THE ÉTALE FUNDAMENTAL GROUP

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1. Introduction

The goal of this paper is to prove the existence of the étale fundamental group and compute a few basic examples. We postpone any technical discussion for later, but approximately speaking, the étale fundamental group $\pi_1(X)$ takes a connected scheme X and produces the profinite completion of what one would expect is the usual topological fundamental group.

As such, $\pi_1(X)$ is able to keep track of some desirable topology. For example, we will be able to show that projective space over an algebraically closed field has vanishing π_1, and we will be able to show that the fundamental group of an elliptic curve (which is essentially a torus) is the profinite completion of $\mathbb{Z}^2$. It is also true, though we will not show it, that π_1 is a truly topological invariant, in that it is invariant under homeomorphism [SP, Proposition 0BQN]. However, the étale fundamental group is interesting beyond what it can recover from topology. For example, for a field k, one has

$$\pi_1(\operatorname{Spec} k) = \operatorname{Gal}(\overline{k}/k),$$

so we are also managing to capture arithmetic information.

Now motivated, we go into a little detail. The étale fundamental group comes from a more abstract theory of Galois categories. We could define Galois categories now, but we will wait until Definition 2.2 so that we can spend the time to provide a few examples as well, but roughly speaking it is a category $\mathcal{C}$ equipped with a special functor $F\colon \mathcal{C} \to \mathrm{FinSet}$. What is remarkable about this theory is that it manages to include not just the construction of the étale fundamental group but also the Galois theory of fields and the theory of finite covering spaces from algebraic topology (though we will not discuss algebraic topology in this paper).

With this in mind, there are two goals for this paper: understand Galois categories, and apply this understanding to scheme theory. As such, our first main result is about Galois categories.

Theorem 1.1. *Let $\mathcal{C}$ be a Galois category with fiber functor F; set $G := \operatorname{Aut} F$. Then $F\colon \mathcal{C} \to \mathrm{FinSet}(G)$ is an equivalence of categories.*

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This is remarkable because, as stated above, Galois categories are present in many contexts, so our abstract theory is able to show that they're all talking about finite G-sets for some explicitly describable profinite group G. Our second result establishes that the built theory applies to schemes.

Theorem 1.2. *Fix a connected scheme X and a geometric point $\overline{x}$ of X. Then the category $\mathrm{F\acute{E}t}(X)$ of finite étale covers of X equipped with the base-change functor $F\colon \mathrm{F\acute{E}t}(X) \to \mathrm{FinSet}$ by*

$$FY := Y_{\overline{x}}$$

forms a Galois category.

1.1. **Layout.** We spend all of section 2 building the theory of Galois categories, culminating in the proof of Theorem 1.1 in section 2.4. We then apply this theory to schemes in section 3. After picking up a few tools, we prove Theorem 1.2 in section 3.3. We then close the paper by computing a few basic examples in section 4.

2. Galois Categories

In this section, we define a Galois category and prove that they are equivalent to $\mathrm{FinSet}(G)$ for a profinite group G in Theorem 1.1.

2.1. **Basic Facts.** Following [SP, Definition 0BMY], we take the following definition of a Galois category.

Definition 2.1 (connected)**.** *Fix a category $\mathcal{C}$. An object $A \in \mathcal{C}$ is connected if and only if it is not initial and has no nontrivial proper subobjects. In other words, A is not an initial, and any monomorphism $B \hookrightarrow A$ is either an isomorphism or has B initial.*

Definition 2.2 (Galois category)**.** *A Galois category is a category $\mathcal{C}$ together with a functor $F\colon \mathcal{C} \to \mathrm{FinSet}$ satisfying the following conditions.*

- *$\mathcal{C}$ has finite limits and colimits.*
- *Every object in $\mathcal{C}$ is the finite coproduct of connected objects in $\mathcal{C}$.*
- *The functor F is exact; i.e., F preserves finite limits and colimits.*
- *The functor F reflects isomorphisms; i.e., for a morphism $f\colon A \to B$, if $Ff\colon FA \to FB$ is an isomorphism, then f is an isomorphism.*

Here, F is called the fiber functor.

Remark 2.3. *This definition is not the standard one; see for example [Cad13, Definition 2.1]. In particular, one often assumes that $\mathcal{C}$ has quotients by finite automorphisms groups instead of assuming that we have all finite colimits. We have chosen the above definition because the above definition is more memorable.*

Here are the chief examples. We will be pretty terse.

Example 2.4. *Fix a profinite group G. Then the category of finite G-sets $\mathrm{FinSet}(G)$ equipped with the forgetful functor $F\colon \mathrm{FinSet}(G) \to \mathrm{FinSet}$ is a Galois category. Let's quickly run the checks.*

- *$\mathrm{FinSet}(G)$ has finite limits and colimits by using the constructions in Set.*
- *Connected objects are transitive G-sets, which implies that any G-set is decomposable into connected objects. Indeed, if A is connected, then the orbit Ga of any element $a \in A$ has the embedding $Ga \hookrightarrow A$, from which $Ga = A$ follows. Conversely, if A is transitive, then any nontrivial subobject $B \hookrightarrow A$ has an element of A and therefore has all A because A is transitive.*
- *Lastly, F is exact and reflects isomorphisms because constructions are inherited from Set.*

Example 2.5. *Fix a field k. Let $\mathcal{C} := \mathrm{SAlg}(k)^{\mathrm{op}}$ denote the opposite category of finite separable k-algebras, and define F by the set of embeddings $FA := \mathrm{Hom}_k(A, k^{\mathrm{sep}})$. Let's quickly run the checks.*

- *The category of k-algebras has an initial object k, fiber coproducts given by $\otimes$, a terminal object 0, and fiber products, so $\mathcal{C}$ has finite limits and colimits.*
- *Note that any finite separable k-algebra is the product of separable field extensions of k, so it suffices to show that separable field extensions ℓ of k are connected objects. Indeed, given an epimorphism $\ell \twoheadrightarrow A$ onto a nonzero k-algebra A, write $A = \prod_{i=1}^n \ell_i$ where ℓ_i/k is finite separable. Then, for each i, we see $\ell \twoheadrightarrow \ell_i$, but also $\ell \hookrightarrow \ell_i$ because ℓ is a field, so $\ell = \ell_i$; lastly $n = 1$ for dimension reasons.*

  In fact, conversely, if A is a connected object, then write $A = \prod_{i=1}^n \ell_i$. The surjections $A \twoheadrightarrow \ell_i$ for each i imply that $n = 1$ and $A \cong \ell_1$ because A is connected.

- *One can compute directly that F is exact by tracking through fiber products and coproducts. Lastly, separability of our extensions implies F reflects isomorphisms.*

Example 2.4 is especially compelling to keep in mind in the following discussion. To set us up, here are some basic facts. The idea here is to turn desirable facts into facts about limits, colimits, and isomorphisms, and then use the required properties of F.

Lemma 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with finite limits and colimits, and let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor which reflects isomorphisms.*

(a) For $f\colon A \to B$ in $\mathcal{C}$, then f is monic or epic if and only if Ff is monic or epic, respectively.
(b) For $A \in \mathcal{C}$, then A is initial or final if and only if FA is monic or epic, respectively.

Proof. We show these individually.

(a) We show the monic case; the epic case follows dually. Now, f is monic if and only if $\Delta_f\colon A \to A \times_B A$ is an isomorphism. The hypothesis on F makes this equivalent to the diagonal map $F\Delta_f\colon FA \to FA \times_{FB} FA$ being an isomorphism, which is equivalent to the map Ff being monic.
(b) We show the initial case; the final case follows dually. If A is initial, then FA is initial because F is exact. Conversely, let I be an initial object. Then $A \in \mathcal{C}$ is initial if and only if $I \cong A$, which by hypothesis on F is equivalent to $FI \cong FA$ being an isomorphism. But FI is initial as just discussed, so $FI \cong FA$ is equivalent to FA being initial. ■

Lemma 2.7. *Let $(\mathcal{C}, F)$ be a Galois category. Then F is faithful.*

Proof. Fix morphisms $f, g\colon X \to Y$ such that $Ff = Fg$. Because $\mathcal{C}$ has limits, we may set $E \coloneqq \operatorname{eq}(f, g)$. Because F is exact, we see $FE = \operatorname{eq}(Ff, Fg)$, but $Ff = Fg$, so $FE = FX$, so $E = X$ because F reflects isomorphisms, so $f = g$ follows. ■

Next, we take a moment to understand how objects decompose into connected objects.

Lemma 2.8. *Let $(\mathcal{C}, F)$ be a Galois category.*

(a) Suppose X is not initial and Y is connected. Then any morphism $f\colon X \to Y$ is epic.
(b) Fix $f, g\colon X \to Y$ such that X is connected. If $Ff(x_0) = Fg(x_0)$ for some $x_0 \in FX$, then $f = g$.
(c) Fix $f\colon X \to Y$ where X is connected and $Y = \bigsqcup_{i=1}^n Y_i$. Then there exists a unique i such that f factors through Y_i.
(d) Decomposition into connected objects is unique up to permutation and isomorphism.

