fully pin down homology as singular homology. However, a relaxation of the dimension axiom is able to produce more exotic homology theories.

Example 3.104. For example, there is some homology arising from cobordism of manifolds: consider maps of manifolds to our space X modulo cobordism, where two maps $f_1 \colon M_1 \to X$ and $f_2 \colon M_2 \to X$ are equivalent modulo cobordism if and only if there is some $F \colon N \to X$ such that $\partial N = M_1 \sqcup M_2$ and $F|_{M_i} = f_i$ for each i.

Remark 3.105. Generally speaking, homology theories provide functors from the homotopy category of topological spaces to the category of graded abelian groups. Isomorphisms between homology theories (perhaps on a subcategory of the homotopy category) amount to natural isomorphisms between these functors. Namely, in all the situations above, we were able to produce isomorphisms between our homology theories essentially on the level of chain complexes, which promises that the induced isomorphisms on the level of homology would be natural. We also remark that changing coefficients is natural.

3.10 November 7

We're falling behind, but everything will be okay.

3.10.1 Homology and the Fundamental Group

Throughout, X is path-connected.

Proposition 3.106 (Hurewicz)**.** Let X be a path-connected space with basepoint $x_0 \in X$. Then there is a natural map $h \colon \pi_1(X, x_0) \to H_1(X)$.

Proof. Consider the path $\alpha \colon S^1 \to X$. This will induce a map

$$H_1(\alpha) \colon H_1\left(S^1\right) \to H_1(X),$$

but $H_1\left(S^1\right)$ is isomorphic to $\mathbb{Z}$ generated by the counterclockwise loop. So we define $h(\alpha)$ as going to $H_1(\alpha)(1)$. Homotopic maps define the same map on homology so h is defined up to homotopy class in $\pi_1(X, x_0)$.

Remark 3.107. An alternate way to think about this map is by viewing S^1 as Δ^1 with endpoints identified, so α produces a singular chain $\Delta^1 \to X$, and with the endpoints identified this is actually a singular cycle, so $[\alpha]$ is a genuine class in $H_1(X)$. Note that this agrees with the above definition by tracking through what the map $H_1\left(S^1\right) \to H_1(X)$ actually is: we send the generating singular cycle $\Delta^1 \to S^1$ to the map $\Delta^1 \to S^1 \xrightarrow{\alpha} X$.

Lastly, we should probably check that our map is a homomorphism. Fix $\alpha, \beta \colon S^1 \to X$. Note that the composite $\alpha \cdot \beta$ can simply be thought of as a map $S^1 \vee S^1 \to X$. But now this looks like the composite

$$S^1 \to S^1 \vee S^1 \to X,$$

which on homology is the map $H_1\left(S^1\right) \to H_1\left(S^1\right) \oplus H_1\left(S^1\right) \to H_1(X)$, which goes $1 \mapsto (1,1) \mapsto h(\alpha) + h(\beta)$. ■

Theorem 3.108. Let X be a path-connected space with basepoint $x_0 \in X$. Then h descends to an isomorphism

$$\pi_1(X, x_0)^{\text{ab}} \to H_1(X).$$

Proof. Note $\ker h$ contains the commutator of $\pi_1(X, x_0)$ because the image is an abelian group, so h does descend to some morphism $\pi_1(X, x_0)^{\text{ab}} \to H_1(X)$. Next, we check that h is surjective: it suffices to show that any cycle lives in the image of h, so consider some cycle $z = \sum_i n_i\sigma_i$. We may assume that $n_i \in \{\pm 1\}$ for each i, and because ∂z must vanish, if any σ_i is not immediately a loop, we may find some σ_j which connects to σ_i to cancel out the endpoints; this then allows us to replace σ_i with $\sigma_i \cdot \sigma_j$ upon removing σ_j. Continuing this process finitely, we may assume that our cycle is a sum of loops. But now each of these loops is in the image of h by translating them to have basepoint at x_0, where the translation is legal because this corresponds to adding a loop which goes directly forwards and backwards (which is of course homologically trivial).

It remains to check injectivity. The point is that two-dimensional homology classes are represented by surfaces. Namely, a 1-cycle is trivial if and only if it is represented by loops which are the boundaries of Δ^2, making a similar argument to the one we gave above. However, such a boundary is a product of commutators because of how these oriented surfaces behave. Essentially, one glues together these 2-cycles to build a surface that embeds into X with boundary equal to the loop, and then one can apply a homotopy of the loop through this surface to trivialize it. ■

3.10.2 Applications of Homology

Here is a nice result which we will use for some applications.

Proposition 3.109. We have the following.

(a) Upon embedding $D^k \subseteq S^n$, we have $\widetilde{H}_i\left(S^n \setminus D^k\right) = 0$ for all i.

(b) If S is a subspace of S^n homeomorphic to S^k for $0 \leqslant k \leqslant n$, we have

$$\widetilde{H}_i\left(S^n \setminus S\right) \cong \begin{cases} \mathbb{Z} & \text{if } i = n-k-1, \\ 0 & \text{else.} \end{cases}$$

Here are some nice applications.

Corollary 3.110 (Jordan curve). Any embedding $f\colon S^1 \to S^2$ separates S^2 into two path-connected components.

Proof. Namely, $\widetilde{H}_0\left(S^2 \setminus S^1\right) \cong \mathbb{Z}$, so $S^2 \setminus S^1$ must have two connected components. ■

Example 3.111. One has

$$\widetilde{H}_i\left(S^3 \setminus S^1\right) \cong \begin{cases} \mathbb{Z} & \text{if } i = 1, \\ 0 & \text{else.} \end{cases}$$

We verified this by hand on the homework by providing $\pi_1\left(S^3 \setminus S^1\right)$ with a presentation, from which the computation for H_1 follows by Theorem 3.108. Note that this is potentially surprising because there are some pretty horrible embeddings $h\colon S^1 \to S^3$; for example, $\pi_1\left(S^3 \setminus h\left(S^1\right)\right)$ need not be well-behaved.

Anyway, let's show Proposition 3.109.

Proof of Proposition 3.109. We show our parts separately.

(a) We induct on k. Identify D^k with its image in S^n, for convenience. If $k = 0$, then we are looking at S^n minus a point, which is homeomorphic to $\mathbb{R}^n$, which is contractible and hence has trivial homology. We now apply the induction. Let $h\colon I^k \to D^k$ be a homeomorphism, for convenience. We would like to use Mayer–Vietoris, so we define

$$A := S^n \backslash h\left(I^{k-1} \times [0/1, 2]\right) \qquad \text{and} \qquad B := S^n \backslash h\left(I^{k-1} \times [1/2, 1]\right).$$

Then $A \cap B = S^n \backslash D^k$ is the desired space, and $A \cup B = S^n \backslash h\left(I^{k-1} \times \{1/2\}\right)$ is of lower dimension and so has vanishing homology by the induction. We now may apply Theorem 3.96, which tells us that

$$\widetilde{H}_i\left(S^n \backslash D^k\right) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B).$$

Now, any nontrivial cycle in $\widetilde{H}_i\left(S^n \backslash D^k\right)$ would imply require nontrivial cycle in $\widetilde{H}_i\left(A\right)$ or $\widetilde{H}_i\left(B\right)$. But now A and B are just some version of $S^n \backslash D^k$ again, so we may continue this subdivision process, and having a nontrivial cycle requires a nontrivial cycle in $\widetilde{H}_i\left(S^n \backslash h\left(I^{k-1} \times J\right)\right)$ for smaller and smaller intervals J, which will eventually converge to a unique point x in all of these intervals J.

We now complete by a compactness argument. Namely, α viewed as a cycle of $S^n \backslash \{x\}$ must be trivial, so we can write $\alpha = \partial\beta$ for some $(i+1)$-cycle β, and because β is the union of compact sets, it will live in some $S^n \backslash h\left(I^{k-1} \times J\right)$ for one of these vary small intervals J (because the union of the $S^n \backslash h\left(I^{k-1} \times J\right)$s is $S^n \backslash \{x\}$), so the equation $\alpha = \partial\beta$ must actually hold in one of the homology groups $\widetilde{H}_i\left(S^n \backslash h\left(I^{k-1} \times J\right)\right)$s, which is a contradiction.

(b) This is also an induction on k. For $k = 0$, we note that $S^n \backslash S^0$ is $\mathbb{R}^n$ minus a point, which is $S^{n-1} \times \mathbb{R}$, which has exactly the correct homology by contracting away the $\mathbb{R}$. To complete the proof, one does some Mayer–Vietoris argument. Namely, write S^n as the hemispheres D_1^k and D_2^k, which union to S and have intersection some space S' homeomorphic to S^{k-1}, from which Theorem 3.96 produces

$$\widetilde{H}_{i+1}\left(S^n \backslash D_1^k\right) \oplus \widetilde{H}_{i+1}\left(S^n \backslash D_2^k\right) \to \widetilde{H}_{i+1}\left(S^n \backslash S'\right) \to \widetilde{H}_i\left(S^n \backslash S\right) \to \widetilde{H}_i\left(S^n \backslash D_1^k\right) \oplus \widetilde{H}_i\left(S^n \backslash D_2^k\right).$$

The left and right terms vanish by (a), so we get an isomorphism of our homology groups, from which the result follows by induction. ■

Here is a surprising application to algebra.

