18.708: Topics in Algebra

Nir Elber

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
de Rham Cohomology in Mixed Characteristic

These talks were given by Alexander Petrov.

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Here we go.

1.1.1 Algebraic de Rham Cohomology

Let's begin by describing what we mean by de Rham cohomology. We will consider a smooth variety X over an algebraically closed field F.

Definition 1.1 (smooth)**.** We say that a variety X over a field F is *smooth* if and only if $\Omega_{X/F}$ is a vector bundle of rank $\dim X$ on each connected component. Here, on an affine open subset $U \subseteq X$, recall that $\Omega_{X/F}(U)$ is spanned by symbols of the form $f\,dg$, where the symbol d is (as usual) F-linear and satisfies the Leibniz rule.

Definition 1.2 (algebraic de Rham cohomology)**.** Fix a smooth variety X over a field F. Then one can iterate the F-linear map $d\colon \mathcal{O}_X \to \Omega_{X/F}$ to a map $d\colon \Omega^i_{X/F} \to \Omega^{i+1}_{X/F}$ for each i, where $\Omega^i_{X/F} \coloneqq \wedge^i \Omega_{X/F}$. We now define the *de Rham complex* to the complex

$$\Omega^\bullet_{X/F}\colon 0 \to \mathcal{O}_X \stackrel{d}{\to} \Omega^1_{X/F} \stackrel{d}{\to} \cdots,$$

and we define the *de Rham cohomology* $\mathrm{H}^n_{\mathrm{dR}}(X/F)$ to be the nth hypercohomology of $\Omega^\bullet_{X/F}$. Here, hypercohomology means the total cohomology of some produced acyclic double complex which resolves the complex (e.g., a Čech resolution). Note that this hypercohomology is merely a vector space over F.

Example 1.3. The map $d\colon \Omega^1_{X/F} \to \Omega^2_{X/F}$ is given by $d(f\,dg) = df \wedge dg$.

Example 1.4. Suppose that X is affine. Then vector bundles are already acyclic, so the hypercohomology does nothing. Thus,

$$\mathrm{H}^n_{\mathrm{dR}}(X/F) = \mathrm{H}^n\left(X; 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/F} \xrightarrow{d} \cdots\right).$$

As usual, this is $\ker(d|_{\Omega^n}) / \operatorname{im}(d|_{\Omega^{n-1}})$.

Remark 1.5. If X is affine and $i > \dim X$, then $\Omega^i_{X/F}$ vanishes, so the algebraic de Rham cohomology also vanishes.

Remark 1.6. A different definition is required for non-smooth X. Roughly speaking, one should embed into a smooth variety and take cohomology there.

Here is one way to convince ourselves that this is a reasonable cohomology theory.

Theorem 1.7 (Grothendieck)**.** Suppose that X is a smooth variety over $\mathbb{C}$. Then there is a canonical isomorphism

$$\mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C}); \mathbb{C}) \to \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}).$$

Here, the left-hand side is Betti cohomology (also called singular cohomology).

Sketch. We argue in the case that X is affine. Then $X(\mathbb{C})$ already has a notion of $\Omega^{i,\mathrm{an}}_{X/\mathbb{C}}$ given by the holomorphic forms. Algebraic forms embed into holomorphic ones, which produces a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \Omega^1_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^2_{X/\mathbb{C}}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{\mathrm{an}}_X(X) & \longrightarrow & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \cdots
\end{array}$$

of complexes. It then turns out that this is an isomorphism on cohomology, so we reduce to comparing analytic de Rham cohomology with singular cohomology.

This is now a problem of analysis. One can pass from holomorphic differentials to smooth differentials via a similar process, which produces another morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{\mathrm{an}}_X(X) & \longrightarrow & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\infty(X(\mathbb{C}), \mathbb{C}) & \longrightarrow & \Omega^1_{C^\infty}(X(\mathbb{C})) & \longrightarrow & \Omega^2_{C^\infty}(X(\mathbb{C})) & \longrightarrow & \cdots
\end{array}$$

of complexes, which is also an isomorphism on complexes. We are now reduced to the setting of de Rham's theorem for real manifolds. ■

Example 1.8. Consider $X := \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = \operatorname{Spec} k[t, 1/t]$.

- Our differential map $d\colon \mathbb{C}[t, 1/t] \to \mathbb{C}[t, 1/t]\,dt$ sends t^n to $nt^{n-1}\,dt$. Thus, $\mathrm{H}^0_{\mathrm{dR}}(X)$ is one-dimensional given by the constants, and $\mathrm{H}^1_{\mathrm{dR}}(X)$ is one-dimensional spanned by dt/t.
- The point above works also for holomorphic differentials. The interesting bit is in degree 1, where the point is that there is no global antiderivative for dx/x.
- On the other hand, $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, so we expect its singular cohomology to be supported in degrees 0 and 1, where it should be one-dimensional.

Corollary 1.9 (Artin vanishing)**.** If X is an affine algebraic complex smooth variety, then $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) = 0$ for $n > \dim X$.

