18.757: Representation Theory of Lie Groups

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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Introduction

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Welcome to the class.

1.1.1 Administrative Notes

Here are some administrative notes.

- There will be problem sets every two weeks, due on Fridays. They are not expected to be too time-consuming.
- Technically, this course is a sequel to 18.745–18.755, but one can get away with a bit less. In particular, we will assume familiarity with some basic notions in Lie theory, things about simple complex Lie algebras (as related to compact Lie groups), the theory of roots and weights, and this theory of finite-dimensional representations. For example, things like the Peter–Weyl theorem may come up.
- We will largely follow Etingof's lecture notes.

This course is about the representations of Lie groups, especially those which are not necessarily compact. For example, we may focus on real reductive Lie groups such as $\mathrm{SL}_n(\mathbb{R})$, and there is a new feature here that we must care about infinite-dimensional representations.

One of our motivations comes from quantum physics, where one finds groups acting on infinite-dimensional Hilbert spaces. Another motivation is number-theoretic: one uses this theory to set up the archimedean theory of automorphic forms.

1.1.2 Finite Groups

Let's recall some background. As one does, let's begin with the representation theory of finite groups. We split this into a few theorems.

> **Theorem 1.1** (Maschke)**.** Let G be a finite group. Then every finite-dimensional complex representation is unitary (by averaging any given Hermitian form) and semisimple.

Theorem 1.2 (Peter–Weyl)**.** Let G be a finite group. Then there is a decomposition

$$\mathbb{C}[G] = \bigoplus_{V \in \mathrm{IrRep}(G)} \mathrm{End}_{\mathbb{C}}(V)$$

of $\mathbb{C}$-algebras. It follows that the characters $\{\mathrm{tr}_V\}_{V \in \mathrm{IrRep}(G)}$ form an orthonormal basis of the class functions $G \to \mathbb{C}$.

Remark 1.3. Let G be a finite group. For $f \in \mathbb{C}[G]$, there is a dimension formula

$$f(1) = \frac{1}{|G|} \sum_{V \in \mathrm{IrRep}(G)} \dim V_i \cdot \mathrm{tr}_{V_i} f.$$

Indeed, this follows from writing $\langle \varphi, \psi \rangle = \sum_V \mathrm{tr}_V(\varphi\psi')$ (where $\psi' \colon g \mapsto \psi(g^{-1})$) and then noting that $\langle \varphi, \psi' \rangle = \frac{1}{|G|}(\varphi * \psi')(1)$.

1.1.3 Compact Groups

We now move up to the representation theory of compact connected Lie groups. Here is a generalization of Maschke's theorem.

Theorem 1.4 (Weyl's unitarian trick)**.** Let G be a compact Lie group. Then the representations of G are semisimple, and the irreducible representations of G are finite-dimensional and unitary.

In order to compute the representations of G, one wants to pass to the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. One needs to be slightly careful about this.

Proposition 1.5. Fix a Lie group G with Lie algebra $\mathfrak{g}$. Consider the functor $F \colon \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$ sending a representation ρ to the differential representation $d\rho_1$.

(a) If G is connected, then F is fully faithful.

(b) If G is connected and simply connected, then F is also essentially surjective and hence an equivalence.

Example 1.6. Take $G = \mathrm{U}(1)$. Because G is abelian, all irreducible representations are one-dimensional. These representations $\mathrm{U}(1) \to \mathbb{C}^\times$ are indexed by $n \in \mathbb{Z}$, given by $z \mapsto z^n$. Note that these are not in bijection with the representations of $\mathrm{Lie}\, G$ because G is not simply connected!

Example 1.7. Take $G = \mathrm{SU}(n)$ so that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. A weight is a character of the maximal torus T, for which we can take to be the subgroup of diagonal matrices. Explicitly,

$$T = \{\mathrm{diag}(z_1, \ldots, z_n) : z_1 \cdots z_n = 1\},$$

so the weight lattice is $\mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1)$, and a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is dominant when the entries are increasing.

Example 1.8. For example, with $\mathrm{SU}(2)$, we have an isomorphism $\mathbb{Z}^2/\mathbb{Z}(1,1) \to \mathbb{Z}$ given by $(\lambda_1, \lambda_2) \mapsto \lambda_2 - \lambda_1$, and the dominant weights are the nonnegative integers. One finds that weight n corresponds to the nth symmetric power of the standard representation of $\mathrm{SU}(2)$.

Here is a generalization of the Peter–Weyl theorem.