Intuitively, (a) tells us that nontrivial mappings to connected objects are surjective; (b) tells us that a single element "drags" along all elements in a morphism from a connected object; and, (c) tells us that mapping out of a connected object should only go to a connected object.

Proof. We show these individually.

(a) Let E be the equalizer of the two inclusions $i_1, i_2\colon Y \to Y \sqcup_X Y$. It suffices to show that $E \cong Y$: indeed, this implies that $i_1 = i_2$, meaning $FY \sqcup_{FX} FY$ must equal the image of $FY \to FY \sqcup_{FX} FY$ by checking on elements, so $Y \sqcup_X Y \cong Y$, so $X \to Y$ is epic.

   Now, $E \to Y$ is monic, so because Y is connected, we see that E is either initial or $E \cong Y$. But E is not initial by Lemma 2.6: note $FE \neq \varnothing$ because we have a map $X \to E$ and so a map $FX \to FE$.
(b) Set $E \coloneqq \operatorname{eq}(f, g)$. We want $E \cong X$. Note X is connected, so because $E \hookrightarrow X$, it suffices to show E is not initial. By Lemma 2.6, it suffices to show $FE \neq \varnothing$, but $x_0 \in \operatorname{eq}(Ff, Fg) = FE$ by hypothesis.
(c) For each i, set $E_i \coloneqq X \times_Y Y_i$. Because $Y_i \hookrightarrow Y$ is monic, $E_i \hookrightarrow X$ is monic; because X is connected, we see that E_i is initial (equivalently, $FE_i = \varnothing$ by Lemma 2.6) or $E_i \cong X$. Passing through F,

$$FE_i = FX \times_{FY} FY_i = \{(x, y) \in FX \times FY_i : Ff(x) = y\}.$$

   Now, fixing any $x_0 \in FX$, find i_0 such that $Ff(x_0) \in FY_{i_0}$, so $FE_i \neq \varnothing$ and $E_i \cong X$. But then f is

$$X \cong E_{i_0} \to Y_{i_0} \to Y.$$

   Lastly, to see that i_0 is unique, note that f factoring through Y_i implies that FE_i is nonempty by the above argument. But only FE_{i_0} is nonempty because $\operatorname{im} Ff \subseteq FY_{i_0}$.

(d) Suppose we have an isomorphism $f\colon \bigsqcup_{i=1}^m X_i \cong \bigsqcup_{j=1}^n Y_j$. Each $f_i\colon X_i \to \bigsqcup_{j=1}^n Y_j$ factors through some $Y_{\sigma i}$ as a surjection $f_i\colon X_i \to Y_{\sigma i}$ by (a) and (c). We want to show that $n = m$, that σ is a permutation, and that each f_i is an isomorphism. Well, Ff is an isomorphism, so

$$\#FY = \#\operatorname{im} Ff \overset{(1)}{\leq} \sum_{j\in\operatorname{im}\sigma}^{m} \#FY_j \overset{(2)}{\leq} \#FX.$$

Because $\#FX = \#FY$, equalities follow everywhere. But equality in (1) only holds if each σ is bijective, and equality in (2) only holds if each Ff_i is injective and thus bijective. ■

2.2. **Galois Objects.** Throughout, fix a Galois category $(\mathcal{C}, F)$. As in Galois theory, we look for objects with a maximal number of automorphisms.

Remark 2.9. *Suppose X is connected, and fix $x_0 \in X$. We note that two automorphisms $f, g\colon X \to X$ are equal as soon as they are equal on $x_0 \in FX$ by Lemma 2.8, so*

$$\#\operatorname{Aut} X = \#\{Ff(x_0) : f \in \operatorname{Aut} X\} \leq \#FX.$$

Definition 2.10 (Galois)**.** *Fix a category $\mathcal{C}$. An object $X \in \mathcal{C}$ is Galois if and only if X is connected and $\#\operatorname{Aut} X = \#FX$.*

By Remark 2.9, we see that a connected object X is Galois if and only if $\{Ff(x_0) : f \in \operatorname{Aut} X\} = FX$ for each $x_0 \in FX$, or equivalently, the action of $\operatorname{Aut} X$ on FX to be transitive. Galois objects will be helpful because it allows us to build automorphisms of X based on FX. Anyway, here are our examples.

Example 2.11. *Let G be a profinite group. As discussed in Example 2.4, connected objects in* $\operatorname{FinSet}(G)$ *are transitive G-sets, which up to isomorphism look like G/H for some open subgroup H. Note that an automorphism $\sigma\colon G/H \to G/H$ must satisfy*

$$\sigma(gH) = g\sigma(H)$$

for any $gH \in G/H$, so it is enough to specify $\sigma(H) = g_0H$. However, we see $\sigma(gH) \coloneqq gg_0H$ is well-defined if and only if $g_0Hg_0^{-1} = H$. Thus, $\operatorname{Aut}_G G/H$ acts transitively on G/H if and only if $g_0Hg_0^{-1} = H$ for all $g_0 \in G$, so the Galois objects look like G/H where H is an open normal subgroup of G.

Example 2.12. *Fix a field k, and let $\mathcal{C} \coloneqq \operatorname{SAlg}(k)^{\mathrm{op}}$ as in Example 2.5. As discussed, connected objects are finite separable field extensions ℓ/k, so ℓ is a Galois object if and only if ℓ/k is Galois: both are equivalent to*

$$\#\operatorname{Hom}_k(\ell, k^{\mathrm{sep}}) = \operatorname{Aut}(\ell/k).$$

A central fact about Galois field extensions is that one can always embed a separable extension into a Galois one. Motivated by this (and Example 2.12), we show the following result.

Proposition 2.13. *Fix a Galois category $(\mathcal{C}, F)$. For any connected object X, there is a Galois object Y equipped with an epimorphism $Y \to X$.*

Proof. By Lemma 2.8, it suffices to exhibit any morphism $Y \to X$. For brevity, set $n \coloneqq \#FX$.

(1) We construct Y. We want $\operatorname{Aut} Y$ to be large, so we will use the S_n-action on X^n for help later.
List the n elements of FX as $\{s_1, \ldots, s_n\}$, where $n > 0$ because FX is nonempty by Lemma 2.6. To make Y interact with the S_n-action on X^n, we choose Y among the connected components of X^n so that $(s_1, \ldots, s_n) \in FY$. We have a map $Y \to X^n \to X$, so it remains to show that Y is Galois.

(2) We claim that any $(t_1, \ldots, t_n) \in FY$ has all the $t_\bullet$ distinct. Indeed, suppose that $t_i = t_j$ for some $i < j$; we claim $t'_i = t'_j$ for any $(t_1, \ldots, t_n) \in FY$, which will finish. Now, consider the diagonal

$$\Delta_{ij}\colon X^{n-1} \to X^n$$

defined by $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n)$. In particular, letting Y' denote the connected component in X^{n-1} such that $(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \in FY$, we have an epimorphism $Y' \to Y$ by Lemma 2.8. But then $FY' \twoheadrightarrow FY$, so $FY \subseteq F\Delta_{ij}$, as claimed.

(3) We show that the action of $\operatorname{Aut} Y$ is transitive on FY. For each $g \in S_n$, the automorphism $g\colon X^n \to X^n$ will send Y to a unique connected component by Lemma 2.8. Define G to be the subgroup of S_n such that each $g \in G$ sends Y gets to Y; note G embeds into $\operatorname{Aut} Y$.
In fact, by the argument in Lemma 2.8, we see that $g \in S_n$ has $g \in G$ if and only if

$$g(s_1, \ldots, s_n) \in FY.$$

This is enough. Indeed, for any $(t_1, \ldots, t_n) \in FY$, the $t_\bullet$ are distinct, so find $g \in S_n$ such that $g(s_1, \ldots, s_n) = (t_1, \ldots, t_n) \in FY$. Then $g \in G$, and $g\colon (s_1, \ldots, s_n) \mapsto (t_1, \ldots, t_n)$, as needed. ∎

2.3. **The Profinite Group.** Throughout, fix a Galois category $(\mathcal{C}, F)$. In this section we motivate and construct the profinite group in Theorem 1.1. The difficulty is recovering the group G from the category $(\mathcal{C}, F)$. As motivation, we have the following remark.

Remark 2.14. *Fix a profinite group G, and let $F\colon \mathrm{FinSet}(G) \to \mathrm{FinSet}$ denote the forgetful functor. We claim $G \cong \operatorname{Aut} F$ by sending $g \in G$ to the automorphism $\eta(g)\colon F \Rightarrow F$ by left multiplication by g. Checking naturality and that η is a homomorphism are straightforward.*

- *Injective: if $\eta(g)\colon F \Rightarrow F$ is id_F, then for any open subgroup $H \subseteq G$, g fixes G/H, so $g \in H$ always.*
- *Surjective: fix an $\eta\colon F \Rightarrow F$. For each open subgroup $H \subseteq G$ find $g_H \in G$ such that $\eta_{G/H}(H) = g_H G$. Naturality of η implies that $\{g_H\}_{H \subseteq G}$ defines an element of $g \in G$, and we can check $\eta = \eta(g)$.*

As such, we hope to recover the desired group G from $\operatorname{Aut} F$. We remark that there is a natural injection

$$\operatorname{Aut} F \to \prod_{X \in \mathcal{C}} \operatorname{Aut} FX \tag{2.1}$$

sending $\eta \in \operatorname{Aut} F$ to $\eta_X\colon FX \to FX$. This makes $\operatorname{Aut} F$ into a profinite group acting on the FX.