Proof. The algebraic de Rham cohomology complex vanishes above $\dim X$. ■

Corollary 1.10. Fix a smooth variety X over $\mathbb{C}$. Then $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$ is finite-dimensional.

Proof. Pass to singular cohomology. ■

Remark 1.11. This corollary still admits algebraic proofs in characteristic zero by working with holonomic $\mathcal{D}$-modules. Pavel Etingof claims that there is an algebraic proof using the fact that the direct image of a holonomic $\mathcal{D}$-module is a holonomic $\mathcal{D}$-module.

We would like to point out that our de Rham cohomology is algebraic but still interesting.

Remark 1.12. Suppose that X is smooth over $\mathbb{Q}$. Base-changing by a field is exact, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}} \cong \mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}).$$

However, Theorem 1.7 grants an isomorphism to $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})_{\mathbb{C}}$. Notably, we then find a lattice and a rational structure over in some complex vector space, but the comparison between the two is quite interesting mathematically (and amounts to the study of periods).

Example 1.13. In the case that $X = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\}$, the comparison between $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}}$ and $\mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})$ is mediated by a constant $2\pi i$. Indeed, once unwinds the de Rham theorem, this amounts to the statement that a contour integral of dx/x going once around the origin is $2\pi i$.

1.1.2 Frobenius Structure

We now pass to positive characteristic. Let k be a perfect field of positive characteristic p, and we may still consider a smooth variety X.

Remark 1.14. If k is perfect, then $\Omega^1_{X/k} = \Omega^1_{X/\mathbb{F}_p}$ by doing some thinking about inseparable extensions. The moral is that

$$y^{1/p}\,dy = d\left((y^{1/p})^p\right),$$

so the coefficients can be brought down when everything is a pth power.

This cohomology is rather strangely behaved.

Example 1.15. Take $X := \mathbb{A}^1_k$. The de Rham cohomology still lives in degrees zero and one, so we would like to study the kernel and cokernel of the k-linear map $d\colon k[t] \to k[t]\,dt$ given by $t^n \mapsto nt^{n-1}$.

- We see that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \ker d$ is spanned by t^{pi} for each i.
- We see that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \operatorname{im} d$ is infinite-dimensional because the differentials $t^{mp-1}\,dt$ fail to be in the image. In fact, these classes form a basis.

Let's try to view these infinite-dimensional groups as a feature instead of a bug. Indeed, it turns out that the de Rham complex has some extra structure. The de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots$$

is merely made of sheaves of k-vector spaces over X. In characteristic zero, this is all the structure present, but in characteristic p, we have more structure.

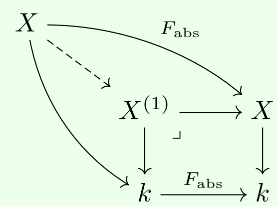
Notation 1.16. Fix a variety X over a field k of characteristic p. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define

$$\mathcal{F}^p := \{f^p : f \in \mathcal{O}_X\}$$

to locally be given by the pth powers.

The moral is that $d(f^p) = 0$ always, so the de Rham complex is in fact $\mathcal{O}_X^p$-linear! Let's attempt to codify this.

Definition 1.17 (relative Frobenius)**.** Fix a scheme X over a field k of characteristic p. Then there is an *absolute Frobenius* $F_{\text{abs}} \colon X \to X$ which is the identity on topological spaces and the pth power on sheaves. This is a morphism of schemes but not of k-schemes (in general). The *relative Frobenius* $F \colon X \to X^{(1)}$ is the morphism fitting into the following diagram.



Remark 1.18. Note that $X^{(p)}$ is isomorphic to X as a scheme but not as a k-scheme! However, we now benefit because the relative Frobenius F is morphism of k-schemes.

Remark 1.19. The relative Frobenius $F \colon X \to X^{(1)}$ is finite flat of degree $p^{\dim X}$

Example 1.20. If $X = \operatorname{Spec} k[t_1, \ldots, t_n]$, then $X^{(1)} = \operatorname{Spec} k\left[t_1^p, \ldots, t_n^p\right]$. Thus, we see that the embedding

$$k\left[t_1^p, \ldots, t_n^p\right] \subseteq k[t_1, \ldots, t_n]$$

is indeed finite flat of degree p^n.

We now see that

$$0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/F} \to \cdots$$

is a complex of quasicoherent sheaves on $X^{(1)}$. In fact, because F is finite flat, these are all vector bundles: $F_*\mathcal{O}_X$ has rank $p^{\dim X}$ and $F_*\Omega^i_{X/k}$ has rank $p^{\dim X}\binom{\dim X}{i}$. Because $\mathcal{O}_{X^{(1)}} = (F_*\mathcal{O}_X)^p$, we see that this complex is in fact $\mathcal{O}_{X^{(1)}}$-linear.