Theorem 1.9 (Peter–Weyl)**.** Let G be a compact Lie group. The canonical map

$$\bigoplus_{\text{dominant } \lambda} V_\lambda \otimes V_\lambda^* \to C^\infty(G)$$

is an embedding with dense image; here V_λ refers to the irreducible representation corresponding to the dominant weight λ.

Example 1.10 (Fourier analysis)**.** With $G = \mathrm{U}(1)$, then one can calculate that $V_n \otimes V_n^* \to C^\infty(G)$ has image given by the nth power map $\mathrm{U}(1) \to \mathrm{U}(1)$. Thus, we are asserting that the collection of such polynomials are dense in the collection of all smooth functions $\mathrm{U}(1) \to \mathbb{C}$. If we identify $\mathrm{U}(1)$ with $\mathbb{R}/\mathbb{Z}$ via the exponential map, then this is asserting that the exponentials $z \mapsto e^{2\pi i n z}$ have dense span in the collection of all smooth functions $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$.

One can characterize the image of the Peter–Weyl map.

Definition 1.11 (finite)**.** A function $f \in C^\infty(G)$ is G-*finite* if and only if the span of $\{gf : g \in G\}$ is finite-dimensional. We let $C_{\mathrm{fin}}(G)$ be the space of G-finite functions.

Remark 1.12. It turns out that the map

$$\bigoplus_{\text{dominant } \lambda} V_\lambda \otimes V_\lambda^* \to C^\infty(G)$$

has image given by the space of G-finite functions. (This is often proven as an input to the proof of the Peter–Weyl theorem; it is much easier to show!)

Remark 1.13. The vector space $C_{\mathrm{fin}}(G)$ has two ring structures: there is pointwise multiplication (in $\mathbb{C}$) and also convolution given by

$$(\varphi * \psi)\colon g \mapsto \int_G \varphi(x)\psi\left(x^{-1}g\right)\,dx,$$

where dx is a Haar measure for G normalized so that $\int_G dx = 1$. (Because G is compact, dx is bi-invariant.)

Remark 1.14. The convolution algebra is non-unital, so one sometimes upgrades to the algebra of distributions, where we have the unit δ_1. Similar remarks hold for $C^\infty(G)$ and even $C(G)$.

Remark 1.15 (algebraic groups)**.** In fact, $C_{\mathrm{fin}}(G)$ also has a comultiplication given by pulling back along the multiplication map $m\colon G \times G \to G$. Namely, the comultiplication is the composite

$$C_{\mathrm{fin}}(G) \stackrel{m^*}{\to} C_{\mathrm{fin}}(G \times G) = C_{\mathrm{fin}}(G) \otimes C_{\mathrm{fin}}(G).$$

Thus, we have a Hopf algebra, which allows us to associate a complex algebraic group G_{alg} to G, and it turns out that unitary representations of G all arise from algebraic representations of G_{alg}. Conversely, one can take a complex algebraic group G_{alg} and then find a maximal compact subgroup $G \subseteq G_{\mathrm{alg}}(\mathbb{C})$ which is unique up to conjugacy.

1.1.4 Unitary Representations

We will be interested in the unitary representations of Lie groups G, which we no longer assume to be compact.

Remark 1.16. If G is simple and not compact, then all unitary representations are infinite-dimensional. Proceeding by contraposition, suppose that G is simple and admits a finite-dimensional unitary representation $\rho\colon G \to \mathrm{SU}(n)$. Because G is simple, this is an embedding, and because $\mathrm{SU}(n)$ is compact, we conclude that G must then also be compact.

Thus, we see that we will be interested in infinite-dimensional representations. Of course, one still must add in topologies everywhere, though this point is more technical now that our vector spaces are not finite-dimensional. For example, for unitary representations, we are looking for actions of G on Hilbert spaces, though we will find occasion to look at more general topological vector spaces.

Our main source of examples of representations arise from more general group actions.

Example 1.17. If G acts on a "geometric space" X, then we receive an induced action of G on classes of functions on X. For example, when X is a reasonably nice topological space, then we can think about G acting on $L^2(X)$; when X is a manifold, we can think about G acting on $C^\infty(X)$.

Example 1.18. The action of G on G itself by left multiplication gives rise to some "regular" representations.

Example 1.19. Many matrix groups such as $\mathrm{SL}(n, \mathbb{R})$ admit standard group action on a vector space. Note that this standard action may not be unitary!

Let's begin building our language.

Definition 1.20 (subrepresentation, irreducible)**.** Fix a group G and topological vector space V. If $\rho\colon G \to \mathrm{GL}(V)$ is a continuous representation, then a *subrepresentation* is a closed subspace $W \subseteq V$ which is G-invariant. We say that ρ is *irreducible* if and only if there are no proper nontrivial subspaces.