Theorem 3.112. The rings $\mathbb{R}$ and $\mathbb{C}$ are the only finite-dimensional commutative division $\mathbb{R}$-algebras.

Proof. Suppose $\mathbb{R}^n$ has been given a commutative division ring structure. There is a map $f\colon S^{n-1} \to S^{n-1}$ by sending $x \mapsto x^2/\left|x^2\right|$, where x^2 refers to the multiplication structure; namely, $x^2 \neq 0$ when $x \neq 0$ because $\mathbb{R}^n$ is a division ring. Further, the product is multilinear and hence extends linearly from a basis, so it is essentially a linear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and hence is continuous, so f is a continuous map. We also note that $f(-x) = f(x)$, so we in fact achieve a map

$$\overline{f}\colon \mathbb{RP}^{n-1} \to S^{n-1}.$$

We also note that $\overline{f}$ is injective because $\mathbb{R}^n$ is commutative: having $x^2 = (\alpha y)^2$ implies that $(x-\alpha y)(x+\alpha y) = 0$ by commutativity, so $x = \pm\alpha y$, so x and y are the same point in $\mathbb{RP}^{n-1}$.

Now, one can show that an injective continuous map from a compact manifold to a connected manifold (both of the same dimension) must be surjective and hence a homeomorphism. So $\overline{f}$ is a homeomorphism when $n \geqslant 2$, but $\mathbb{RP}^{n-1}$ and S^{n-1} fails to be a homeomorphism for $n > 2$ because (say) they have different fundamental groups.

So we are left with the cases $n = 1$ and $n = 2$. When $n = 1$, there is nothing to say because it is an $\mathbb{R}$-algebra already and hence must be $\mathbb{R}$. Lastly, one must show that a 2-dimensional commutative division $\mathbb{R}$-algebra must be $\mathbb{C}$, which is just algebra and hence omitted. ■

Here is a more topological application.

Corollary 3.113 (Borsuk–Ulam)**.** Each map $g\colon S^n \to \mathbb{R}^n$ must have a point $x \in S^n$ such that $g(x) = g(-x)$.

In the case of $n = 1$, this is some kind of intermediate value theorem. We will prove this in the general case via cohomology later.

THEME 4

COHOMOLOGY

4.1 November 9

Today we will start talking about cohomology.

Remark 4.1. Let's begin with some motivational remarks.

- Historically, de Rham cohomology came first, arising from the generalized Stokes' theorem.
- Cohomology has a ring structure called the cup product, which will also prove to be a useful invariant for us.
- Cohomology is required to discuss Poincaré duality.
- Elements of the cohomology groups $H^2(G, A) = H^2(K(G, 1), A)$ represent group extensions of G by A.

4.1.1 Cochains and Cohomology

We go ahead and define cohomology now.

Definition 4.2 (cochain complex)**.** A *cochain complex* $(C^\bullet, \partial^\bullet)$ is a sequence of maps

$$\cdots \overset{\partial^{n-1}}{\to} C^{n-1} \overset{\partial^n}{\to} C^n \overset{\partial^{n+1}}{\to} C^{n+1} \overset{\partial^{n+2}}{\to} \cdots,$$

where we require $\partial^2 = 0$. The *cohomology groups* are

$$H^i(C^\bullet) := \frac{\ker \partial^{i+1}}{\operatorname{im} \partial^i}.$$

Definition 4.3 (dual chain complex)**.** Fix a chain complex (C, ∂) of free abelian groups. Then given an abelian group A, there is a *dual cochain complex* (C^*, ∂^*)

$$\cdots \to \operatorname{Hom}_{\mathbb{Z}}(C_{n-1}, G) \to \operatorname{Hom}_{\mathbb{Z}}(C_n, G) \to \operatorname{Hom}_{\mathbb{Z}}(C_{n+1}, G) \to \cdots.$$

Here, the boundary map $\operatorname{Hom}_{\mathbb{Z}}(C_n, G) \to \operatorname{Hom}_{\mathbb{Z}}(C_{n+1}, G)$ is defined by $f \mapsto (f \circ \partial)$. By abuse of notation, we let $H^n(C_\bullet; G)$ denote the cohomology groups of this dual cochain complex.

It turns out that one can recover cohomology from homology, which is what we will focus on today.

Example 4.4. Using $\mathbb{Z}$ as our dualizing object, the chain complex $0 \to \mathbb{Z} \to 0$ dualizes to $0 \leftarrow \mathbb{Z} \leftarrow 0$.

Example 4.5. Use $\mathbb{Z}$ as our dualizing object again, and consider the chain complex $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$, where $m \neq 0$. Then the dual cochain complex is simply $0 \leftarrow \mathbb{Z} \xleftarrow{m} \mathbb{Z} \leftarrow 0$, which we find by identifying $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ with $\mathbb{Z}$ via $f \mapsto f(1)$ and then tracking through what the coboundary map is.

Remark 4.6. One can show that a finite chain complex of finitely generated free abelian abelian groups will break into a direct sum of chain complexes of the form $0 \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$ where m is a nonzero integer. This is an exercise in Hatcher.

4.1.2 Primer on the Universal Coefficients Theorem

We now investigate how cohomology and homology interact.

Proposition 4.7. Fix a chain complex $(C_\bullet, \partial_\bullet)$. Then there is a natural map

$$H^n(C_\bullet; G) \to \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G).$$

In fact, this map is surjective if the $C_\bullet$ are free abelian groups.

Proof. For brevity, define $Z_n := \ker \partial_n$ to be our cycles for any n, and let $B_n = \operatorname{im} \partial_{n+1}$ to be our boundaries for any n. Now, a class $[\varphi] \in H^n(C_\bullet; G)$ is represented by a $\varphi \colon C_n \to G$ such that $\varphi \circ \partial = \partial^*(\varphi) = 0$, which equivalently means that φ vanishes on restriction to B_n. Anyway, the point is that we can take $z \in Z_n$ and simply output $\varphi(z_n)$, and we see that this is well-defined up to z_n because φ vanishes on B_n. Further, this is well-defined up to φ because the image of ∂^* in $\operatorname{Hom}_{\mathbb{Z}}(C_n, G)$ will vanish on z_n because all such morphisms take the form $\psi \circ \partial$, and $\psi(\partial z_n) = 0$ (because z_n is a cycle).

It remains to show that our map is surjective provided the $C_\bullet$ are free abelian groups. The point is that we have the short exact sequence

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

by definition of these objects, and because $B_{n-1} \subseteq C_{n-1}$ is free, this will actually split, so $C_n \cong Z_n \oplus B_{n-1}$ (albeit non-canonically). Thus, given some map $H_n(G) \to G$, we see that this lifts to a map $\varphi \colon Z_n \to G$, which can then be extended via the splitting to a full map $\widetilde{\varphi} \colon C_n \to G$ vanishing on the image of B_n. Namely, $\widetilde{\varphi}$ has $\partial^*(\varphi) = \varphi \circ \partial = 0$, so $\widetilde{\varphi}$ represents some class in $H^n(C_\bullet; G)$. By construction, $\widetilde{\varphi}$ will restrict to φ on Z_n, so we are in fact hitting the correct map $\varphi \colon H_n(G) \to G$. ■

Remark 4.8. Given a homomorphism $\psi \colon \pi_1(X) \to \mathbb{Z}$, we can descend to a map $\psi \colon H_1(X) \to \mathbb{Z}$. In light of this, we can view some $[\varphi] \in H^1(X, \mathbb{Z})$ producing an "integration map" taking such loops ψ.

Remark 4.9. The end of the proof constructs a splitting $\varphi \mapsto \widetilde{\varphi}$ of $H^n(C_\bullet; G) \to \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G)$.

It remains to compute the kernel of the map in Proposition 4.7. This needs some work; continue in the context of the proof. We begin by drawing the following short exact sequence of complexes.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & B_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

These exact sequences are $\mathbb{Z}$-split currently, so dualizing keeps them $\mathbb{Z}$-split, so we end up with the following short exact sequence of dual cochain complexes.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & Z_{n+1}^* & \longleftarrow & C_{n+1}^* & \xleftarrow{\partial^*} & B_n^* & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longleftarrow & Z_n^* & \longleftarrow & C_n^* & \xleftarrow{\partial^*} & B_{n-1}^* & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Here, the asterisk denotes dualizing. This produces a long exact sequence in cohomology

$$\cdots \leftarrow B_n^* \leftarrow Z_n^* \leftarrow H^n(C_\bullet; G) \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^* \leftarrow \cdots .$$

Now, let $i_n \colon B_n \to Z_n$ denote the inclusion, and we see that we get

$$0 \leftarrow \ker i_n^* \leftarrow H^n(C_\bullet; G) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0.$$

The short exact sequence

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

dualizes to tell us that $\ker i_n^* = \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet; G), G)$, so it remains to compute whatever $\operatorname{coker} i_{n-1}^*$ is. Well, as with $\ker i_n^*$, we see that

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$$

dualizes to

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(H_{n-1}(C_\bullet), G) \to Z_{n-1}^* \to B_{n-1}^*. \tag{4.1}$$

This can be extended to a full free abelian resolution using some homological algebra nonsense, and then the quotient $\operatorname{coker} i_{n-1}^*$ is simply an Ext-group.