Example 1.21. Take $X = \operatorname{Spec} k[t]$. Then $X^{(1)} := \operatorname{Spec} k\left[t^p\right]$, and $d \colon k[t] \to k[t]\,dt$ is $k\left[t^p\right]$-linear! Thus, $\mathrm{H}^i_{\mathrm{dR}}(X/k)$ was required to be given by $k\left[t^p\right]$-modules, which explains why we received vector spaces of infinite dimension.

Note that passing through F_* is not going to adjust the underlying k-vector spaces, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \mathbb{H}^n_{\mathrm{Zar}}\left(X^{(1)}; 0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/k} \xrightarrow{d} \cdots\right).$$

To see why this has globalized the $\mathcal{O}_X^p$-linearity, we need the Cartier isomorphism.

Theorem 1.22 (Cartier isomorphism)**.** Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism

$$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$

Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Remark 1.23. This is a reason why characteristic p may be more convenient than characteristic 0: one could still try to understand $\mathcal{H}^i(\Omega^\bullet_{X/k})$ when $\operatorname{char} k = 0$, but this has no easy answer.

Example 1.24. Consider $X = \mathbb{A}^1_k$. Then $\mathcal{H}^1$ is given by the module

$$\frac{k[t]\,dt}{d(k[t])},$$

which our formalism now remembers is a $k\left[t^p\right]$-module. And indeed, we can show that this is isomorphic to $k\left[t^p\right] \cdot t^{p-1}\,dt$. Setting $s := t^p$, we know that $\Omega^1_{X^{(1)}/k}$ is given by the module $k[s]\,ds$, so our isomorphism of modules is given by sending ds to $t^{p-1}\,dt$. One can even check that this isomorphism is canonical in the sense that it will not change under automorphisms of $\mathbb{A}^1$.

We will prove Theorem 1.22 later after a detour.

1.1.3 Crystalline Cohomology

We continue with our perfect field k of positive characteristic p. Our story so far has taken a variety X over a field k, and then we have produced some (total) complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$ in the derived category $D(\mathrm{Vec}_k)$. Crystalline cohomology will allow us to produce an answer in characteristic 0 instead of characteristic p. The idea is to "choose" a lift to characteristic p and then check that the answer is independent of the lift.

The correct formalism for this lifting is that of a "formal scheme."

Definition 1.25 (Witt ring)**.** Fix a perfect field k of characteristic p. Then there is a ring $W(k)$ satisfying that

- $W(k)$ is p-torsion-free,
- $W(k)/p \cong k$, and
- $W(k)$ is the limit of the $W(k)/p^n$ as $n \to \infty$.

This ring $W(k)$ turns out to be unique up to unique isomorphism. We may write $W_n(k) := W(k)/p^n$.

Example 1.26. One can see that $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\overline{\mathbb{F}}_p)$ is its unramified closure.

Remark 1.27. There is a completely explicit construction of $W(k)$, but it is rather involved: given a p-torsion-free ring R, we identify $W(R) := R^{\mathbb{N}}$ but with ring structure chosen so that

$$(a_0, a_1, a_2, \ldots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$$

is a ring homomorphism $W(R) \to R$. It turns out that this ring structure is given by some polynomials (called "ghost coordinates"), so we are allowed to define $W(k)$. From a higher level, it turns out that $W(k)$ is the unique deformation of k, which exists because $\Omega^1_{k/\mathbb{F}_p} = 0$.

Definition 1.28 (formal scheme)**.** Fix a perfect field k of characteristic p. A *p-adic formal scheme* X is a collection of schemes X_n over $W_n(k)$ equipped with isomorphisms

$$X_{n+1} \times_{W_{n+1}(k)} W_n(k) \to X_n.$$

The structure sheaf $\widehat{\mathcal{O}}_X$ is the inverse limit of the $\mathcal{O}_{X_n}$s.

Example 1.29. Given a scheme X over $W(k)$, we can produce a formal scheme $\widehat{Y}$ with $\widehat{Y}_n := Y \times_{W(k)} W_n(k)$ and the induced internal isomorphisms.

Remark 1.30. We can even define $\widehat{\Omega}X^1_{\widetilde{X}}$.

We can now describe crystalline cohomology.

Theorem 1.31. Fix a perfect field k of positive characteristic p. Then there is a functor sending smooth k-varieties X to a complex $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ in the derived category $D(\mathrm{Mod}_{W(k)})$ satisfying the following.

(a) There is a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} k \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$.

(b) If $\widetilde{X}$ is a smooth formal scheme over $W(k)$ (meaning that $\widetilde{X}_n$ is smooth over $W_n(k)$ for all n), then

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X_1/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{Zar}}\left(X; \widehat{\mathcal{O}}_{\widetilde{X}} \xrightarrow{d} \widehat{\Omega}^1_{\widetilde{X}} \xrightarrow{d} \cdots\right)$$

Remark 1.32. Here, (a) immediately tells us that the cohomology of $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ is not expected to be finitely generated.

Remark 1.33. There is something remarkable here, which is that choosing two different lifts of X to a smooth formal scheme produces the same cohomology!

Remark 1.34. It turns out that flatness is equivalent to smoothness in this context.

BIBLIOGRAPHY

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List of Definitions