Lemma 2.15. *Fix a Galois category $(\mathcal{C}, F)$. Then* (2.1) *makes $\operatorname{Aut} F$ into a closed subgroup of the product, where each finite set $\operatorname{Aut} FX$ has been given the discrete topology. Thus, $\operatorname{Aut} F$ is a profinite group.*

Proof. The last sentence does follow from the previous one: the infinite product of compact, Hausdorff, totally disconnected spaces (for example, discrete ones) retains these properties. Thus, taking the closed subset $\operatorname{Aut} F$ will continue to enjoy these properties.

Intuitively, the naturality condition on automorphisms of F is a list of equations we require an object in $\prod_{X \in \mathcal{C}} \operatorname{Aut} FX$ to satisfy, and subsets cut out by equations are closed. To rigorize, we show the complement of $\operatorname{Aut} F$ is open: if $(\eta_X)_{X \in \mathcal{C}} \in \prod_{X \in \mathcal{C}} \operatorname{Aut} FX$ is not in the image of $\operatorname{Aut} F$, then it fails naturality, so there is a morphism $f\colon X \to Y$ and element $x \in FX$ such that $Ff(\eta_X(x)) \neq \eta_Y(Ff(x))$. Thus, define

$$U_X \coloneqq \{\varphi \in \operatorname{Aut} FX : \varphi(x) = \eta_X(x)\} \qquad \text{and} \qquad U_Y \coloneqq \{\varphi \in \operatorname{Aut} FY : \varphi(Ff(x)) = \eta_Y(Ff(x))\}.$$

Then setting $U_Z \coloneqq \operatorname{Aut} FZ$ for each object $Z \notin \{X, Y\}$, we see that $U \coloneqq \prod_{Z \in \mathcal{C}} U_Z \subseteq \prod_{X \in \mathcal{C}} \operatorname{Aut} FX$ is open, contains η, and is disjoint from the image of $\operatorname{Aut} F$ because any $\eta' \in U$ has $Ff(\eta'_X(x)) \neq \eta'_Y(Ff(x))$. ∎

Remark 2.16. *Another reason that $G \coloneqq \operatorname{Aut} F$ is a good candidate is that $F\colon \mathcal{C} \to \mathrm{FinSet}$ naturally upgrades to a functor $F\colon \mathcal{C} \to \mathrm{FinSet}(G)$. For example, each $X \in \mathcal{C}$ has FX a G-set via the map $G \to \operatorname{Aut} FX$ in* (2.1). *To be functorial, note a morphism $f\colon X \to Y$ makes Ff G-linear: for any $x \in FX$ and $\sigma \in G$, note*

$$Ff(F\sigma_X(x)) = F\sigma_Y(Ff(x)).$$

2.4. **The Main Theorem.** One difficulty in Theorem 1.1 is that the example $\mathrm{SAlg}(k)^{\mathrm{op}}$ forces us to expect G to encode all automorphisms of $\operatorname{Aut} X$ for each connected object X. In other words, $\operatorname{Aut} F$ should have lots of action on connected objects. Let's show this.

Proposition 2.17. *Fix a Galois category $(\mathcal{C}, F)$ and Galois object X. Then $\operatorname{Aut} F$ acts transitively on FX.*

Proof. The primary difficulty here is to describe F in a way more internal to $\mathcal{C}$. We proceed in steps.

(1) We set some notation. Let Λ index the collection of isomorphism classes of Galois objects, and let X_α be a representative of $\alpha \in \Lambda$; to move morphisms around, fix some $x_\alpha \in FX_\alpha$.

We now give Λ a partial order by $\alpha \geq \beta$ if and only if there is a map $X_\alpha \to X_\beta$. Anytime we have a morphism $X_\alpha \to X_\beta$, the transitive action of $\operatorname{Aut} X_\beta$ on FX_β grant us $f_{\beta\alpha}\colon X_\alpha \to X_\beta$ such that

$$Ff_{\beta\alpha}(x_\alpha) = x_\beta.$$

Lemma 2.8 implies $f_{\beta\alpha}\colon X_\beta \to X_\alpha$ is unique; notably, this implies $\alpha \geq \beta \geq \gamma$ has $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}$.

Lastly, Λ is a directed set: for any $\alpha, \beta \in \Lambda$, use Proposition 2.13 to find some X_γ mapping to any connected component of $X_\alpha \times X_\beta$, so we will have maps $X_\gamma \to X_\alpha$ and $X_\gamma \to X_\beta$.

(2) Acknowledging the difficulty, we show that F is "pro-representable." Define $F'\colon \mathcal{C} \to \mathrm{FinSet}$ by

$$F' \coloneqq \operatorname*{colim}_{\alpha\in\Lambda} \mathrm{Mor}_{\mathcal{C}}(X_\alpha, -).$$

Intuitively, we want to move the colimit inside Mor to say that F' is represented by some limit in $\mathcal{C}$, but $\mathcal{C}$ might not have this limit. Anyway, we claim that $F' \cong F$ by $\eta\colon F' \Rightarrow F$ defined by $\eta_X(f_\alpha) = Ff_\alpha(x_\alpha)$. We omit the check that η is well-defined and natural because these are purely formal. For injectivity, note because Λ is directed, it suffices to show $\eta_X(f_\alpha) = \eta_X(g_\alpha)$ implies $f_\alpha = g_\alpha$ for any maps $f_\alpha, g_\alpha\colon X_\alpha \to X$ in $F'X$, but this comes from Lemma 2.8.

Lastly, for surjectivity, we show η_X is surjective for any $X \in \mathcal{C}$. Well, fix $x \in FX$. If $X = X_\alpha$ is Galois (for some $\alpha \in \Lambda$), note $\operatorname{Aut} X_\alpha$ acts transitively on FX_α, so we may find some $f_\alpha\colon X_\alpha \to X_\alpha$ such that $\eta_{X_\alpha}(f_\alpha) = Ff(x_\alpha) = x$. In the general case, use Proposition 2.13 to find some Galois X' surjecting onto the connected component $Z \hookrightarrow X$ with $x \in FZ$, so the naturality of η implies it is enough to show that $\eta_{X'}$ is surjective and thus hits a point in the fiber of x in FZ.

(3) As in the proof of Proposition 2.13, we want a subgroup of $\operatorname{Aut} F$ to witness our transitivity; we now build this subgroup. The previous step more or less us tells us that it suffices to think about the automorphism groups $A_\alpha \coloneqq \operatorname{Aut} X_\alpha$ for $\alpha \in \Lambda$. We will take a limit of these A_α.

To define this limit, we want surjections $A_\alpha \to A_\beta$ commuting with the actions on X_α and X_β. In other words, whenever $\alpha \geq \beta$, we claim that there is a unique map $t_{\beta\alpha}\colon A_\alpha \to A_\beta$ such that

$$f_{\beta\alpha} \circ \sigma_\alpha = t_{\beta\alpha}(\sigma_\alpha) \circ f_{\beta\alpha} \tag{2.2}$$

for each $\sigma_\alpha \in A_\alpha$. Because X_α is connected, plugging in x_α implies the map $t_{\beta\alpha}(\sigma_\alpha)$ is unique if it exists by Lemma 2.8. In fact, because X_α is connected, it suffices to check that $Ft_{\beta\alpha}(\sigma_\alpha)(x_\beta) = Ff_{\beta\alpha}(F\sigma_\alpha(x_\alpha))$. But now certainly $t_{\beta\alpha}(\sigma_\alpha)$ exists because X_β is Galois.

Uniqueness of the $t_{\beta\alpha}$ implies that $\alpha \geq \beta \geq \gamma$ yields $t_{\gamma\beta} \circ t_{\beta\alpha} = t_{\gamma\alpha}$. Further, $t_{\beta\alpha}$ is surjective. Indeed, for any automorphism $\sigma_\beta \in A_\beta$, note $f_{\beta\alpha}\colon X_\alpha \to X_\beta$ is surjective, so pick $x'_\alpha \in f_{\beta\alpha}^{-1}(\{\sigma_\beta(x_\beta)\})$. Then find σ_α such that $F\sigma_\alpha(x_\alpha) = x'_\alpha$, so

$$Ff_{\beta\alpha}(F\sigma_\alpha(x_\alpha)) = Ff_{\beta\alpha}(x'_\alpha) = F\sigma_\beta(Ff_{\beta\alpha}(x_\alpha)).$$

It follows $f_{\beta\alpha} \circ \sigma_\alpha = \sigma_\beta \circ f_{\beta\alpha}$, so $t_{\beta\alpha}(\sigma_\alpha) = \sigma_\beta$ by uniqueness of $t_{\beta\alpha}$.