> **Remark 4.10.** Professor Agol tried to provide a full construction of Ext in like ten minutes. I have not recorded his attempt.

4.2 November 14

Today we continue with our proof of the Universal Coefficients Theorem.

4.2.1 Homological Algebra taken from Math 250B

Last class we discussed the following notions, which I am taking from my notes from Math 250B.

Definition 4.11 (resolution)**.** Given an R-module M a *resolution of* M is a chain complex (P, ∂) such that

$$P_i = 0 \qquad \text{for } i < 0.$$

Additionally, we require an augmentation map $\varepsilon : P_0 \twoheadrightarrow M$ so that

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

is an exact sequence. We call the above complex the *augmented resolution*, and we notate it by $P \to M$.

Definition 4.12 (projective resolutions)**.** Fix an R-module M with a resolution (P, ∂).

- The resolution is *projective* if and only if P_i is projective for $i \geqslant 0$.
- The resolution is *free* if any only if P_i is free over $i \geqslant 0$.

Note that we have the following coherence check.

Lemma 4.13. Every R-module M has a free resolution and therefore a projective resolution.

Proof. We build the augmented resolution $P \to M$, which we callously call P (so that $P_{-1} = M$). We produce our injective resolution inductively. To start our resolution (P, ∂), we start as required with

$$P_i = \begin{cases} M & i = -1, \\ 0 & i < -1, \end{cases}$$

and $\partial_i = 0$ for $i \leqslant -1$. We now claim that, for any $n \in \mathbb{N}$, we can construct projective modules $\{P_i\}_{i=0}^n$ with maps $\partial_i : P_i \to P_{i-1}$ such that

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$$

is an exact sequence. This induction will finish the proof.[1]

For $n = 0$, we can find a free module P_0 which surjects onto M as $\partial_0 : P_0 \twoheadrightarrow M$, for example by taking $P_0 := \bigoplus_{m \in M} R$. Thus,

$$P_0 \xrightarrow{\partial_0} M \to 0$$

is exact at M because the kernel of $0 : M \to 0$ is all of M, which is precisely the image of $\partial_0 : P_0 \twoheadrightarrow M$.

For the inductive step, we begin with our exact sequence

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$$

and extend it by P_{n+1}. Indeed, as before, we can find a free module P_{n+1} with a surjection $\partial_{n+1} : P_{n+1} \twoheadrightarrow \ker \partial_n$. Tacking this on the front, we have the sequence

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0.$$

It remains to show that this sequence is exact. Well, by the inductive hypothesis, we already have exactness at everyone in $\{P_{n-1}, P_{n-2}, \ldots, P_1, P_0, M\}$. It remains to show exactness at P_n. Well, by construction of ∂_{n+1}, we see that

$$\operatorname{im} \partial_{n+1} = \ker \partial_n,$$

which is exactly the exactness condition at P_n. ■

[1] Technically, one might want to use something like Zorn's lemma to actually go get the projective resolution with infinitely many terms, but we won't do this here.

We would now like to discuss uniqueness of these resolutions. To begin, we note that we can extend morphisms of objects to morphisms of the resolutions.

> **Lemma 4.14.** Suppose that $P := \overline{P} \to M$ and $Q := \overline{Q} \to N$ are projective resolutions for the R-modules M and N, respectively. Then an R-module homomorphism $\varphi : M \to N$ can be extended to a chain morphism $\varphi : P \to Q$.

Proof. The point is to use the fact our modules are projective to extend the morphism $\varphi_{-1} : P_{-1} \to Q_{-1}$ backwards. In particular, for $i < -1$, we set $\varphi_i = 0$ so that the following diagram commutes for any $i \leqslant -1$.

$$\begin{array}{ccc} P_i & \xrightarrow{\partial_i^P} & P_{i-1} \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_{i-1}} \\ Q_i & \xrightarrow[\partial_i^Q]{} & Q_{i-1} \end{array}$$

Namely, the top and bottom arrows are both 0s, so the diagram commutes for free.

Because we have φ_i for $i \leqslant -1$, it suffices exhibit the φ_i for $i \geqslant 0$ inductively, assuming that we have φ_{i-1}; this will finish by muttering something about Zorn's lemma. Namely, we need to induce φ_i to make the following diagram commute.

$$\begin{array}{ccccc} P_i & \xrightarrow{\partial_i^P} & P_{i-1} & \xrightarrow{\partial_{i-1}^P} & P_{i-2} \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_{i-1}} & & \downarrow{\scriptstyle\varphi_{i-2}} \\ Q_i & \xrightarrow[\partial_i^Q]{} & Q_{i-1} & \xrightarrow[\partial_{i-1}^Q]{} & Q_{i-2} \end{array}$$

We would like the fact that P_i is projective in order to induce this arrow, but ∂_i^Q is not a surjection. However, ∂_i^Q does surject onto $\operatorname{im} \partial_i^Q = \ker \partial_{i-1}^Q$ (by exactness), so we would like $\varphi_{i-1} \circ \partial_i^P$ to map into this kernel. Well, we can use the commutativity of the right square to write

$$\partial_{i-1}^Q \circ \varphi_{i-1} \circ \partial_i^P = \varphi_{i-2} \circ \partial_{i-1}^P \circ \partial_i^P = \varphi_{i-2} \circ 0 = 0,$$

so $\operatorname{im}(\varphi_{i-1} \circ \partial_i^P) \subseteq \ker \partial_{i-1}^Q = \operatorname{im} \partial_i^Q$. Thus, the following diagram is well-defined.

$$\begin{array}{ccc} P_i & & \\ {\scriptstyle\varphi_i}\downarrow & \searrow^{\varphi_{i-1}\circ\partial_i^P} & \\ Q_i & \xrightarrow[\partial_i^Q]{} & \operatorname{im} \partial_i^Q \end{array}$$

So, because P_i is projective, we are promised an induced morphism $\varphi_i : P_i \to Q_i$ such that $\partial_i^Q \circ \varphi_i = \varphi_{i-1} \circ \partial_i^P$, which is what we wanted. ∎

Then we get uniqueness of these morphisms up to chain homotopy.

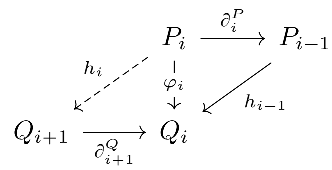
> **Lemma 4.15.** Suppose that $P := \overline{P} \to M$ and $Q := \overline{Q} \to N$ are augmented projective resolutions for the R-modules M and N, respectively. Further, fix two chain morphisms $\alpha, \beta : P \to Q$ such that $\alpha_{-1} = \beta_{-1}$; i.e., the restrictions of α and β to $M \to N$ are equal. Then α and β are homotopic.

Proof. It suffices to show that $\alpha - \beta \sim 0$, so set $\varphi := \alpha - \beta$. In particular, we know that $\varphi_{-1} = \alpha_{-1} - \beta_{-1} = 0$, and we would like to extend this to $\varphi \sim 0$.

Unsurprisingly, we construct our chain homotopy h to witness $\varphi \sim 0$ inductively; i.e., we want $\varphi_i = h_{i-1}\partial_i^P + \partial_{i+1}^Q h_i$ for each i. To start off, we set $h_i = 0$ for $i \leqslant -1$ because this is a morphism $h_i : P_i \to Q_{i+1}$, which must be the zero morphism anyways. Observe that $i \leqslant -1$ will then have

$$\varphi_i = 0 = h_{i-1}\partial_i^P + \partial_{i+1}^Q h_i$$

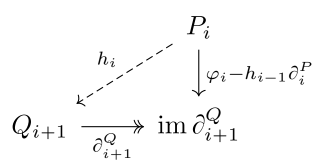
because everything involved is 0. For the inductive step, we have $i \geqslant 0$ and are trying to induce the arrow h_i in the following diagram which does not commute.



As usual, we would like to induce h_i using the fact that P_i is projective. The main point is to show that $\varphi_i - h_{i-1}\partial_i^P$ maps into $\operatorname{im}\partial_{i+1}^Q = \ker\partial_i^Q$. Well, because $\varphi_{i-1} = h_{i-2}\partial_{i-1}^P + \partial_i^Q h_{i-1}$ already, we compute

$$\begin{aligned}\partial_i^Q\left(\varphi_i - h_{i-1}\partial_i^P\right) &= \partial_i^Q\varphi_i - \partial_i^Q h_{i-1}\partial_i^P \\ &= \partial_i^Q\varphi_i - \left(\varphi_{i-1} - h_{i-2}\partial_{i-1}^P\right)\partial_i^P \\ &= \left(\partial_i^Q\varphi_i - \varphi_{i-1}\partial_i^P\right) + h_{i-2}\partial_{i-1}^P\partial_i^P.\end{aligned}$$

The left term here is zero because φ is a chain morphism; the right term is zero by exactness of P. Thus, $\operatorname{im}\left(\varphi_i - h_{i-1}\partial_i^P\right) \subseteq \ker\partial_i^Q = \operatorname{im}\partial_{i+1}^Q$, so the following diagram makes sense.



