18.755: Lie Groups and Lie Algebras II

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

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Here we go.

1.1.1 Review of Lie Groups

We start with some quick review. Here are our groups.

Definition 1.1 (Lie group)**.** A *Lie group* is a group object G in the category of manifolds. One may specify a "real" or "complex" Lie group, which means that we are taking the category of real or complex manifolds. Explicitly, we are asking for G to be equipped with regular maps $m\colon G \times G \to G$, $i\colon G \to G$, and an identity. A *homomorphism of Lie groups* is a morphism of the group objects.

Example 1.2. One has the usual examples: $\mathbb{R}^n$, $\mathrm{U}(n)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{O}(p,q)$, and $\mathrm{SU}(n)$ are all real Lie groups.

Example 1.3. There are classical groups over $\mathbb{C}$, such as $\mathrm{SL}_n(\mathbb{C})$, which are all Lie groups.

Definition 1.4. If G is a Lie group, then its connected component G° is a normal Lie subgroup.

Remark 1.5. The quotient $\pi_0 G \coloneqq G/G^\circ$ is a discrete topological group.

Remark 1.6. Given a Lie group G, the universal cover $\widetilde{G} \to G$ can be checked to a Lie group via some universal properties, so we receive a homomorphism $\pi\colon \widetilde{G} \to G$. It turns out that the kernel is a central discrete subgroup $Z \subseteq \widetilde{G}$. It notably follows that $\pi_1(G)$ is abelian.

Remark 1.7. One can check that G° is generated by any open neighborhood of the identity. Indeed, the generated subgroup can be seen to be both open and closed.

Example 1.8. With $G = S^1$, we have the universal cover $\widetilde{G} = \mathbb{R}$, and the kernel is $\mathbb{Z} \subseteq \mathbb{R}$.

We also have subgroups.

Definition 1.9 (Lie subgroup)**.** A *Lie subgroup* is an immersed submanifold $H \subseteq G$ which is also a subgroup, meaning that $H \hookrightarrow G$ admits injective differentials. A *closed Lie subgroup* is an embedded submanifold $H \subseteq G$ which is also a subgroup.

Remark 1.10. It turns out that closed Lie subgroups are in fact closed subsets, which can be checked locally.

Example 1.11. The subgroup $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is a Lie subgroup, but it is not a closed Lie subgroup. The only closed Li subgroups are vector spaces.

Example 1.12. The subgroup $\mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is a closed real Lie subgroup.

Remark 1.13. It turns out that a closed subgroup of G is in fact a closed Lie subgroup. We will prove this later in the semester.

Definition 1.14 (quotient)**.** Fix a closed Lie subgroup $H \subseteq G$. Then G/H is a manifold with transitive G-action. If H is normal, then G/H is further a Lie group.

Remark 1.15. In general, if G acts transitively on a manifold X, then for any $x \in X$, $\mathrm{Stab}_G(x) \subseteq G$ is a closed Lie subgroup, and the quotient is isomorphic to X.

Remark 1.16. If G acts on a space X which is not transitive, then for any $x \in X$, the subset $Gx \subseteq X$ is at least an immersed submanifold.

Example 1.17. The group $\mathbb{R}$ has an action on $\mathbb{R}^2/\mathbb{Z}^2$ by $t\colon x \mapsto tx$. The orbit of (say), $(1/2, \sqrt{2}/2)$ is an immersed but not closed submanifold.

Definition 1.18 (representation)**.** Fix a Lie group G. A *representation* of a Lie group is a homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

Example 1.19. Let G act on itself by conjugation. Then each $g \in G$ acts on $T_1G \to T_1G$, so we receive an adjoint representation $\mathrm{Ad}_\bullet\colon G \to \mathrm{GL}(T_1G)$.

As usual, one can define morphisms of representations, subrepresentations, direct sums, duals, tensor products, irreducible representations, and so on. We also have a Schur's lemma.

Lemma 1.20. Fix irreducible representations V and W of G.

(a) Then a G-equivariant map $\varphi\colon V \to W$ is either zero or an isomorphism.

(b) Any G-equivariant map $A \to A$ is a scalar.