In total, we have produced an inverse system $\{A_\alpha\}_{\alpha\in\Lambda}$ with surjective transition maps, so the limit $A \coloneqq \lim_{\alpha\in\Lambda} A_\alpha$ is a profinite group with surjective quotients $A \to A_\alpha$.

(4) We now map $A^{\mathrm{op}} \to \operatorname{Aut} F' \cong \operatorname{Aut} F$ to finish. Indeed, send $\sigma \in A$ to $\sigma\colon F \Rightarrow F$ by

$$\sigma(f_\beta) \coloneqq f_\beta \circ \sigma_\beta.$$

Uniqueness of the transition maps in (2.2) shows $\sigma\colon F' \Rightarrow F'$ is well-defined. Checking naturality and that $A^{\mathrm{op}} \to \operatorname{Aut} F'$ is a homomorphism are purely formal, so we omit those checks.

Wrapping up, for some $\alpha \in \Lambda$, the action of $\sigma \in A$ on FX_α can either come from $\sigma_\alpha \in \operatorname{Aut} X_\alpha$ or via $A^{\mathrm{op}} \to \operatorname{Aut} F' \to \operatorname{Aut} F \to \operatorname{Aut} FX_\alpha$, but one can verify these are the same. As such, the action of $A \twoheadrightarrow A_\alpha$ on X_α is transitive because X_α is Galois, so $\operatorname{Aut} F$ also acts transitively on X_α. ■

Remark 2.18. *Fix a profinite group G. The above proof shows any two fiber functors $F_1, F_2\colon \mathrm{FinSet}(G) \to$ FinSet are naturally isomorphic; this tells us "Galois" is a property of the category, not the fiber functor. Indeed, using the notation above, Λ is the set of quotients G/H where $H \subseteq G$ is an open normal subgroup. By choosing x_α judiciously, we can make the $f_{\alpha\beta}$ into the natural projections $G/H \twoheadrightarrow G/H'$ whenever $H \subseteq H'$. But then* (2) *above tells us that F_1 and F_2 are both isomorphic to*

$$\operatorname*{colim}_{\alpha\in\Lambda} \mathrm{Mor}_{\mathcal{C}}(X_\alpha, -).$$

Remark 2.19. *In fact, the map $A^{\mathrm{op}} \to \operatorname{Aut} F$ is an isomorphism, where the inverse map sends some $g \in \operatorname{Aut} F$ to the component isomorphisms $g_{X_\alpha} \in A_\alpha$. The fact that the action of A^{op} on FX_α arises from the map $A^{\mathrm{op}} \to \operatorname{Aut} F \to \operatorname{Aut} FX_\alpha$ shows that the map $A^{\mathrm{op}} \to \operatorname{Aut} F \to A^{\mathrm{op}}$ is the identity. Conversely, the fact that we can determine an element of* $\operatorname{Aut} F$ *on the connected objects and hence on the Galois objects (by Proposition 2.13) assures us that* $\operatorname{Aut} F \to A^{\mathrm{op}} \to \operatorname{Aut} F$ *is also the identity.*

Corollary 2.20. *Fix a Galois category $(\mathcal{C}, F)$. Then $G \coloneqq \operatorname{Aut} F$ acts transitively on FX for any connected object X. In other words, if X is connected, then $FX \in \mathrm{FinSet}(G)$ is connected (recall Remark 2.16).*

Proof. The last sentence follows from the previous one because connected objects in $\mathrm{FinSet}(G)$ are exactly the transitive G-sets by Example 2.4. Now, by Proposition 2.13, find a Galois object Y with epimorphism $f\colon Y \to X$. The transitivity of G acting on FY from Proposition 2.17 will translate into transitivity on FX. Explicitly, fix elements $x, x' \in FX$, and find lifts $y, y' \in FY$ of them. Transitivity of the G-action of FY promises $\sigma \in G$ such that $F\sigma_Y(y) = y'$, so we can calculate $F\sigma_X(x) = x'$. ■

Corollary 2.21. *Fix a Galois category $(\mathcal{C}, F)$. Set $G \coloneqq \operatorname{Aut} F$. If $X \in \mathcal{C}$ is Galois, then $FX \in \mathrm{FinSet}(G)$ is Galois (recall Remark 2.16).*

Proof. Let $X \in \mathcal{C}$ be Galois. For example, X is connected, so FX is connected by Corollary 2.20, so we may write $FX = G/H$ for some open subgroup $H \subseteq G$. Now, $\operatorname{Aut} X$ acts transitively on G/H via $F\colon \operatorname{Aut} X \to \operatorname{Aut}_G(G/H)$, so we see that $\operatorname{Aut}_G(G/H)$ is acting transitively on G/H, as needed. ■

We are now ready to prove our main theorem.

Theorem 1.1. *Let $\mathcal{C}$ be a Galois category with fiber functor F; set $G \coloneqq \operatorname{Aut} F$. Then $F\colon \mathcal{C} \to \mathrm{FinSet}(G)$ is an equivalence of categories.*

Proof. We showed that this F makes sense in Remark 2.16. The difficulties in this proof are that F is full and essentially surjective. Namely, note F is faithful by Lemma 2.7.

We now show that F is full. Fix some G-linear map $s\colon FX \to FY$, and we will show that $s = Ff$ for some $f\colon X \to Y$. The main point will be to use the fact that F reflects isomorphisms. To set up, set

$$\operatorname{Graph}(s) \coloneqq \{(x, y) \in FX \times FY : y = s(x)\}.$$

Note that $\operatorname{Graph}(s)$ is a G-set because s is G-linear. Decomposing $X \times Y$ into connected objects $\bigsqcup_{i=1}^n Z_i$,

$$FX \times FY = \bigsqcup_{i=1}^n FZ_i,$$

so each FZ_i is connected by Corollary 2.20. Matching the decomposition of $\operatorname{Graph}(s)$ up with various connected components in the above decomposition, we produce some subobject $Z \subseteq X \times Y$ such that $FZ = \operatorname{Graph}(s)$. Now, the projection $p_X\colon \operatorname{Graph}(s) \to FX$ is an isomorphism, so it arises from an isomorphism $p\colon Z \to X$! In total, $s\colon FX \to FY$ is the composite

$$FX \overset{Fp}{\leftarrow} FZ = \operatorname{Graph}(s) \overset{Fp_Y}{\to} FY,$$

and each of these morphisms arise from morphisms in $\mathcal{C}$. Thus, s is the image of a morphism $X \to Y$.

Lastly, we show F is essentially surjective. Because $\mathrm{FinSet}(G)$ is already a Galois category by Example 2.4, it suffices to show that any connected object G/H (where $H \subseteq G$ is an open subgroup) is isomorphic to FX for some $X \in \mathcal{C}$. The hard part is to build some Galois X' with a map $FX' \to G/H$. Using the topology on G, we know that there is a basic open set around $\{e\}$ in H, so we can find objects $\{X_1, \ldots, X_n\} \subseteq \mathcal{C}$ (with $n > 0$) such that

$$\{g \in G : g_{X_i} = e_{X_i} = \mathrm{id}_{FX_i} \text{ for } i = 1, 2, \ldots, n\} \subseteq H.$$

Note that we may assume the X_i are connected: indeed, if $g_X = \mathrm{id}_{FX}$ and $g_Y = \mathrm{id}_{FY}$, then $g_{X \sqcup Y} = \mathrm{id}_{X \sqcup Y}$, so we may decompose each X_i into connected components. We now define X' via Proposition 2.13 to be a Galois object equipped with an epimorphism onto some connected component of $\prod_{i=1}^n X_i$.

Connectivity of the X_i implies that the induced maps $X' \to X_i$ and hence $FX' \to FX_i$ are epic. Note FX' is Galois by Corollary 2.21, so we may write $FX' = G/H'$ for some open normal subgroup $H' \subseteq G$. Notably, any $\sigma \in H'$ fixes FY and so fixes each FX_i, so $\sigma \in H$ by the construction of the X_i. Thus, we have a surjection $FX' = G/H' \twoheadrightarrow G/H$. To finish, let X be the quotient of X' by the subgroup of

$$(H/H')^{\mathrm{op}} \subseteq (G/H')^{\mathrm{op}} = \operatorname{Aut}_G(G/H') = \operatorname{Aut}_G FX' = \operatorname{Aut} X'.$$

(We are taking opposite groups because an element $g \in G$ acts on G/H' by $\sigma_g\colon g_0H' \mapsto g_0gH'$ as discussed in Example 2.11.) Because F is exact, it follows that $FX = (FX')/(H/H')^{\mathrm{op}} = (G/H')/(H/H')^{\mathrm{op}} = G/H$. ■

Remark 2.22. *Combining Theorem 1.1 with Remark 2.18, we see that the fiber functor of any Galois category is roughly unique. In particular, $\operatorname{Aut} F$ depends only on the category $\mathcal{C}$ itself.*

3. Finite Étale Covers

In this section, we prove Theorem 1.2, which tells us that the category of finite étale covers of a connected scheme forms a Galois category. This allows us to define the étale fundamental group in Definition 3.15.