In particular, the fact that P_i is projective grants us a morphism h_i such that

$$\partial_{i+1}^Q h_i = \varphi_i - h_{i-1}\partial_i^P,$$

which is what we wanted. ■

This then gives the uniqueness of the resolutions, in the following sense.

> **Lemma 4.16.** Suppose that $P := \overline{P} \to M$ and $Q := \overline{Q} \to M$ are augmented projective resolutions for an R-module M. Then there are chain morphisms $\alpha : P \to Q$ and $\beta : Q \to B$ such that $\alpha\beta \sim \mathrm{id}_Q$ and $\beta\alpha \sim \mathrm{id}_P$.

Proof. To start, we use Lemma 4.14 to construct chain morphisms $\alpha : P \to Q$ and $\beta : Q \to P$ such that $\alpha_{-1} = \beta_{-1} = \mathrm{id}_M$.

By symmetry, it suffices to show that $\alpha\beta \sim \mathrm{id}_Q$. Well, $\alpha\beta : Q \to Q$ is a chain morphism such that

$$(\alpha\beta)_{-1} = \alpha_{-1}\beta_{-1} = \mathrm{id}_M\,\mathrm{id}_M = \mathrm{id}_M,$$

and $\mathrm{id}_Q : Q \to Q$ is also a chain morphism such that $(\mathrm{id}_Q)_{-1} = \mathrm{id}_M$. This finishes by Lemma 4.15. ■

Corollary 4.17. Suppose that $P := \overline{P} \to M$ and $Q := \overline{Q} \to M$ are augmented projective resolutions for an R-module M. Then $H^n(\overline{P}; G) = H^n(\overline{Q}; G)$ for any abelian group G, where this cohomology refers to the cohomology on the complex given by dualizing via G.

Proof. The above results produce maps $\alpha\colon P \to Q$ and $\beta\colon Q \to P$ extending $\mathrm{id}_M\colon M \to M$, and we know that $\alpha\beta$ and $\beta\alpha$ are both homotopic to the identity. This will remain true upon dualizing (by functoriality of dualizing), meaning that the corresponding maps $H^n(\alpha; G)$ and $H^n(\beta; G)$ are inverse morphisms because homotopic maps induce the same map on homology, completing the argument. ■

This gives the following definition.

Definition 4.18 (Ext)**.** We define the group $\operatorname{Ext}^i(H, G)$ as $H^i(P; G)$ where P is a projective resolution of H.

4.2.2 Back to Universal Coefficients

We are now ready to provide the following theorem.

Theorem 4.19 (Universal coefficients)**.** Fix a chain complex (C, ∂) of free abelian groups, and let G be an abelian group. Then there is a (split) short exact sequence

$$0 \to \operatorname{Ext}^1(H_{n-1}(C), G) \to H^n(C; G) \to \operatorname{Hom}_{\mathbb{Z}}(H_n(C), G) \to 0.$$

Proof. Surjectivity on the right is Proposition 4.7. The computation of the kernel we saw came from wanting the cokernel from the right of (4.1), which is exactly the needed Ext-group upon noting that (4.1) is in fact what we get upon dualizing the free resolution $B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$. ■

In light of Theorem 4.19, it will be beneficial for us to be able to compute the Ext-groups.

Lemma 4.20. We have the following.

(a) $\operatorname{Ext}^i(H \oplus H', G) \cong \operatorname{Ext}^i(H, G) \oplus \operatorname{Ext}^i(H', G)$.

(b) If H is projective, then $\operatorname{Ext}^i(H, G) = 0$ for all $i > 0$.

(c) We have $\operatorname{Ext}^1(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$ and 0 for higher indices.

(d) If H is finitely generated, then $\operatorname{Ext}^1(H, \mathbb{Z}) = H_{\mathrm{tor}}$ is the torsion subgroup of H.

Proof. Here we go.

(a) Taking a projective resolution for H, and a projective resolution for H', their direct sum produces a projective resolution for $H \oplus H'$, so dualizing preserves the direct sum, and taking homology will still preserve this direct sum.

(b) Note that H has a projective resolution $0 \to H \to H \to 0$, which we can then dualize and compute that all the images of the boundary morphisms are zero, so the cohomology is zero.

(c) Take the free resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

which dualizes to

$$0 \to \operatorname{Hom}(\mathbb{Z}/n\mathbb{Z}, G) \to G \xrightarrow{n} G,$$

so $\operatorname{Ext}^1(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$. At higher indices, the resolution is simply zero everywhere, so our cohomology vanishes.

(d) This follows from combining the previous parts plus the fact that any finitely generated abelian group H is the direct sum of $\mathbb{Z}$s and $\mathbb{Z}/n\mathbb{Z}$s. ■

Corollary 4.21. Suppose (C, ∂) is a chain complex of free abelian groups. If $H_n(C)$ and $H_{n-1}(C)$ are finitely generated, then

$$H^n(C; \mathbb{Z}) \cong H_n(C)/H_n(C)_{\text{tor}} \oplus H_{n-1}(C)_{\text{tor}}.$$

Proof. Apply Theorem 4.19, nothing that the sequence splits and that $\operatorname{Ext}^1(H_{n-1}(C), \mathbb{Z})$ is $H_{n-1}(C)_{\text{tor}}$ by Lemma 4.20, and $H_n(C)/H_n(C)_{\text{tor}} = \operatorname{Hom}_{\mathbb{Z}}(H_n(C), G)$. ■

We close our discussion by noting that Theorem 4.19 is natural.

Proposition 4.22. Fix a morphism $\alpha\colon (C, \partial) \to (C', \partial')$ of chain complexes of free abelian groups, and let G be an abelian group. Then there is a morphism of short exact sequences as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^1(H_{n-1}(C), G) & \longrightarrow & H^n(C, G) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_n(C), G) & \longrightarrow & 0 \\ & & \alpha\big\uparrow & & \alpha\big\uparrow & & \alpha\big\uparrow & & \\ 0 & \longrightarrow & \operatorname{Ext}^1(H_{n-1}(C'), G) & \longrightarrow & H^n(C', G) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_n(C'), G) & \longrightarrow & 0 \end{array}$$

Proof. Everything in sight is functorial, so all the maps are at least well-defined. The main point is that the right square commutes by tracking through the construction of the horizontal maps: indeed, the map simply sends a class in $H^n(C, G)$ to its evaluation on a cycle. This then induces a map on the kernels, which is the left-hand map above. ■

4.2.3 Cohomology of Spaces

We now apply the abstract machinery we built to topological spaces X. In particular, we now build singular cohomology. Let $(C_n(X), \partial)$ denote the singular chain complex, which then dualizes to a complex $(C^n(X), \delta)$, where $C^n(X, G) \coloneqq \operatorname{Hom}_{\mathbb{Z}}(C_n(X), G)$, and the boundary δ sends $\varphi \mapsto (\varphi \circ \partial)$. It is worth our time to describe this a little more explicitly. Given a singular simplex $\sigma\colon \Delta^{n+1} \to X$ and some $\varphi \in C^n(X, G)$, we can compute that

$$(\delta\varphi)(\sigma) = \varphi(\partial\sigma) = \sum_{i=0}^{n+1} (-1)^i \varphi([v_0, \ldots, \widehat{v}_i, \ldots, v_{n+1}]),$$

where we are notating $\Delta^{n+1} = [v_0, \ldots, v_{n+1}]$.

We now list some properties of our cohomology groups.

- Negating indices as desired, one sees that short exact sequences of cochain complexes induce long exact sequences of cohomology; the main point is that a cochain complex is essentially a chain complex where one negates the indices, so the arguments of Proposition 3.38 apply.

- Relative cohomology: given a pair (X, A), one has the short exact sequence of chain complexes

  $$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0,$$

  and because these are chains of free abelian groups, this produces a short exact sequence of cochain complexes

  $$0 \to C_\bullet(X, A; G) \to C_\bullet(X; G) \to C_\bullet(A; G) \to 0,$$

and the cohomology of $C_\bullet(X,A;G)$ will be denoted $H^n(X,A;G)$; one can see that the map of Proposition 4.7 will have $H^n(X,A;G)$ output to $H_n(X,A)$. Anyway, this thus fits into the long exact sequence

$$\cdots \longrightarrow H^n(X,A;G) \longrightarrow H^n(X;G) \longrightarrow H^n(A;G) \longrightarrow H^{n+1}(X,A;G) \longrightarrow H^{n+1}(X;G) \longrightarrow H^n(X;G) \longrightarrow \cdots$$

as before.

- The boundary maps of our long exact sequences commute. Namely, the morphisms of Proposition 4.7 fit into the following commuting square.

$$\begin{array}{ccc} H^n(A;G) & \longrightarrow & H^{n+1}(X,A;G) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathbb{Z}}(H_n(A),G) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_{n+1}(X,A),G) \end{array}$$

To prove this, one should track through all the boundary morphisms, which I cannot be bothered to do.

- Functoriality: as usual, we see that a continuous map $f\colon X \to Y$ induces a map $C_n(f)\colon C_n(X) \to C_n(Y)$, which then induces a map $C^n(f)\colon C^n(Y;G) \to C^n(X;G)$, which then will induce a morphism on homology $H^n(f)\colon H^n(Y;G) \to H^n(X;G)$. This is essentially the composite of many functorial constructions, so the total thing is functorial.