Proof. Omitted. ■

Definition 1.21 (unitary). A *unitary representation* is one admitting a G-invariant positive-definite Hermitian form.

Remark 1.22. Any unitary representation admits a decomposition into irreducible representations by taking orthogonal complements.

Non-Example 1.23. Let $B \subseteq \mathrm{GL}_2(\mathbb{C})$ be the subgroup of upper-triangular matrices. Then the standard representation of B does not admit a decomposition into irreducibles, so it cannot be made unitary.

Example 1.24. If G is finite, then any representation V admits a unitary structure: given any unitary structure $\langle -, - \rangle_0$, one can define an invariant unitary structure

$$\langle v, w \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle_0,$$

where dg is a choice of Haar measure.

Theorem 1.25 (Maschke). Fix a finite group G. Then all representations admit decomposition into irreducible representations.

Proof. This follows from Example 1.24. ■

1.1.2 Review of Lie Algebras

We now linearize our story.

Remark 1.26. Note that G acts on itself by left translations ℓ_g, so the tangent bundle TG can be given a global frame by the induced isomorphisms $d\ell_g \colon T_1 G \to T_g G$.

Notation 1.27. For each $a \in T_1 G$, we define the vector field L_a by

$$L_a \coloneqq ga \in T_a G.$$

Remark 1.28. One can check that all left-invariant vector fields take the form L_a.

Definition 1.29 (commutator). Fix a Lie group G. For each $a, b \in T_1 G$, we may take the commutator $[L_a, L_b]$ to produce another left-invariant vector field, which we label $L_{[a,b]}$.

Remark 1.30. The formalism of the commutator tells us that $[-,-]$ is antisymmetric and satisfies the Jacobi identity

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

Definition 1.31 (Lie algebra). Fix a vector space $\mathfrak{g}$ over a field k. Then a *Lie algebra* is such a vector space $\mathfrak{g}$ equipped with an antisymmetric pairing $[-,-] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

Example 1.32. For any Lie group G, we have seen that we may equip $\operatorname{Lie} G := T_1 G$ with the structure of a Lie group.

Example 1.33. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathfrak{g} = M_n(\mathbb{C})$, and one can check that $[X, Y] = XY - YX$.

We now define Lie subalgebras and morphisms of Lie algebras in the expected way.

Definition 1.34 (Lie ideal)**.** Fix a Lie algebra $\mathfrak{g}$. Then a *Lie ideal* $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace for which $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Example 1.35. For any closed Lie subgroup $H \subseteq G$, we see that $\operatorname{Lie} H \subseteq \operatorname{Lie} G$ is a Lie subalgebra. If H is normal, then $\operatorname{Lie} H$ is a Lie ideal.

As expected, there is some representation theory.

Definition 1.36. Fix a Lie algebra $\mathfrak{g}$ over a field k. Then a *representation* of $\mathfrak{g}$ is a morphism $\mathfrak{g} \to \mathfrak{gl}_n(k)$.

One can relate $\operatorname{Lie} G$ to G more directly via exponentiation.

Definition 1.37 (exponential)**.** Fix a Lie group G with Lie algebra $\mathfrak{g}$. We define a map $\exp\colon \mathfrak{g} \to G$ as follows. For each $a \in \mathfrak{g}$, one can check that the differential equation

$$\begin{cases} e'(t) = e(t) \cdot a, \\ e(0) = 1, \end{cases}$$

admits a unique solution; we then define $\exp(ta) := e(t)$. (This is independent of the choice of t.) It turns out that $t \mapsto \exp(ta)$ is a group homomorphism.

Example 1.38. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\exp\colon M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is the usual matrix exponential.

Remark 1.39. It turns out that $\exp$ is a local diffeomorphism (though not necessarily injective), so there is a local inverse $\log\colon U \to \mathfrak{g}$, where U is some open neighborhood of the identity.

Remark 1.40. For small a and b, it turns out that

$$\log(\exp(a)\exp(b)) = a + b + \frac{1}{2}[a, b] + \cdots,$$

where $\cdots$ denotes cubic terms. For example, if G is commutative, then we see that the Lie bracket $[-, -]$ vanishes; conversely, if $[-, -]$ vanishes, then G can be checked to commute in an open neighborhood of the identity, so G commutes.