3.1. Étale and Totally Split Morphisms.

In this subsection, we review properties of étale morphisms and friends.

Definition 3.1 (unramified)**.** *A scheme morphism $f\colon X \to S$ locally of finite presentation is unramified if and only if one of the following equivalent conditions are satisfied.*

- $\Omega_{X/S} = 0$.
- *The diagonal $\Delta_f\colon X \to X \times_S X$ is an open embedding.*
- *For any $x \in X$, we have $\mathfrak{m}_{f(x)}\mathcal{O}_{f(x)} = \mathfrak{m}_x$ and the residue field extension $k(x)/k(f(x))$ is finite separable.*

We showed that these properties are equivalent in class; a proof is recorded in [SP, Lemma 02GF].

Definition 3.2 (flat)**.** *A scheme morphism $f\colon X \to S$ is flat at $x \in X$ if and only if the ring map $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ is flat. Because exactness is checked stalk-locally, it is equivalent to require the following: for any affine open subschemes* $\operatorname{Spec} A \subseteq S$ *and* $\operatorname{Spec} B \subseteq f^{-1}(\operatorname{Spec} A)$*, the ring extension $f^\sharp\colon A \to B$ is flat.*

Definition 3.3 (étale)**.** *A scheme morphism $f\colon X \to S$ locally of finite presentation is étale if and only if it is flat and unramified.*

The moral of this subsection is that finite étale maps are analogous to (finite) covering spaces in algebraic topology. To justify this, we begin by making an analogous definition for a trivial cover and then show that our finite étale maps are "étale-locally" trivial. Here are our trivial covers.

Definition 3.4 (totally split)**.** *A scheme morphism $f\colon X \to S$ is totally split if and only if we can decompose $S = \bigsqcup_{\alpha\in\Lambda} S_\alpha$ into open subschemes such that $f^{-1}(S_\alpha) \cong S_\alpha \sqcup \cdots \sqcup S_\alpha$ (for some finite number of S_αs) and $f^{-1}(S_\alpha) \to S_\alpha$ is the natural projection.*

Proposition 3.5. *Fix a finite étale map $f\colon X \to S$. Then there is a finite faithfully flat map $S' \to S$ so that the base-change $f'\colon X' \to S'$ is totally split, where $X' \coloneqq X \times_S S'$.*

Proof. To begin, decompose S into connected components $\bigsqcup_{\alpha\in\Lambda} S_\alpha$. If we can show $f^{-1}(S_\alpha) \to S_\alpha$ is totally split for each $\alpha \in \Lambda$, then we can zipper these pieces together. Thus, we may assume that S is connected.

Because f is finite and flat, it is locally free, so because S is connected, the degree of f at each point in S is constant. Thus, we may induct on $n \coloneqq \deg f$. If $n = 0$, then there is nothing to say because this requires $X = \varnothing$. Otherwise, take $n > 0$, and X is nonempty. Because f is separated and unramified, the diagonal $\Delta_f\colon X \to X \times_S X$ is both an open and closed embedding, so we can write

$$X \times_S X = X \sqcup Y$$

for some open and closed subscheme $Y \subseteq X \times_S X$. Degree of a locally free morphism is preserved by base-change, so the projection $p_2\colon X \times_S X \to X$ has degree n. However, one element in any fiber of p_2 will come from the image of $X \subseteq X \times_S X$, so the other $n-1$ elements must come from Y, meaning that the composite $Y \subseteq X \times_S X \to X$ has degree $n-1$. Thus, the inductive hypothesis promises a finite faithfully flat X-scheme S' such that the base-change $Y \times_X S' \to S'$ is totally split.

We now claim that S' is the desired S-scheme. To visualize, we build the following diagram.

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d, "f'"] & X \times_S X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

To see that $X \times_S S' \to S'$ is totally split, note fiber products commute with disjoint unions, so

$$X \times_S S' = (X \times_S X) \times_X S' = (X \sqcup Y) \times_X S' = (X \times_X S') \sqcup (Y \times_S S') \to S'$$

remains totally split because the disjoint union of totally split morphisms is totally split. Lastly, $S' \to S$ is finite and faithfully flat because the maps $S' \to X$ and $X \to S$ are both finite and faithfully flat. In particular, $f\colon X \to S$ is surjective because the degree n is locally constant and positive. ■

3.2. **Affine Descent.** We require a few descent results. It would take us much too far afield to prove these results in their correct context, so we will pick up only what we need. Approximately speaking, our end goal is to show that being finite étale can be checked after faithfully flat base-change.

Lemma 3.6. *Let $f\colon S' \to S$ be a quasicompact faithfully flat map. Then $U \subseteq S$ is open if and only if $\varphi^{-1}(U) \subseteq S'$ is open.*

Proof. The forward direction is by continuity of f. To continue, we make some reductions. Taking complements, it suffices to show $Z' := f^{-1}(Z) \subseteq S$ is closed implies $Z \subseteq S$ is closed. Because f is surjective, we see that $Z = f(Z')$, so giving Z' the reduced scheme structure, it remains to show that $\varphi(Z') \subseteq S$ is closed.

Well, by [Har77, Lemma II.4.5], it is enough to show $f(Z')$ is stable under specialization. However, going up for flat extensions [Eis95, Lemma 10.11] implies $f(U')$ is stable under generalization for any open $U' \subseteq S'$, so $S \setminus f(Z') = f(S' \setminus Z')$ is stable under generalization, so $f(Z')$ is stable under specialization. ■

Proposition 3.7. *Let $f\colon X \to S$ be an affine morphism, and let $p\colon S' \to S$ be an affine faithfully flat morphism. Set $X' := X \times_S S'$ and let $f'\colon X' \to S'$ and $p'\colon X' \to X$ be the projections.*

- *(a) If f' is finite, then f is also finite.*
- *(b) If f' is flat, then f is flat.*
- *(c) If f' is an isomorphism, then f is also an isomorphism.*
- *(d) If f' is an open embedding, then f is also an open embedding.*
- *(e) Suppose that f is locally of finite presentation. If f' is unramified, then f is also unramified.*
- *(f) If f' is finite étale, then f is finite étale.*

Proof. For a few parts, we will want to work affine-locally, so we build the diagram now: choosing some $\operatorname{Spec} A \subseteq S$, we produce the following pullback square.

$$\begin{tikzcd} \operatorname{Spec} B' \arrow[r, "f'"] \arrow[d, "p'"'] & \operatorname{Spec} A' \arrow[d, "p"] \\ \operatorname{Spec} B \arrow[r, "f"] & \operatorname{Spec} A \end{tikzcd} \tag{3.1}$$

Here, $\operatorname{Spec} B = f^{-1}(\operatorname{Spec} A)$ and $\operatorname{Spec} A' = p^{-1}(\operatorname{Spec} A)$ and $B' = B \otimes_A A'$.

- (a) We work affine-locally, with (3.1). We are given that B' is finite over A, and we want to show that B is finite over A. Well, find some finitely many generators for $B' = B \otimes_A A'$ as an A'-module, which without loss of generality may take the form $\{b_i \otimes 1\}_{i=1}^n$. As such, we have a map $A^n \to B$ such that
$$A^n \otimes_A A' \to B \otimes_A A' \to 0$$
is exact. It follows that $A^n \to B \to 0$ is exact, so B is finite over A.
- (b) We work affine-locally, with (3.1). We are given that B' is a flat A'-algebra, and we want to show that B is a flat A-algebra. Well, suppose that we have an exact sequence
$$M_1 \to M_2 \to M_3$$
of A-modules. Because A' is flat over A, and $B' = B \otimes_A A'$ is flat over A', we get the exact sequence
$$(B \otimes_A A') \otimes_{A'} (M_1 \otimes_A A') \to (B \otimes_A A') \otimes_{A'} (M_2 \otimes_A A') \to (B \otimes_A A') \otimes_{A'} (M_3 \otimes_A A').$$
Factoring out the A' and removing it by faithful flatness finishes the proof that B is flat over A.
- (c) We work affine-locally, with (3.1). Because f' is an isomorphism, we see that
$$0 \to A \otimes_A A' \to B \otimes_A A' \to 0$$
is exact, so it follows that $0 \to A \to B \to 0$ is exact, so the result follows.
- (d) We follow [Eme]. To begin, note that the surjectivity of p and p' implies that
$$f'(X') = p^{-1}(p(f'(X'))) = p^{-1}(f(p'(X'))) = p^{-1}(f(X)),$$
so $p^{-1}(f(X))$ is open in S', so $f(X)$ is open in S by Lemma 3.6. It remains to show that $X \cong U$, where $U := f(X)$. Well, setting $U' := f'(X')$, note $f\colon X \to U$ base-changes to the isomorphism $f'\colon X' \to U'$, so (c) finishes.