- Homotopy invariance: exactly as in the proof of homology, homotopic maps $f,g\colon (X,A) \to (Y,B)$ induce isomorphisms $H^n(Y,B;G) \to H^n(X,A;G)$. The proof is entirely dual, the main point being that one can take the chain homotopy produced in that proof and then take its dual to produce the needed chain homotopy here.

- Excision: there is an excision statement using relative cohomology exactly given as one would expect. Its proof is dual to the case of homology.

- Axioms: one can axiomatize cohomology theories as one would expect. Here are some axioms for CW-complexes. These are functors $\widetilde{h}^n$ satisfying the following properties.
  - Homotopic maps produce the same map on cohomology.
  - Excision: there is a long exact sequence for CW-pairs (X,A) given by

$$\cdots \to \widetilde{h}^n(X/A) \to \widetilde{h}^n(X) \to \widetilde{h}^n(A) \to \widetilde{h}^n(X/A) \to \cdots .$$

  - Wedge sums: given $X = \bigvee_\alpha X_\alpha$ with embeddings $i_\alpha\colon X_\alpha \to X$, the induced map

$$\widetilde{h}_n(X) \xrightarrow{\prod_\alpha \widetilde{h}_n(i_\alpha)} \prod_\alpha \widetilde{h}_n(X_\alpha)$$

    is an isomorphism.

- Simplicial cohomology: Δ-complexes have $H^n_\Delta(X,A;G)$ defined as the cohomology given by dualizing the chain complex $C^\Delta_n(X,A)$. One can show, as in the homology situation, that $H^n_\Delta(X,A;G) \cong H^n(X,A;G)$.

- Cellular cohomology: as in the situation with homology, one has the following complex from a CW-complex X, where the diagonal maps are produced by repeatedly applying excision.

$$\begin{array}{ccccccc} & & H^k\left(X^k;G\right) & & & & \\ & \nearrow & & \searrow & & & \\ H^k\left(X^k,X^{k+1};G\right) & & \xrightarrow{\partial^k} & & H^{k+1}\left(X^{k+1},X^k;G\right) & \xrightarrow{\partial^{k+1}} & H^{k+1}\left(X^{k+1},X^k;G\right) \\ & & & & & \searrow \quad \nearrow & \\ & & & & & H^{k+1}\left(X^k\right) & \end{array}$$

Then the cohomology of this cochain complex is called cellular cohomology and agrees with the usual cohomology. Alternating, one can just appeal to the case with homology: note that Theorem 4.19 tells us that

$$H^\bullet\left(X^n,X^{n-1};G\right)\cong\operatorname{Hom}_{\mathbb{Z}}\left(H_\bullet\left(X^n,X^{n-1}\right),G\right)$$

because $H_k\left(X^n,X^{n-1}\right)$ is always $\mathbb{Z}$-free and thus has vanishing Ext. So the cellular homology complex simply dualizes.

4.3 November 16

Today we will discuss the cup product.

4.3.1 The Cup Product

In the discussion that follows, we choose coefficients in a ring R, which we will assume is commutative and has unity and so on.

Definition 4.23. Fix a topological space X and a ring R. Given $\varphi\in C^k(X;R)$ and $\psi\in C^\ell(X;R)$, we define the *cup product* as the cochain $(\varphi\cup\psi)\in C^{k+\ell}(X;R)$

$$(\varphi\cup\psi)(\sigma)=\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,v_n]}),$$

where $\sigma\colon[v_0,\ldots,v_n]\to X$ (with $n=k+\ell$) is a singular simplex.

Extending linearly, we see that this defines a map

$$C^k(X;R)\otimes_R C^\ell(X;R)\to C^{k+\ell}(X).$$

For example, we would like this to agree with the boundary map.

Lemma 4.24. Fix a topological space X and a ring R. Given $\varphi\in C^k(X;R)$ and $\psi\in C^\ell(X;R)$, we have

$$\partial(\varphi\cup\psi)=\partial\varphi\cup\psi+(-1)^k\varphi\cup\partial\psi,$$

where ∂ is the boundary map.

Proof. We check on a single singular simplex $\sigma\colon[v_0,\ldots,v_n]\to X$, where $n=k+\ell$. Indeed, we directly compute

$$\begin{aligned}(\partial\varphi\cup\psi)(\sigma)&=\partial\varphi(\sigma|_{[v_0,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{n+1}]})\\&=\sum_{i=0}^{k+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{n+1}]}),\end{aligned}$$

and

$$\begin{aligned}(-1)^k(\varphi \cup \partial\psi)(\sigma) &= (-1)^k\varphi(\sigma|_{[v_0,\ldots,v_k]})\partial\psi(\sigma|_{[v_k,\ldots,v_{n+1}]}) \\ &= \sum_{i=k}^{n+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\widehat{v}_i,\ldots,v_{n+1}]}),\end{aligned}$$

and

$$\begin{aligned}\partial(\varphi \cup \psi)(\sigma) &= \sum_{i=0}^{n+1}(-1)^i(\varphi \cup \psi)(\sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{n+1}]}) \\ &= \sum_{i=0}^{k}(-1)^i\varphi(\sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{n+1}]}) \\ &\quad + \sum_{i=k+1}^{n+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_{k+1},\ldots,\widehat{v}_i,\ldots,v_{n+1}]}).\end{aligned}$$

Collecting the terms completes the proof; notably, the last term of the $(\partial\varphi \cup \psi)$ sum cancels with the first term of the $(-1)^k(\varphi \cup \partial\psi)$ sum, making the total number of terms agree. ■

Corollary 4.25. Fix a topological space X and a ring R. Given $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, if φ and ψ are cocycles, then so is $\varphi \cup \psi$.

Proof. Note that $\partial\varphi = 0$ and $\partial\psi = 0$, so the result follows from Lemma 4.24. ■

Corollary 4.26. Fix a topological space X and a ring R. Given $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, if φ or ψ is a coboundary and the other is a cocycle, then so is $\varphi \cup \psi$.

Proof. For example, if φ is a coboundary and ψ is a cocycle, then write $\varphi = \partial\varphi_0$, so

$$\varphi \cup \partial = \partial\varphi_0 \cup \psi = \partial(\varphi_0 \cup \psi) - (-1)^k\varphi_0 \cup \underbrace{\partial\psi}_{0} = \partial(\varphi_0 \cup \psi).$$

The other argument is similar. ■

The point now is that we have induced a multiplication structure

$$\cup\colon H^k(X;R) \otimes_R H^\ell(X;R) \to H^{k+\ell}(X;R)$$

because we send cocycles to cocycles and coboundaries to coboundaries; a direct computation shows that $\cup$ is associative and distributes over addition, so we are in fact getting a graded ring structure, perhaps without unity and perhaps not commutative. Let's see some examples.

Example 4.27. Consider the genus-2 surface M, which can be visualized as an octagon with every other edge identified in the opposite orientation. Now, there is a map $H^1(M) \times H^1(M) \to H^2(M)$. Recall that we computed $H_i(M)$ is always free abelian, with ranks 1, 4, and 1 in degrees 0, 1, and 2. Then Theorem 4.19 tells us that we may identify $H^i(M)$ with $\operatorname{Hom}_{\mathbb{Z}}(H_i(M), \mathbb{Z})$; for example, distinguish generators a_1, a_2, b_1, and b_2 for $H_1(M)$, and then we let the corresponding indicators (i.e., the dual basis) be α_1, α_2, β_1, and β_2. This produces a cocycle "for free" by Theorem 4.19, but one can also just check it directly. For example, up to labeling the edges appropriately, $[\alpha_1] \cup [\beta_1]$ will be nonzero on a single 2-simplex by a direct computation; a dual computation (with opposing signs) explains that $[\beta_1] \cup [\alpha_1]$ is exactly the negative of this.

Cup products also come with a naturality.

Proposition 4.28. Fix a continuous map $f: X \to Y$. Then the maps $H^\bullet(f)$ induce a homomorphism of graded rings

$$H^\bullet(f): H^\bullet(Y;R) \to H^\bullet(X;R).$$

Proof. This is a direct computation. Of course $H^\bullet(f)$ is already additive, so it only remains to show that it is multiplicative. As usual, we fix some $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(Y;R)$ along with some singular simplex $\sigma: \Delta^n \to X$ where $n := k + \ell$. Then

$$\begin{aligned} H_n(f)(\varphi \cup \psi)(\sigma) &= (\varphi \cup \psi)(f \circ \sigma) \\ &= \varphi(f \circ \sigma|_{[v_0,\ldots,v_k]})\psi(f \circ \sigma|_{[v_k,\ldots,v_n]}) \\ &= H_n(f)(\varphi)(\sigma|_{[v_0,\ldots,v_k]})H_n(f)(\psi|_{[v_k,\ldots,v_n]}) \\ &= H_n(f)(\varphi) \cup H_n(f)(\psi), \end{aligned}$$

as desired. ■

Remark 4.29. Let's take a moment to provide a geometric interpretation of the cup product if two 1-cocycles $\alpha, \beta \in H^1(X;\mathbb{Z})$. By Theorem 4.19, we are computing the product of two elements of

$$H^1(X;\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_1(X),\mathbb{Z}) = \operatorname{Hom}(\pi_1(X),\mathbb{Z}),$$

where in the last equality we have used the fact that $\pi_1(X)^{\text{ab}} = H_1(X)$. So we may choose maps $a: X \to S^1$ and $b: X \to S^1$ such that the induced maps $\pi_1(X) \to \pi_1(\mathbb{Z})$ and $\pi_1(X) \to \pi_1(\mathbb{Z})$ are simply α and β. Now, taking the product of $a \times b$ produces a map $X \to S^1 \times S^1$ for which $\pi_1(a \times b)$ projects down to α and β. From here, an explicit computation (using the above result) can show $\alpha \cup \beta$ is simply given by $\pi_1(a \times b)(x \cup y)$ where $x, y \in H^2\left(S^1 \times S^1\right)$ are the generators by the edges of the corresponding square diagram.