1.1.3 Fundamental Theorems

In a first course, one checks the following two fundamental theorems.

Theorem 1.41. Fix a Lie group G. Then there is a bijection between connected closed Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \operatorname{Lie} G$.

Theorem 1.42. Fix Lie groups G and K, with G simply connected. Then taking the differential

$$\operatorname{Hom}(G, K) \to \operatorname{Hom}(\operatorname{Lie} G, \operatorname{Lie} K)$$

is an isomorphism.

There is a third fundamental theorem, which we will prove later.

Theorem 1.43. For any finite-dimensional Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$), then there is a Lie group G with $\operatorname{Lie} G \cong \mathfrak{g}$.

The three theorems provide an equivalence between the category of simply connected Lie groups and the category of Lie algebras, thereby classifying the former.

Remark 1.44. It follows that one may classify connected Lie groups as quotients of simply connected Lie groups by discrete central subgroups.

1.1.4 Representations of Lie Algebras

Let's start with the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 1.45. Fix the usual basis $e := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $f := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, and $h := [e, f]$ of $\mathfrak{sl}_2(\mathbb{C})$.

(a) Then all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ can be parameterized as $\{V_n\}_{n\geq 0}$, where V_n is the representation of homogeneous polynomials in x and y of degree n.

(b) Every representation is a direct sum of irreducible representations.

(c) Clebsch–Gordon rule: for any n and m, we have

$$V_n \otimes V_m = \bigoplus_{i=0}^{\min\{m,n\}} V_{|m-n|+2i}.$$

It will be helpful to turn representation theory of Lie algebras into a module category.

Definition 1.46 (universal enveloping algebra)**.** Fix a Lie algebra $\mathfrak{g}$. Then we define $U\mathfrak{g}$ as the quotient of the tensor algebra by the relation

$$[x, y] = x \otimes y - y \otimes x.$$

Remark 1.47. It turns out that $\operatorname{Rep} \mathfrak{g}$ is the same category as $\operatorname{Mod} U\mathfrak{g}$.

Even though we have taken a quotient by an inhomogeneous relation, $U\mathfrak{g}$ still receives a natural filtration by degree.

Theorem 1.48 (Poincaré–Birkhoff–Witt)**.** Fix a Lie algebra $\mathfrak{g}$, and equip $U\mathfrak{g}$ with the natural filtration. For any basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$, the ordered monomials in the basis form a basis of $U\mathfrak{g}$.

To continue our story, we need some adjectives for Lie algebras.

Definition 1.49 (solvable)**.** A Lie algebra $\mathfrak{g}$ is *solvable* if and only if the derived series eventually vanishes. Here, the derived series is defined inductively by $D^0(\mathfrak{g}) := \mathfrak{g}$ and $D^{n+1}(\mathfrak{g}) := [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$ for each $n \geq 0$.

Definition 1.50 (nilpotent)**.** A Lie algebra $\mathfrak{g}$ is *nilpotent* if and only if the lower central series eventually vanishes. Here, the derived series is defined inductively by $L_0(\mathfrak{g}) := \mathfrak{g}$ and $L_{n+1}(\mathfrak{g}) := [L_n(\mathfrak{g}), \mathfrak{g}]$ for each $n \geq 0$.

Remark 1.51. One can see that nilpotent implies solvable.

The representation theory of solvable Lie algebras is quite easy.

Theorem 1.52 (Lie)**.** Fix a finite-dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero.

(a) Then every irreducible representation of $\mathfrak{g}$ is one-dimensional.

(b) Every representation admits a basis on which $\mathfrak{g}$ acts by upper-triangular matrices.

Theorem 1.53 (Engel)**.** Fix a finite-dimensional Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ is nilpotent for all $X \in \mathfrak{g}$.

Thus, we see that we will want to ignore solvable and nilpotent pieces.

Definition 1.54 (radical)**.** Fix a Lie algebra $\mathfrak{g}$. Then the *radical* $\operatorname{rad}\mathfrak{g}$ is the sum of all solvable ideals of $\mathfrak{g}$.