(e) To set us up, note that an affine morphism $f\colon X \to S$ has closed and hence affine diagonal Δ_f.
Now, by definition, f is unramified if and only if the diagonal $\Delta_f\colon X \to X \times_S X$ is an open embedding. However, base-changing by $p\colon S' \to S$, we know that f' is unramified, so $\Delta_{f'}\colon X' \to X' \times_{S'} X'$ is an open embedding, so (d) tells us that $\Delta_f\colon X \to X \times_S X$ is an open embedding.

(f) By (a), f is finite. Then (b) and (e) imply f is étale. ■

3.3. **The Main Theorem.** We are now ready to prove Theorem 1.2. Here is the relevant category.

Definition 3.8. *Fix a scheme X. Then $\mathrm{F\acute{E}t}(X)$ is the category of morphisms $f\colon Y \to X$ of finite étale maps to X. We will frequently identify the objects f with their codomains Y.*

Remark 3.9. *Because étale morphisms and finite morphisms satisfy cancellation, any morphism $f\colon Y \to Y'$ of X-schemes in $\mathrm{F\acute{E}t}(X)$ will be finite étale. Indeed, the Cancellation theorem [Vak17, Theorem 11.2.1] allows us to merely check that the diagonal of a finite étale map is finite étale, which is true because the diagonal of a finite map is a closed embedding and the diagonal of an étale map is an open embedding.*

The goal is to show that $\mathrm{F\acute{E}t}(X)$ is Galois, where the fiber functor is given by base-change to a geometric point. We begin with the checks internal to the category.

Proposition 3.10. *Fix a scheme X. The category $\mathrm{F\acute{E}t}(X)$ has finite limits and colimits.*

Proof. Showing finite limits is easier, so we check these first. The terminal object in $\mathrm{F\acute{E}t}(X)$ is $\mathrm{id}_X\colon X \to X$. Additionally, the category of X-schemes has fiber products given by the usual fiber products, and these remain the fiber products in $\mathrm{F\acute{E}t}(X)$. It follows that $\mathrm{F\acute{E}t}(X)$ has all finite limits.

We now show finite colimits. It suffices to show that we have coproducts and coequalizers. The category of schemes has coproducts given by disjoint union, which remain the coproducts in $\mathrm{F\acute{E}t}(X)$. Lastly, we turn to coequalizers. As previously, we would like to retrieve these coequalizers from some subcategory of Sch, and here we work in the category $\mathrm{Aff}(X)$ of affine X-schemes. In particular, by [Har77, Exercise II.5.17] or [SP, Lemma 01SA], we see $\mathrm{Aff}(X)^{\mathrm{op}}$ is equivalent to the category $\mathrm{QCohAlg}(\mathcal{O}_X)$ of quasicoherent $\mathcal{O}_X$-algebras, so $\mathrm{Aff}(X)$ has coequalizers because $\mathrm{QCohAlg}(\mathcal{O}_X)$ has equalizers. To be explicit, fix finite étale morphisms $f, g\colon Y_1 \to Y_2$ for which we would like to construct a coequalizer. Then Y_1 and Y_2 are affine over X, so we see $Y_1 = \operatorname{Spec}_X \mathcal{A}_1$ and $Y_2 = \operatorname{Spec}_X \mathcal{A}_2$ for $\mathcal{O}_X$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Notably, because Y_1 and Y_2 are finite and flat over X, we see $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite locally free $\mathcal{O}_X$-algebras.

Now, define the $\mathcal{O}_X$-algebra $\mathcal{C} := \mathrm{eq}(f^\sharp, g^\sharp)$. The anti-equivalence described in the previous paragraph promises that $\operatorname{Spec}_X \mathcal{C}$ is the coequalizer of $f, g\colon Y_1 \to Y_2$, so it suffices to show $\operatorname{Spec}_X \mathcal{C} \to X$ is finite étale. This map is already affine, so Proposition 3.7 allows us to check that we remain finite étale after any affine faithfully flat base-change. Well, applying Proposition 3.5 to the maps $Y_1 \to X$ and $Y_2 \to X$ (each) lets us assume that $Y_1 \to X$ and $Y_2 \to X$ are totally split.

Continuing with reductions, we may check that $\operatorname{Spec}_X \mathcal{C} \to X$ is finite étale affine-locally on the target, so we replace X with a connected component to assume that X is connected. In this case, $Y_i \cong X_1 \sqcup \cdots \sqcup X_{n_i}$ where $X_j = X$ for each j, where $Y_i \to X$ is the natural projection. As such, $f\colon Y_1 \to Y_2$ looks like

$$\bigsqcup_{i=1}^{n_1} X_i \to \bigsqcup_{j=1}^{n_2} X_j.$$

Now, each X_i in the source is connected and thus can only map to a single X_j in the target. In fact, the factored map $X_i \to X_j$ must be the identity because it is an X-morphism. The same argument for g implies g is also the coproduct of identities. Inverting everything, we see that

$$f^\sharp, g^\sharp\colon \prod_{j=1}^{n_2} \mathcal{O}_{X_j} \to \prod_{i=1}^{n_1} \mathcal{O}_{X_i}$$

factors through identities $\mathcal{O}_X = \mathcal{O}_X$ in each coordinate. Thus, we can compute that the equalizer $\mathcal{C} = \mathrm{eq}(f^\sharp, g^\sharp)$ is indeed finite locally free. ■

Lemma 3.11. *Fix a scheme X. An object Y in $\mathrm{F\acute{E}t}(X)$ is connected if and only if Y is connected as a scheme. Thus, any object in $\mathrm{F\acute{E}t}(X)$ is the finite coproduct of connected objects.*

Proof. In one direction, suppose that Y is a disconnected scheme so that we can write $Y = Y_1 \sqcup Y_2$. Then the composites $Y_1, Y_2 \to Y \to X$ make Y_1 and Y_2 into objects in $\mathrm{F\acute{E}t}(X)$. Further, the maps $Y_1, Y_2 \to Y$ are open embeddings and hence monic, so Y has nontrivial subobjects, meaning Y is not connected in $\mathrm{F\acute{E}t}(X)$.

Conversely, suppose Y is a connected scheme, and suppose we have a monomorphism $f\colon Y' \to Y$ in $\mathrm{F\acute{E}t}(X)$ from a non-initial (i.e., nonempty) scheme Y'; we show that f is an open embedding, which will finish because f has closed image and Y is connected. Well, the finite limits in $\mathrm{Sch}(X)$ and $\mathrm{F\acute{E}t}(X)$ agree by Proposition 3.10, so f being monic in $\mathrm{F\acute{E}t}(X)$ is equivalent to the diagonal $\Delta_f\colon Y' \to Y' \times_Y Y'$ being an isomorphism, so f is monic in $\mathrm{Sch}(X)$.

Now, f is universally injective, topologically open, and continuous, so f is a homeomorphism onto an open subset $Y_0 \coloneqq f(X) \subseteq Y$. Thus, replacing Y by Y_0, we may assume that f is faithfully flat, and we want to show that f is an isomorphism. Well, f is affine, so Proposition 3.7 lets us check that f is an isomorphism after base-change by an affine faithfully flat map, so we base-change f by itself (!) to note that the projection $\pi_2\colon Y' \times_Y Y' \to Y'$ is an isomorphism because the composite

$$Y' \overset{\Delta_f}{\to} Y' \times_Y Y' \to Y'$$

is the identity, and Δ_f is an isomorphism (because f is monic). ■

Proposition 3.12. *Fix a connected scheme X. Then any object in $\mathrm{F\acute{E}t}(X)$ is the finite coproduct of connected objects.*

Proof. Fix an object $f\colon Y \to X$ in $\mathrm{F\acute{E}t}(X)$. By Lemma 3.11, the desired decomposition is the disjoint union of connected components, so it suffices to show that Y has only finitely many connected components.

Well, fix any $y \in Y$ and set $x \coloneqq f(y)$. Because f is finite, the fiber $f^{-1}(\{x\})$ has only finitely many points; let $Y_1, \ldots, Y_n$ be the connected components of each element in the fiber $f^{-1}(\{x\})$. We claim that these are all the connected components of Y, which will finish. Indeed, fix some connected component Y' in Y. Now, f is open and closed, so $f(Y') \subseteq X$ is nonempty and clopen, so $f(Y') = X$ because X is connected. So $x \in f(Y')$, so $f^{-1}(\{x\}) \cap Y' \neq \varnothing$, so Y' is equal to one of the Y_i. ■

As mentioned previously, our fiber functor will arise from base-change, so let's understand base-change.

Lemma 3.13. *Let $X' \to X$ be a scheme morphism. Then the base-change functor $\mathrm{F\acute{E}t}(X) \to \mathrm{F\acute{E}t}(X')$ sending $Y \mapsto Y \times_X X'$ is exact.*

Proof. Note that the functor is well-defined because finite étale morphisms are preserved by base-change.