Remark 4.30. We take a moment to note that there are relative cup products

$$H^k(X,A;R) \times H^\ell(X,B;R) \to H^{k+\ell}(X,A \cup B;R).$$

The point is that φ vanishing on A and ψ vanishing on B will have $\varphi \cup \psi$ vanishing on their union by Lemma 4.24. From here, we note that we then get another graded ring structure on $H^\bullet(X,A;R)$.

Example 4.31. We can compute that $H^\bullet\left(\mathbb{RP}^n, \mathbb{F}_2\right)$ is isomorphic to $\mathbb{F}_2[x]/\left(x^{n+1}\right)$ where x has degree 1. Similarly, we can compute that $H^\bullet\left(\mathbb{RP}^\infty, \mathbb{F}_2\right)$ is isomorphic to $\mathbb{F}_2[x]$ where x has degree 1.

Proof. We will work on $\mathbb{RP}^n$ only. The point is that $\mathbb{RP}^n$ can be given a triangulation by viewing it as $S^n/\sim$ where $\sim$ is the antipodal equivalence relation. Now, taking joins via $*$, we note that $S^n = S^0 * \cdots * S^n$, which provides S^n with a simplicial structure. Explicitly, realizing S^0 on an axis of $\mathbb{R}^{n+1}$, we see that we can view S^n as sitting inside $\mathbb{R}^{n+1}$ as connecting the dots of the form $(0,\ldots,\pm 1,\ldots,0)$; then modding out by $\sim$, we receive a simplicial structure on $\mathbb{RP}^n$ with $n+1$ vertices.

We now acknowledge that Theorem 4.19 tells us that $H^k(\mathbb{RP}^n, \mathbb{F}_2) = \mathbb{F}_2$ for $0 \leqslant k \leqslant n$, so we are at least correct on the level of abelian groups. It remains to compute the cup product, where we must show that a generator of $H^k(\mathbb{RP}^n, \mathbb{F}_2)$ cupped with a generator of $H^\ell(\mathbb{RP}^n, \mathbb{F}_2)$ produces a generator of $H^{k+\ell}(\mathbb{RP}^n, \mathbb{F}_2)$. One must make some choice of generator, so we choose a generator of $H^1(\mathbb{RP}^n, \mathbb{F}_2)$ by being 1 on each edge meeting the plane $x_0 + \cdots + x_n = 0$ and 0 elsewhere. Then we compute the cup product with itself a few times to conclude. ■

Remark 4.32. One can similarly compute for $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$; the cohomology turns out to be the same "ring" but with the generator x in degree 2 so that the ring is in fact commutative.

4.4 November 28

Today we will continue talking about the cup product. Homework has been posted.

4.4.1 More on Projective Space

Let's make a few remarks on projective space. Last time we computed the cohomology ring of $H^\bullet(\mathbb{RP}^n, \mathbb{F}_2)$ fairly explicitly as $\mathbb{F}_2[\alpha]/(\alpha^{n+1})$, and by taking the direct limit with $n \to \infty$, we get $H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]$. We note that we can recover $H^\bullet(\mathbb{RP}^\infty, \mathbb{Z})$ in the following way. The map $\mathbb{Z} \to \mathbb{F}_2$ induces a map on cellular cohomology chain complexes as follows.

$$\begin{array}{ccccccccccc}
\cdots & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \longleftarrow & 0
\end{array}$$

By computing the cohomology, we see that the ring map $H^\bullet(\mathbb{RP}^\infty, \mathbb{Z}) \to H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2)$ is an isomorphism in positive degree, from which we can pull back to get

$$H^\bullet(\mathbb{RP}^\infty, \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha]}{(2\alpha)}$$

where α has degree 2. There is a similar computation for $\mathbb{CP}^\infty$.

4.4.2 More on Cup Products

Let's go ahead and prove that the cup product is graded commutative.

Proposition 4.33. Fix pairs (X, A) and (X, B) with classes $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$. Then

$$\alpha \cup \beta = (-1)^{k\ell}(\beta \cup \alpha).$$

Remark 4.34. Roughly speaking, one expects this anticommutativity from differential geometry and in particular the anticommutativity of the wedge product for differential forms.

Proof. We take $A = B = \varnothing$; the general case can be derived from this. The main point is that trying to compute $\beta \cup \alpha$ wants us to reverse $[v_0, \ldots, v_n]$ to $[v_n, \ldots, v_0]$. Thus, given a singular n-simplex $\sigma\colon [v_0, \ldots, v_n] \to X$, we will define $\overline{\sigma}\colon [v_0, \ldots, v_n] \to X$ by extending $\overline{\sigma}(v_i) := v_{n-i}$ linearly. One can then extend $\sigma \mapsto \overline{\sigma}$ linearly to all n-cycles. However, we kind of are introducing a sign here because $(v_0, \ldots, v_n) \mapsto (v_n, \ldots, v_0)$ is a permutation of sign $\varepsilon_n := (-1)^{n(n+1)/2}$ by explicitly computing the number of needed transpositions, so we will actually define $\rho\colon C_\bullet(X) \to C_\bullet(X)$ by extending

$$\rho(\sigma) := \varepsilon_n \overline{\sigma}$$

linearly to all n-cycles. By a short computation with boundaries, we see that ρ is actually a chain map, and we note that ρ squares to the identity. In fact, one can write down an explicit chain homotopy from ρ to the identity; alternatively, one can use the Eilenberg–Steenrod axioms in order to show that "switching the vertices" of all our n-simplices is producing a naturally isomorphic cohomology theory.

From here, we can now compute

$$\begin{aligned}(\rho^*\varphi \cup \rho^*\psi)(\sigma) &= \varphi(\varepsilon_k\sigma|_{[v_k,\ldots,v_0]})\psi(\varepsilon_\ell\sigma|_{[v_n,\ldots,v_k]}) \\ \rho^*(\varphi \cup \psi)(\sigma) &= \varepsilon_{k+\ell}\psi(\sigma|_{[v_n,\ldots,v_k]})\varphi(\sigma|_{[v_k,\ldots,v_0]}),\end{aligned}$$

where $n = k + \ell$. Passing to cohomology will make ρ^* be the identity as discussed above, so the above equalities imply $\varepsilon_k[\varphi] \cup \varepsilon_\ell[\psi] = \varepsilon_{k+\ell}([\psi] \cup [\varphi])$, which is the result upon counting our signs. ■

Remark 4.35. For a surface Σ, we note that the cup product induces an antisymmetric form

$$H^1(\Sigma;\mathbb{Q}) \otimes_{\mathbb{Q}} H^1(\Sigma;\mathbb{Q}) \to H^2(\Sigma;\mathbb{Q}) \cong \mathbb{Q},$$

which shows up in differential geometry.

4.4.3 The Künneth Formula

Given two graded rings R and S, we can form a graded ring $R \otimes S$ in the usual way. However, due to our graded anticommutativity, we will require that our multiplication introduce the sign

$$(r \otimes s)(r' \otimes s') = (-1)^{(\deg s)(\deg r')}(rr' \otimes ss')$$

to account for switching s and r'.

Example 4.36. Take the graded polynomial ring $\Lambda_R[\alpha_1, \ldots, \alpha_n]$ where the α_i have degree $2i + 1$. Note that $\alpha_i^2 = 0$ for each α_i. One sees that

$$H^\bullet(S^n;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_n]$$

by an explicit computation.

With this in mind, we define the cross product.

Definition 4.37 (cross product)**.** Fix spaces X and Y. Then we define the *cross product* $\times\colon H^\bullet(X;R) \otimes H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$ by extending

$$(\alpha \times \beta) := p_X(\alpha) \otimes p_Y(\beta)$$

R-linearly to the entire tensor product.

Remark 4.38. One can recover the cup product from the cross product by using the diagonal embedding $\Delta\colon X \to X \times X$. Then the composite

$$H^\bullet(X;R) \otimes H^\bullet(X;R) \xrightarrow{\times} H^\bullet(X \times X;R) \overset{\times}{\Delta}\!\!\to H^\bullet(X;R)$$

is equal to the cup product. Indeed, the main point is that Δ composed with either projection is simply the identity.

Remark 4.39. In fact one can directly define a cross product by defining a chain map

$$C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$$

by taking two singular simplices $\sigma_X\colon \Delta^k \to X$ and $\sigma_Y\colon \Delta^\ell \to Y$ by producing a map $\Delta^k \times \Delta^\ell \to X \times Y$, essentially by viewing everything as a cube.