Remark 1.55. One can check that $\operatorname{rad}\mathfrak{g}$ is a solvable ideal, so it is automatically the largest solvable ideal.

Definition 1.56 (semisimple)**.** Fix a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is *semisimple* if and only if $\operatorname{rad}\mathfrak{g} = 0$.

Remark 1.57. It turns out that

$$\mathfrak{g}_{\mathrm{ss}} := \frac{\mathfrak{g}}{\operatorname{rad}\mathfrak{g}}$$

is always semisimple. It turns out that the induced exact sequence splits, so there is a decomposition $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \ltimes \operatorname{rad}\mathfrak{g}$, which is known as the Levi decomposition; we will prove this later.

Having defined semisimple, we should define "simple."

Definition 1.58 (simple)**.** A Lie algebra $\mathfrak{g}$ is *simple* if and only if its only ideals are 0 and $\mathfrak{g}$.

Remark 1.59. One can check that semisimple Lie algebras are precisely the sums of simple Lie algebras.

It turns out to be convenient to allow a little radical.

Definition 1.60 (reductive)**.** A Lie algebra $\mathfrak{g}$ is *reductive* if and only if its radical is its center.

Example 1.61. One can check that $\mathfrak{sl}_n(\mathbb{C})$ is simple, and $\mathfrak{gl}_n(\mathbb{C})$ is reductive.

To test for a Lie algebra being semisimple (and other adjectives), we introduce the Killing form.

Definition 1.62 (Killing form)**.** Fix a Lie algebra $\mathfrak{g}$. Then we define the *Killing form* by

$$K(x, y) := \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y).$$

Remark 1.63. One can check that K is $\mathfrak{g}$-invariant.

Theorem 1.64 (Cartan criteria)**.** Fix a Lie algebra $\mathfrak{g}$.

(a) $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subseteq K$.

(b) $\mathfrak{g}$ is semisimple if and only if K is non-degenerate.

Proposition 1.65. A Lie algebra $\mathfrak{g}$ is reductive if and only if it admits a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for which the bilinear form

$$B_V(X, Y) \coloneqq \operatorname{tr}(\rho_X \circ \rho_Y)$$

is non-degenerate.

We may as well state one of the main theorems of our representation theory.

Theorem 1.66. Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.

1.1.5 Structure Theory of Lie Algebras

Here is another piece of structure theory.

Definition 1.67 (adjoint)**.** Fix a semisimple Lie algebra $\mathfrak{g}$. Then we define the *adjoint* Lie group G^{ad} by $G^{\mathrm{ad}} \coloneqq \operatorname{Aut}(\mathfrak{g})^\circ \subseteq \operatorname{GL}(\mathfrak{g})$.

Remark 1.68. It turns out that $\operatorname{Lie} G^{\mathrm{ad}} = \mathfrak{g}$.

In our setting, one can generalize the Jordan decomposition.

Definition 1.69 (semisimple, nilpotent)**.** An element $X \in \mathfrak{g}$ is *semisimple* or *nilpotent* if and only if the operator $\operatorname{ad}_X X$ is.

Theorem 1.70. Fix a Lie algebra $\mathfrak{g}$. Then any $X \in \mathfrak{g}$ can be written uniquely as a sum of a semisimple and nilpotent element.

Remark 1.71. It turns out that semisimple elements always act semisimply on representations, and nilpotent elements always act nilpotently on representations.

The notion of semisimple elements is important to define Cartan subalgebras.

Definition 1.72 (Cartan)**.** Fix a semisimple Lie algebra $\mathfrak{g}$. Then a *Cartan subalgebra* is a maximal commutative subalgebra of

Proposition 1.73. Fix a semisimple Lie algebra $\mathfrak{g}$. All Cartan subalgebras are conjugate by G^{ad}.

Definition 1.74. Fix a semisimple Lie algebra $\mathfrak{g}$. Then the *rank* of $\mathfrak{g}$ is the dimension of the Cartan subalgebras.

A choice of Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ produces a root decomposition, which we write as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.$$

Definition 1.75 (root system)**.** Fix a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then the *root system* of $\mathfrak{g}$ consists of those nonzero eigenvalues $\alpha \in \mathfrak{h}^*$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We write $\mathfrak{g}_\alpha$ for this eigenspace, and we write $\Phi(G)$ for the root system.