As usual, limits are easier. For left-exactness, we note that the base-change of the terminal object $\mathrm{id}_X \in \mathrm{F\acute{E}t}(X)$ is the terminal object $\mathrm{id}_{X'} \in \mathrm{F\acute{E}t}(X')$. Further, taking fiber products commutes with base-change: for morphisms $Y_1, Y_2 \to Y$ in $\mathrm{F\acute{E}t}(X)$, we can check

$$(Y_1 \times_Y Y_2) \times_X X' = (Y_1 \times_X X') \times_{(Y \times_X X')} (Y_2 \times_X X').$$

For right-exactness, note that the base-change of a disjoint union $Y_1 \sqcup Y_2$ remains becomes the disjoint union $(Y_1 \times_X X') \sqcup (Y_2 \times_X X')$. To check that coequalizers are preserved by base-change, we run through the construction of coequalizers in Proposition 3.10. Given morphisms $f, g\colon Y_1 \to Y_2$, we defined $\operatorname{coeq}(f, g) \coloneqq \operatorname{Spec}_X \operatorname{eq}(f^\sharp, g^\sharp)$. Now, the universal property ensures a map

$$\operatorname{coeq}(f, g)_{X'} \to \operatorname{coeq}(f_{X'}, g_{X'}),$$

where the subscript denotes base-change. We want to show that this map is an isomorphism. Well, we argue as in Proposition 3.10: by Proposition 3.7, it suffices to check that this is an isomorphism after base-change by a faithfully flat affine morphism, so we may assume that all morphisms are totally split. Also, we may check isomorphisms affine-locally, so we may assume X and X' are both connected. But in this case we saw coeq arose from some equalizer of finite free algebras, which commutes with base-change. ■

In fact, we are trying to understand base-change to a geometric point, so the following example will be helpful.

Example 3.14. *Let k be an algebraically closed field, and set $X \coloneqq \operatorname{Spec} k$. Then we claim $\mathrm{F\acute{E}t}(X) \cong \mathrm{FinSet}$ by sending the finite étale cover $f\colon Y \to X$ to the set Y, which is finite because f is quasifinite. In particular, $\mathcal{O}_Y(Y)$ is a finite separable k-algebra, which is a finite product of ks because k is algebraically closed, and points in Y correspond to factors in the product. This description allows us to check we have an equivalence.*

Theorem 1.2. *Fix a connected scheme X and a geometric point $\overline{x}$ of X. Then the category $\mathrm{F\acute{E}t}(X)$ of finite étale covers of X equipped with the base-change functor $F\colon \mathrm{F\acute{E}t}(X) \to \mathrm{FinSet}$ by*

$$FY \coloneqq Y_{\overline{x}}$$

forms a Galois category.

Proof. Quickly, we note that F is actually the base-change functor to $\overline{x}$ follows by the equivalence to FinSet given in Example 3.14, so in particular F is well-defined and exact by Lemma 3.13. For a few other checks, note Proposition 3.10 implies that $\mathrm{F\acute{E}t}(X)$ has finite limits and colimits, and Proposition 3.12 implies that objects are the finite coproduct of connected objects.

It remains to show that F reflects isomorphisms. Well, fix a morphism $f\colon Y \to Y'$ in $\mathrm{F\acute{E}t}(X)$ such that $Ff\colon Y_{\overline{x}} \to Y'_{\overline{x}}$ is an isomorphism. Now, f is finite étale, so $\mathcal{O}_Y$ is a finite locally free $\mathcal{O}_{Y'}$-algebra. As such, to check that $\mathcal{O}_Y$ is rank-1 over $\mathcal{O}_{Y'}$, but this rank can be computed after base-change by $\overline{x}$, where we know the ranks coincide because Ff is an isomorphism. ■

At long last, here is our definition.

Definition 3.15 (étale fundamental group)**.** *Fix a connected scheme X and a geometric point $\overline{x}$ of X. Then the étale fundamental group $\pi_1(X, \overline{x})$ is the automorphism group of the base-change functor $F\colon \mathrm{F\acute{E}t}(X) \to$ FinSet defined by $FY \coloneqq Y_{\overline{x}}$. In light of the following remark, we might abbreviate $\pi_1(X, \overline{x})$ to just $\pi_1(X)$.*

Remark 3.16. *It might look like $\pi_1(X, \overline{x})$ depends on $\overline{x}$, but Remark 2.22 assures us that it does not.*

To prove that we can actually compute this, here is a basic example.

Example 3.17. *Fix a field k. Then $\pi_1(\operatorname{Spec} k) = \mathrm{Gal}(\overline{k}/k)$.*

Proof. We have essentially already done this. Indeed, a finite étale cover $X \to \operatorname{Spec} k$ must make $X = \operatorname{Spec} A$ where A is a finite separable k-algebra. Thus, $\mathrm{F\acute{E}t}(X) \cong \mathrm{SAlg}(k)^{\mathrm{op}}$, so we are in the context of Example 2.5.

To complete the argument, we use Remark 2.19. Our fiber functor $F\colon\ \mathrm{SAlg}(k)^{\mathrm{op}} \to \mathrm{FinSet}$ is by $FA \coloneqq \mathrm{Hom}_k(A, \overline{k})$ for a fixed algebraic closure $\overline{k}$. As in Example 2.12, the Galois objects of $\mathrm{SAlg}(k)^{\mathrm{op}}$ are the Galois field extensions ℓ/k, whose automorphism group is $\mathrm{Gal}(\ell/k)$. Fixing an algebraic closure $\overline{k}$ chooses a specific element $\ell \hookrightarrow \overline{k}$ for each Galois ℓ, so we now see that

$$\pi_1(\operatorname{Spec} k) = \operatorname{Aut} F = \lim_{\text{Galois } \ell/k} \mathrm{Gal}(\ell/k) = \mathrm{Gal}(\overline{k}/k),$$

as desired. ■

4. Examples

We close this paper with a few example computations, for fun. Our examples increase in difficulty.

4.1. **Some Curves.**

We will work with an algebraically closed field, essentially in order to use the Riemann–Hurwitz formula [Har77, Corollary IV.2.4].

Example 4.1. *Fix an algebraically closed field k. Then $\pi_1(\mathbb{P}^1_k) = 0$.*

Proof. We claim that any connected finite étale map $f\colon Y \to \mathbb{P}^1_k$ is $f\colon Y \cong \mathbb{P}^1_k$. This implies that $\mathrm{F\acute{E}t}(X)$ has no non-terminal Galois objects and thus trivial π_1 by Remark 2.19.

For the claim, we use the Riemann–Hurwitz formula. Indeed, f ensures that Y is a smooth (because f is étale) and projective (because f is proper) curve (because f is integral) over an algebraically closed field k. Further, f is separable and unramified because it is étale, so the Riemann–Hurwitz formula enforces

$$2g(Y) - 2 = (\deg f)\left(2g\left(\mathbb{P}^1_k\right) - 2\right) = -2\deg f \le -2.$$

Thus, $g(Y) = 0$ and $\deg f = 1$, so $f^\sharp\colon K(\mathbb{P}^1_k) \to K(Y)$ is an isomorphism, so $f\colon Y \cong \mathbb{P}^1_k$. ■

Example 4.2. *Fix an algebraically closed field k of characteristic 0. Then $\pi_1\left(\mathbb{A}^1_k\right) = 0$.*

Proof. As in Example 4.1, we claim that any connected finite étale map $f\colon Y \to \mathbb{A}^1_k$ is $f\colon Y \cong \mathbb{A}^1_k$, which implies that π_1 is trivial by Remark 2.19.

To set up, note that Y is birational to a smooth projective k-curve $\overline{Y}$ by [Har77, Corollary I.6.11], which extends to a full embedding $Y \hookrightarrow \overline{Y}$ by [Har77, Proposition I.6.8]. Further, the rational map $\overline{Y} \dashrightarrow Y \to \mathbb{A}^1_k \to \mathbb{P}^1_k$ extends to a full map $\overline{f}\colon \overline{Y} \to \mathbb{P}^1_k$ by [Har77, Proposition I.6.8] again.