The construction of the graded anticommutativity above assures that $\times$ is in fact a ring homomorphism. Indeed, we compute

$$\begin{aligned}(\alpha\times\beta)\cup(\alpha'\cup\beta') &= p_X(\alpha)\cup p_Y(\beta)\cup p_X(\alpha')\cup p_Y(\beta')\\ &= (-1)^{(\deg\alpha')(\deg\beta)}p_X(\alpha)\cup p_X(\alpha')\cup p_Y(\beta)\cup p_Y(\beta')\\ &= (-1)^{(\deg\alpha')(\deg\beta)}p_X(\alpha\cup\alpha')\cup p_Y(\beta\cup\beta')\\ &= (-1)^{(\deg\alpha')(\deg\beta)}(\alpha\cup\alpha')\times(\beta\cup\beta').\end{aligned}$$

In simple cases, the cross product map defines an isomorphism.

Theorem 4.40. Fix CW-complexes X and Y. If $H^\ell(Y;R)$ is a finitely generated free R-module for all ℓ, then

$$\times\colon H^\bullet(X;R)\otimes H^\bullet(Y;R)\to H^\bullet(X\times Y;R)$$

is an isomorphism.

Proof. We use the Eilenberg–Steenrod axioms. Define the cohomology theories

$$\begin{aligned}h^n(X,A) &:= \bigoplus_{i+j=n} H^i(X,A;R)\otimes_R H^j(Y;R)\\ k^n(X,A) &:= H^n(X\times Y, A\times Y;R).\end{aligned}$$

Note that there is a natural transformation $\mu\colon h^n\to k^n$ given by the cross product. Now, one can check that h^n and k^n are both cohomology theories, and μ_n is an isomorphism on the point, so one can see purely formally that μ_n will be an isomorphism on all CW pairs (X,A).

Let's give a few of the details here.

- We note that μ is natural in the topological spaces automatically, and it is also natural in the excision long exact sequence by an explicit computation of the boundary maps.
- Being an isomorphism on the point extends to all CW complexes approximately as follows: one gets contractible spaces immediately, and then the wedge sum axiom allows us to get the skeleton to any finite-dimensional CW-complex. Then cellular homology allows us to get an isomorphism for any finite-dimensional CW-complex. One then gets the general case by taking some kind of limit.
- The axioms for $h^\bullet$ and $k^\bullet$ are checked rather immediately from the axioms for $H^\bullet$. ■

Let's give a quick application to division rings.

Proposition 4.41. If D is a finite-dimensional division $\mathbb{R}$-algebra, then $\dim_{\mathbb{R}} K$ is a power of 2.

Proof. Say $D=\mathbb{R}^n$, and we want to show that n is a power of 2; take $n\geqslant 2$. Now, define $g\colon S^{n-1}\times S^{n-1}\to S^{n-1}$ by

$$g(x,y):=\frac{x\cdot y}{|x\cdot y|},$$

where the point is that $x\cdot y$ is always nonzero when x and y are nonzero because D is a division algebra. Now, having $(-x)y=-(xy)=x(-y)$ implies that $g(-x,y)=-g(x,y)=g(x,-y)$, so we descend to a map

$$\overline{g}\colon \mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1}\to\mathbb{RP}^{n-1}.$$

This then produces a ring homomorphism

$$H^\bullet\left(\mathbb{RP}^{n-1},\mathbb{F}_2\right)\to H^\bullet\left(\mathbb{RP}^{n-1},\mathbb{F}_2\right)\otimes H^\bullet\left(\mathbb{RP}^{n-1},\mathbb{F}_2\right).$$

Let the generators (in degree 1) of the above three rings be γ, α, and β, respectively. A topological computation reveals that γ goes to $\alpha+\beta$, but then having $\gamma^n=0$ will force $(\alpha+\beta)^n$ to vanish, upon which expanding by the binomial theorem will enforce n to be a power of 2. ■

4.5 November 30

Today we're going to discuss orientations.

4.5.1 Primer on Poincaré Duality

Poincaré duality is a relationship between the homology and cohomology of a manifold. Historically, what happened is that we realized that Betti numbers were symmetric, which were then realized via homology ad cohomology, from which duality was seen. For example, one expects to have $H_n(M;F) \cong H^n(M;F) \cong F$ for any field F (where M is a closed orientable n-manifold). From here, one also expects to have a perfect pairing

$$H^i(M;F) \times H^{n-i}(M;F) \to H^n(M;F) \cong F,$$

which is a nice statement of Poincaré duality. This is in some sense our end goal.

Instead of showing this directly, we will produce a non-degenerate map

$$H^i(X;R) \to \operatorname{Hom}_R(H_i(X), R),$$

which we will call the "cap product." In some sense, we are trying to take an i-cocycle and a j-cycle to produce an $(i-j)$-cocycle.

4.5.2 Manifolds

Before we begin any rigorous discussion of Poincaré duality, we must define manifolds and provide some discussion of their homology.

Definition 4.42 (manifold)**.** Fix a nonnegative integer n. Then an n-*manifold* is a second-countable Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$. (Here, locally homeomorphic to $\mathbb{R}^n$ means that any point has an open neighborhood isomorphic to $\mathbb{R}^n$.)

Let's put the local homeomorphic to good use.

Notation 4.43. Fix an n-manifold M. For a subset $A \subseteq M$, we define $H_i(M|A)$ to mean $HIi(M, M\backslash A)$.

Lemma 4.44. Fix an n-manifold M. For any $x \in M$, we have

$$H_i(M|x;R) \cong \begin{cases} R & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

Proof. Find an open neighborhood $U \subseteq M$ around x homeomorphic to $\mathbb{R}^n$. Then excision followed by the long exact sequence in homology assures us that

$$H_i(M|x;R) \cong H^i(U|x;R) \cong \widetilde{H}_{i-1}(\mathbb{R}^n\backslash\{0\};R) \cong \widetilde{H}^i(S^{n-1}\times\mathbb{R};R) \cong \widetilde{H}^i\left(S^{n-1};R\right),$$

from which the result follows. ■

Remark 4.45. The same argument will show that $H_i(M|A)$ is the same for any open ball $A \subseteq M$ isomorphic to an open ball in a neighborhood of M isomorphic to $\mathbb{R}^n$.

We would now like to add in compactness.

Definition 4.46 (closed)**.** An n-manifold M is *closed* if and only if it is compact.

Note that we are not talking about manifolds with boundary anywhere in our discussion.

4.5.3 Orientations of Manifolds

With Lemma 4.44 in mind, we take the following definition.

Definition 4.47 (orientation)**.** Fix an n-manifold M. A *local orientation* at $x \in M$ is a choice of generator $\mu_x \in H^n(M|x)$. To make these orientations cohere with each other, we define an *orientation of* M to be a choice of local orientations $x \mapsto \mu_x$ for each $x \in X$ which is locally constant in the following sense: any point in M has an open neighborhood $U \subseteq M$ homeomorphic to $\mathbb{R}^n$ and open neighborhood ball $B \subseteq U$ homeomorphic to a ball of finite radius with a choice of $\mu_B \in H_n(M|B)$ such that μ_y is the image of μ_B in $H_n(M|y)$ for each $y \in B$.

Lastly, M is called *orientable* if and only if an orientation on M exists.

We will write $\widetilde{M}$ for the collection of local orientations μ_x as $x \in M$ varies. Note that we have a 2-to-1 map $\widetilde{M} \to M$ because every $x \in M$ has two choices for generator $\mu_x \in H_n(M|x)$. We can also give $\widetilde{M}$ a topology to make this map into a covering space: on any $U \subseteq M$ isomorphic to M, there are exactly two ways to choose orientations on U, so the pre-image up in $\widetilde{M}$ may as well be two disjoint copies of U. Asserting that we have defined a local homeomorphism on these basic open subsets provides us with a topology on $\widetilde{M}$.

Remark 4.48. From here, one can see that a connected n-manifold M is orientable if and only if $\widetilde{M}$ has two connected components. If we did have an orientation, then $\widetilde{M}$ separates into the two choices of orientations; conversely, if $\widetilde{M}$ separates into two components, then each component yields an orientation.

Remark 4.49. There is a generalization of $\widetilde{M}$ up to M_R by choosing generators of $\mu_x \in H_n(M|x;R)$ for each $x \in M$, and we can again produce a covering space $M_R \to M$ via the projection $\mu_x \mapsto x$.

Example 4.50. The Möbius strip fails to be orientable: if we did have orientation, then we could go "around" the strip (keeping the same generator locally) to flip the given orientation, which is a contradiction.

Example 4.51. One can show that $\mathbb{RP}^n$ is orientable only in odd dimensions. For example, $\mathbb{RP}^1$ is basically a circle, which is orientable.

We have been taking $\mathbb{Z}$ coefficients everywhere in the previous discussion, but we might as well take R coefficients instead.

Definition 4.52 (R-orientation)**.** Fix an n-manifold M. Then an *R-orientation* on M is a choice of generators $\mu_x \in H_n(M|x;R)$ for each $x \in M$ such that the map $x \mapsto \mu_x$ is locally constant.

Remark 4.53. One can check that every manifold is $\mathbb{F}_2$-orientable. This essentially follows from the above discussion and a careful tracking through of the definitions.