Remark 1.76. One can check that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{[\alpha,\beta]}$. In fact, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal for the Killing form except when $\alpha = -\beta$, where it is a perfect pairing.

Remark 1.77. It turns out that $\dim \mathfrak{g}_\alpha = 1$ for each α. It follows that

$$\#\Phi(\mathfrak{g}) = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}.$$

Remark 1.78. There are the usual pictures of root systems of various types.

1.1.6 Root Systems

It is useful to write down what properties are satisfied by these root systems.

Definition 1.79 (root system)**.** Fix a Euclidean space E. Then a finite subset $\Phi \subseteq E$ is a *root system* if and only if

(a) Φ spans E,

(b) for each $\alpha, \beta \in \Phi$, the number

$$n_{\alpha\beta} := \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer,

(c) for each $\alpha, \beta \in \Phi$, the reflection

$$s_\alpha(\beta) := \beta - n_{\alpha\beta}\beta$$

is in Φ.

We say that Φ is *reduced* if and only if $\alpha \in \Phi$ implies that $2\alpha \notin \Phi$.

Definition 1.80. A root system Φ is *reducible* if and only if it can be written as a disjoint union of root systems coming from a decomposition of the Euclidean space into a product of Euclidean spaces.

The reflections are important enough to be placed into a group.

Definition 1.81 (Weyl group)**.** Fix a root system $\Phi \subseteq E$. Then the *Weyl group* W is the subgroup of $\mathrm{GL}(E)$ generated by the reflections.

Example 1.82. The Weyl group associated to the root system of $\mathfrak{sl}_{n+1}$ consists of the permutation matrices in $\mathrm{GL}_n(\mathbb{R})$. Indeed, each reflection corresponds to a transposition. This root system is said to be of type A_n, where n refers to the rank.

Example 1.83. The root system associated to $\mathfrak{so}_{2n+1}$ is B_n. The root system associated to $\mathfrak{sp}_{2n}$ is C_n. Lastly, the root system associated to $\mathfrak{so}_{2n}$ is D_n.

Remark 1.84. There are also various exceptional reduced root systems, which we may say something about later.

We can even break down irreducible root systems into more controlled pieces.

Definition 1.85 (positive)**.** Fix a root system $\Phi \subseteq E$. For a choice of $t \in E$ for which $(t, \alpha) \neq 0$ for all $\alpha \in E$, we say that a root in Φ is *positive* if and only if $(t, \alpha) > 0$. Similarly, α is *negative* if and only if $(t, \alpha) < 0$. We let Φ^+ and Φ^- denote the sets of positive and negative roots, respectively.

Definition 1.86. Fix a root system $\Phi \subseteq E$. A positive root is *simple* if and only if it is not a sum of other positive roots (with positive integer coefficients). We let Π denote the set of simple roots.

Proposition 1.87. Fix a root system $\Phi \subseteq E$. Then Π is a basis, and every positive root $\alpha \in \Phi^+$ can be written as a unique sum of elements of Π with positive integer coefficients.

Each root system also admits a dual.

Definition 1.88 (dual root system)**.** Fix a root system $\Phi \subseteq E$. Then we define the *dual root system* $\Phi^\vee \subseteq E^\vee$ to be given by the points

$$\alpha^\vee = \frac{2(\alpha, -)}{(\alpha, \alpha)}$$

for each $\alpha \in \Phi$.

Remark 1.89. The reduced root system B_n is dual to C_n.

It will be helpful to have some lattices from our root systems.

Definition 1.90. Fix a root system $\Phi \subseteq E$.

- The *root lattice* Q is spanned by ϕ.
- The *coroot lattice* $Q^\vee$ is spanned by the $\alpha^\vee$.
- The lattice $P \subseteq E$ is $(Q^\vee)^*$.
- The *weight lattice* $P^\vee \subseteq E^*$ is Q^*.

In general, $Q \subseteq P$, but equality does not have to hold.

Example 1.91. For $\mathfrak{sl}_n$, the quotient P/Q is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

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List of Definitions