Now, $\overline{f}$ is finite (because $\overline{Y}$ and $\mathbb{P}^1_k$ are proper) and separable (because $\operatorname{char} k = 0$), and $\overline{f}$ is determined by Y at all but finitely many points, so $\overline{f}$ must be unramified away from $\infty \in \mathbb{P}^1_k$. Well, let the fiber over ∞ in $\overline{f}$ be $\{y_1, \ldots, y_r\}$ (note $r > 1$ because $\operatorname{im} \overline{f} \subseteq \overline{Y}$ is dense and closed), where y_i has ramification index e_i. All ramification is tame because $\operatorname{char} k = 0$, so the Riemann–Hurwitz formula grants

$$2g(\overline{Y}) - 2 = (\deg f)\left(g\left(\mathbb{P}^1_k\right) - 2\right) + \sum_{i=1}^{r}(e_i - 1) = -\deg f - r \leq -2,$$

so we again must have $g(\overline{Y}) = 0$ and then $\deg \overline{f} = 1$, so $\overline{f} \colon \overline{Y} \to \mathbb{P}^1_k$ is an isomorphism. After identifying $Y \subseteq \mathbb{P}^1_k$ with $\mathbb{A}^1_k$ (namely, re-parameterizing $Y \subseteq \mathbb{P}^1_k$), we conclude $f \colon Y \to \mathbb{A}^1_k$ is the identity on points and thus the identity because everything is a k-variety. ■

Example 4.3. *Fix an algebraically closed field k of characteristic 0. Then $\pi_1(\mathbb{G}_m) = \widehat{\mathbb{Z}}$, where $\mathbb{G}_m = \operatorname{Spec} k\left[x, x^{-1}\right] = \mathbb{A}^1_k \setminus \{0\}$.*

Proof. We will still be able to classify connected finite étale maps $f \colon Y \to \mathbb{G}_m$. Arguing as in Example 4.2, embed Y into some smooth projective k-curve $\overline{Y}$, and the rational map $\overline{Y} \dashrightarrow \mathbb{P}^1_k$ can be extended to a full finite map $\overline{f} \colon \overline{Y} \to \mathbb{P}^1_k$ by [Har77, Proposition I.6.8].

Because $\overline{f}$ is determined by its étale behavior as $f \colon Y \to \mathbb{G}_m$, we see that $\overline{f}$ is ramified at worst in $\{0, \infty\}$. So let the fiber over $p \in \{0, \infty\}$ be $\{y_1^p, \ldots, y_{r^p}^p\}$, where y_i has ramification index e_i; note $r^0, r^\infty \geq 1$ because $\overline{f}$ is surjective. All ramification is tame because $\operatorname{char} k = 0$, so the Riemann–Hurwitz formula grants

$$2g(\overline{Y}) - 2 = (\deg f)\left(g(\mathbb{P}^1_k) - 2\right) + \sum_{i=1}^{r^0}\left(e_i^0 - 1\right) + \sum_{i=1}^{r^\infty}\left(e_i^\infty - 1\right) = -r^0 - r^\infty.$$

Thus, we must have $g(\overline{Y}) = 0$ and $r^0 = r^\infty = 1$, where the single point lying over 0 (and ∞) will have ramification index $\deg f$. In particular, we may set $\overline{Y} = \mathbb{P}^1_k$, and adjusting this isomorphism allows to assume that the fiber over 0 is $\{0\}$ and the fiber over ∞ is $\{\infty\}$.

Now, set $n := \deg f = \deg \overline{f}$. The map $\overline{f} \colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ fixes ∞ and thus arises from a map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ of degree n such that the fiber over 0 is $\{0\}$. But the only polynomial of degree n whose only root is 0 is $z \mapsto \lambda z^n$. Adjusting $\overline{Y}$ up to isomorphism again, we may thus assume that $\overline{f} \colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ is the nth-power map. Now, Y surjects onto $\mathbb{G}_m$, and each fiber must have exactly n elements because $Y \twoheadrightarrow \mathbb{G}_m$ is finite étale of degree n, so $Y \subseteq \mathbb{G}_m$ must actually have all the points, and because everything is reduced, we conclude $Y = \mathbb{G}_m$, and $Y \to \mathbb{G}_m$ is the nth-power map.

Thus, we have classified our connected étale covers. We now let our connected cover $Y_n \to \mathbb{G}_m$ denote the nth power map. Note the automorphisms of $Y_n = \operatorname{Spec} k\left[x, x^{-1}\right]$ are given by $x \mapsto \zeta x$ for some nth root of unity ζ, so these objects are actually Galois, so Remark 2.19 implies

$$\pi_1(\mathbb{G}_m) = \lim_{n \in \mathbb{Z}} \operatorname{Aut} Y_n \cong \lim_{n \in \mathbb{Z}} \mu_n(\overline{k}) \cong \lim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}},$$

as desired. ■

4.2. **Elliptic Curves.** We now discuss the étale fundamental group of an elliptic curve over an algebraically closed field of characteristic 0. With more effort, one can make this argument go through for abelian varieties over an arbitrary field [EGM, Corollary 10.37], but we will do this special case because we can still communicate the main idea while arguing along the lines of Example 4.3. Building off of the development of isogenies in my fall term paper [Elb22, Section 2.3], we need the following facts; we follow [EGM].

Lemma 4.4. *Fix an isogeny $\beta \colon B \to C$ of abelian k-varieties. If homomorphisms $\alpha_1, \alpha_2 \colon A \to B$ have $\beta \circ \alpha_1 = \beta \circ \alpha_2$, then $\alpha_1 = \alpha_2$.*

Proof. Noting $\beta \circ (\alpha_1 - \alpha_2) = 0$, so $\alpha_1 - \alpha_2$ must factor through the fiber $\gamma^{-1}(0_D) = \ker \gamma$. However, B is connected, so $\beta_1 - \beta_2$ must send it to a connected scheme, so $\beta_1 - \beta_2$ maps B to the connected component $\{0_C\} \subseteq \ker \gamma$. Lastly, B is reduced, so $\beta_1 - \beta_2 = 0$ follows. ■

Proposition 4.5. *Fix an isogeny $\varphi \colon A \to B$ of abelian k-varieties, and set $d := \deg \varphi$. Suppose* $\operatorname{char} k = 0$. *Then there exists an isogeny $\psi \colon B \to A$ of degree d such that $\varphi \circ \psi = [d]_B$ and $\psi \circ \varphi = [d]_A$.*

Proof. This proof requires the notion of fppf quotients, which we will not introduce here; as such, we will be quite sketchy. One can check that the isogeny $\varphi\colon A \to B$ identifies B with the fppf quotient $A/\ker\varphi$. Further, one knows that $\ker\varphi$ has size d because φ is separable, so is annihilated by $[d]_A$, so $[d]_A$ will factor through φ as

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} A.$$

Thus, $\psi \circ \varphi = [d]_A$. Lastly, $\varphi \circ \psi \circ \varphi = \varphi \circ [d]_A = [d]_B \circ \varphi$, so $\varphi \circ \psi = [d]_B$ follows from Lemma 4.4. ■

Remark 4.6. *We only used* $\operatorname{char} k = 0$ *above to know that* φ *is separable, implying* $\ker\varphi$ *is killed by* $[d]_A$*. One can remove this hypothesis and still show* $\ker\varphi$ *is killed by* $[d]_A$ *if one admits more theory of algebraic groups; in particular, see [EGM, Exercise 4.4].*

Example 4.7. *Fix an algebraically closed field* k *of characteristic* 0*. Then* $\pi_1(E) = \widehat{\mathbb{Z}}^2$ *for any elliptic* k*-curve* E*. Here,* $\widehat{\mathbb{Z}}$ *is the profinite completion of* $\mathbb{Z}$*.*

Proof. As usual, we want to classify connected étale covers $f\colon Y \to E$. Because E is a smooth projective k-curve, we see that Y is also a smooth proper k-curve, as we argued in Example 4.1. We claim $g(Y) = 1$: indeed, f is smooth and in particular unramified, so the Riemann–Hurwitz formula grants

$$2g(Y) - 2 = (\deg f)(2g(E) - 2) = 0,$$

so $g(Y) = 1$. Let $0_E \in E$ be the base-point of E. Choosing a k-point $0_Y \in Y$ lying over 0_E, we acknowledge that Y is a genus-1 curve with a marked point 0_E, so it is an elliptic k-curve and in particular an abelian variety. Now, the map $f\colon Y \to E$ is the composition of a homomorphism and a translation by [Elb22, Corollary 2.7], so $f(0_Y) = 0_E$ (by definition of 0_Y) ensures that f is a homomorphism. Lastly, we see that f is quasifinite and dominant (because finite), so f is an isogeny.

Thus, we can apply Proposition 4.5 to be promised some isogeny $g\colon E \to Y$ such that $f \circ g = [n]$ for some $n > 0$. The point is that any connected object in $\mathrm{F\acute{E}t}(E)$ has a cover by some $[n]\colon E \to E$; for brevity, call this object $[n]\colon E_n \to E$ where $E_n = E$. Quickly, note that translations by $E[n](k)$ are automorphisms of E_n, but E_n has at most $\deg[n] = \#E[n](k)$ total automorphisms, so we conclude $\operatorname{Aut} E_n = E[n](k)$. Thus, because any Galois cover can be covered by E_n, we compute

$$\pi_1(E) = \lim_{n\in\mathbb{Z}} \operatorname{Aut} E_n = \lim_{n\in\mathbb{Z}} E[n](k) = \prod_{p \text{ prime}} \lim_{\nu\geq 1} E\left[p^\nu\right](k) = \prod_{p \text{ prime}} T_pE \cong \prod_{p \text{ prime}} \mathbb{Z}_p^2 = \widehat{\mathbb{Z}}^2,$$

where we have used the facts about torsion of abelian varieties established in [Elb22]. ■

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