Here is the main result on R-orientability.

Theorem 4.54. Fix a closed connected n-manifold.

(a) If M is R-orientable, then the map $H_n(M;R) \to H_n(M|x;R)$ is an isomorphism for all $x \in M$.

(b) If M fails to be R-orientable, then the map $H_n(M;R) \to H_n(M|x;R)$ is injective with image exactly $\{r \in R : 2r = 0\}$. In particular, we recall that $H_n(M|x;R) \cong R$ here.

(c) We have $H_i(M;R) = 0$ for $i > n$ (even if M is not closed).

Remark 4.55. It is true that M has a CW-structure with cells of dimension at most n, which would prove (c) easily. However, showing this is somewhat difficult; for example, one must get around the fact that not every manifold has a simplicial structure.

Remark 4.56. Parts (a) and (b) show that one can detect if M is orientable via $H_n(M;\mathbb{Z})$. However, $H_n(M;\mathbb{F}_2) = \mathbb{F}_2$ always.

To prove Theorem 4.54, we will instead prove the following more technical lemma, from which the theorem will follow quickly. Approximately, speaking, the lemma is a version of the statement where we allow non-compact manifolds.

Lemma 4.57. Fix an n-manifold M, and let $A \subseteq M$ be a compact subspace.

(a) Given a locally constant section $x \mapsto \alpha_x$ of the projection $M_R \to M$, there exists a unique class $\alpha_R \in H_n(M|A;R)$ whose image in $M_n(M|x;R)$ is simply α_x.

(b) $H_i(M|A;R) = 0$ for $i > n$.

Proof. We proceed in steps. For brevity, we abbreviate the ring R everywhere.

1. We remark that an induction via Mayer–Vietoris implies that if the statement is true for A and B and $A \cap B$, then we also get the statement for $A \cup B$. For example, this allows us to divide up the compact set A into pieces contained in open balls locally homeomorphic to $\mathbb{R}^n$, so we may assume that A is contained in such an open ball.

2. We show the result for $\mathbb{R}^n$ and $A = B$ where B is a compact ball. Here, we know that $H_n(\mathbb{R}^n|B) \to H^n(\mathbb{R}^n|x)$ is always an isomorphism for any $x \in B$, which produces uniqueness of the needed class in (a). For existence, at any point $y \in B$, choose some generator, but then there is an open neighborhood U of y so that we can lift μ_y to some $\mu_U \in H_n(M|U)$ via excision. Then for any two points $x, y \in M$, a path connecting them will enforce that the orientations cohere into a single class up in $H^n(M|B)$.

3. We show the result for $\mathbb{R}^n$ and A a general compact set. To show that the class exists, just use a very large simplex containing A and then reduce to the previous case. For uniqueness, take a difference and apply excision and the Mayer–Vietoris reduction cleverly in order to produce the result. ■

And now here is the proof of the theorem from the lemma.

Proof of Theorem 4.54. Take $A = M$; part (c) is immediate. Note that the set of sections $M_R \to M$ is an R-module; call this R-module $\Gamma_R(M)$. Then there is a homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the corresponding section $x \mapsto \alpha_x$; part (a) of the lemma tells us that this map is an isomorphism, from which parts (a) and (b) of the theorem follow quickly from an understanding of the covering map $M_R \to M$. ■

4.6 December 7

Today we will discuss Poincaré duality.

4.6.1 Statement of Poincaré Duality

Theorem 4.54 allows us to make the following definition.

Definition 4.58 (fundamental class)**.** Fix a closed R-orientable n-manifold. Then there is a class $[M] \in H_n(M;R)$, called the *fundamental R-class*, such that the image of $[M]$ under the maps $H_n(M;R) \to H_n(M|x;R)$ goes to a generator.

Remark 4.59. If the manifold M is a Δ-complex, then $[M]$ can simply be defined as the sum of the n-simplices: each point $x \in M$ will live in (roughly speaking) one of these n-simplices, so the image of $[M]$ will indeed go to a generator because the only n-simplex in $[M]$ which fails to vanish is the one containing x.

Remark 4.60. Further, suppose that M has a triangulation, making it piecewise linear. Then one can give M a dual cell structure, from which Poincaré duality can be seen. Namely, an i-cycle essentially assigns a number to each cell, but then this will simply define an $(n-i)$-cocycle via the dual cell structure.

The above remark can be seen as a concrete construction of the "cap product."

Definition 4.61 (cap product)**.** Fix a topological space X. We define the *cap product* $\frown\colon C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ for $k \geqslant \ell$ by extending the relation

$$(\sigma \frown \varphi)(\sigma) := \varphi(\sigma|_{[v_0,\ldots,v_\ell]})\sigma|_{[v_\ell,\ldots,v_k]}$$

bilinearly.

One can check that $\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \partial\varphi)$ by an explicit computation, so the cap product of a cycle and a cocycle will be a cycle. The main point is that $\frown$ descends to

$$H_k(X;R) \times H^\ell(X;R) \to H_{k-\ell}(X;R).$$

A direct computation shows that the following diagram commutes for any continuous map $f\colon X \to Y$.

$$\begin{array}{ccccc}
H_k(X;R) & \times & H^\ell(X;R) & \xrightarrow{\frown} & H_{k-\ell}(X;R) \\
{\scriptstyle H_k(f)}\downarrow & & \uparrow{\scriptstyle H^\ell(f)} & & \downarrow{\scriptstyle H_{k-\ell}(f)} \\
H_k(Y;R) & \times & H^\ell(Y;R) & \xrightarrow{\frown} & H_{k-\ell}(Y;R)
\end{array}$$

So our cap product is natural. We are now able to state Poincaré duality.

Theorem 4.62 (Poincaré duality)**.** Fix a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$. Then there is an isomorphism $D\colon H^k(M;R) \to H_{n-k}(M;R)$ given by $[M] \frown -$.

Remark 4.63. If R is a field, then we see that $H^n(M;R) = H_0(M;R) \cong R$ when M is connected. As such, roughly speaking, Poincaré duality says that we have a non-degenerate pairing

$$H^k(M;R) \times H^{n-k}(M;R) \to H^n(M;R) \cong R.$$

Theorem 4.62 is proven by going up to a stronger statement for non-compact manifolds; this will allow us to prove the statement by induction. This will require a new cohomology theory.

4.6.2 Cohomology with Supports

Here is our new cohomology theory.

Definition 4.64. Fix a topological space X. Then we define $C^i_c(X;G)$ to be the subgroup of $C^i(X;G)$ of cochains $\varphi\colon C_i(X) \to G$ such that there is a compact K with $\varphi|_{C_i(X\setminus K)} = 0$. In other words,

$$C^i_c(X;G) = \varinjlim_{K\subseteq X} C^i(X, X\setminus K; G).$$

Now, given an R-oriented n-manifold M, we note that we have a unique $[M_K] \in H_n(M|K;R)$ for each compact K, and so we can let $\varphi \in C^k_c(M;R)$ be a cochain vanishing on $C_k(M\setminus K;R)$. Then we see that $[M_K] \frown \varphi$ and thus gives a homomorphism

$$D_M\colon H^k_c(M;R) \to H_{n-k}(M;R)$$

by taking the colimit of the maps $H^k(M|K;R) \to H_{n-k}(M|K;R)$ over compacts K. (One has to check that the cup product coheres with this restriction of compacts defining the direct limit, but this is no issue.) This setting now generalizes our earlier Theorem 4.62 into the following theorem.

Theorem 4.65. Fix an R-orientable n-manifold M, the map

$$D_M\colon H^k_c(M;R) \to H_{n-k}(M;R)$$

given as above is an isomorphism.

We prove the above theorem by induction; note that it generalizes Theorem 4.62 by taking M to be compact, where the point is that M being compact forces cohomology with compact support to simply agree with regular cohomology.

Remark 4.66. There are various inductive approaches which "almost" work provided we had some extra structure. For example, if M is homeomorphic to a Δ-complex, then one can build the preceding theorem by gluing together a discussion in the compact case, proving the needed isomorphism. For example, this approach will work for $M = \mathbb{R}^n$ as well as any surface.

We now sketch Theorem 4.65. We will use the following technical result.

Lemma 4.67. Suppose that an orientable n-manifold M is the union of two open orientable n-manifolds U and V. Then the following diagram (with rows given by Mayer–Vietoris) commutes.

$$\begin{array}{ccccccc}
H^k_c(U\cap V) & \longrightarrow & H^k_c(U)\oplus H^k_c(V) & \longrightarrow & H^k_c(M) & \longrightarrow & H^{k+1}_c(U\cap V) \\
\downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U\oplus -D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} \\
H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U\cap V)
\end{array}$$

One can take coefficients in any ring.

Proof. One does a long and tedious computation. I cannot be bothered to write out the details today. Essentially, one checks the result by replacing H_c with an explicit compact $K \subseteq U$ and $L \subseteq V$ and then pass to the colimit to produce the result. ■

From here, one proves Theorem 4.65 by induction: with $M = U \cap V$, induction will allow us to assume that D_U, D_V, and $D_{U\cap V}$ are all isomorphisms, from which it follows that D_M is an isomorphism. It is not totally clear what we induct on or what our base case is, which is the remaining content of the proof.

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List of Definitions