Note the hypothesis that $W \subseteq V$ is closed for our subrepresentations!

Example 1.21. The action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathbb{R}^n$ has no nontrivial proper subrepresentations and hence is irreducible. However, this representation is not unitary.

Remark 1.22. The action of G on itself makes $L^2(G)$ a representation of G. However, this representation frequently fails to be irreducible. For example, $L^2(G)$ has many automorphisms, so it cannot be irreducible by a suitable version of Schur's lemma. In some cases, we can see this more concretely: taking $G = \mathbb{R}$, then we know $\mathbb{R}$ is isomorphic to its dual, so $L^2(\mathbb{R}) \cong L^2\left(\mathbb{R}^{2\vee}\right)$, and this right-hand side has more obvious subrepresentations given by the subspace of functions which vanish on a given subset of positive measure.

1.2 September 9

Today, we will discuss some general nonsense of topological vector spaces.

1.2.1 Examples of Representations

Last time, we ended with the following example, which we recall here.

Example 1.23 (Heisenberg group)**.** Fix a positive integer $n \geq 1$. Then we define the Heisenberg group H_n as the matrix group

$$H_n := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbb{R}^{1\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R} \right\}.$$

It turns out that H_n admits a natural action on $L^2(\mathbb{R}^n)$, which is an irreducible representation. Quickly, the a-coordinate will act by translation on the $\mathbb{R}^n$, and the b-coordinate will act by a character $b \mapsto e^{2\pi i\langle b,-\rangle}$. One finds a similar action on $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.

Check it

Remark 1.24. There is a finite analogue, where all the $\mathbb{R}$s are replaced with a finite field $\mathbb{F}_p$, and the character $b \mapsto e^{2\pi i\langle b,-\rangle}$ is replaced with $b \mapsto e^{2\pi i\langle b,-\rangle/p}$. Equivalently, we find $H_n(\mathbb{F}_p) = \mathbb{F}_p \times V \times V^\vee$ (where $V = \mathbb{F}_p^n$); then $H_n(\mathbb{F}_p)$ admits a natural action on V, where V acts by translation, and $\xi \in V^\vee$ acts by multiplying with the function $\psi_\xi(x) := e^{2\pi i\langle x,\xi\rangle/p}$. The central $\mathbb{F}_p \subseteq H$ now acts by scalar multiplication as $a \mapsto e^{2\pi i a/p}$.

Let's check that this representation is irreducible. It is enough to check that $\operatorname{End}_{H_n}(\mathbb{C}[V])$ is $\mathbb{C}$. Well, commuting with the V leaves us with

$$\operatorname{End}_V(\mathbb{C}[V]) = \operatorname{End}_{\mathbb{C}[V]}(\mathbb{C}[V]) = \mathbb{C}[V].$$

Further, one sees that commuting with the scalar multiplication by elements in $V^\vee$ restricts the possible endomorphisms all the way down to scalars.

Example 1.23 appears in the book, and the argument is not too different from the one given in the remark.

Example 1.25. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ and therefore has a unitary representation on $L^2(\mathbb{R}^2)$. This cannot possibly be irreducible because $\mathrm{SL}_2(\mathbb{R})$ commutes with the extra scalar action on $\mathbb{R}^2$, so $L^2(\mathbb{R}^2)$ has too many endomorphisms. To make this representation smaller, we can choose $s \in \mathbb{C}$, which produces a character on $\mathbb{R}^+$ by $\chi_s \colon t \mapsto t^s$; then we can define $L^2(\mathbb{R}^2, \chi_s)$ to be the functions with commute with this character (namely, $f(t^{-1}x) = t^s f(x)$). (Geometrically, this is basically the sections of a line bundle on $\mathbb{RP}^1$ given by the character.) We will soon see that almost all s produces an irreducible representation.

For example, for $s = 0$, then $L^2(\mathbb{R}^2, \chi_0)$ consists of the functions on $\mathbb{RP}^1$. This is not irreducible because it has a subrepresentation given by the constant functions. But $L^2(\mathbb{R}^2, \chi_0)/(\mathbb{C}\cdot 1)$ is still not irreducible: it turns out to be the sum of two irreducible representations L^+ and L^-, where L^+ is the closure of $z\mathbb{C}[z]$, and L^- is the closure of $z^{-1}\mathbb{C}[z^{-1}]$, where z is a standard coordinate on $\mathbb{RP}^1$. This can be related to Fourier series by embedding $\mathbb{RP}^1$ into $\mathbb{CP}^1$, which is basically a circle. We will prove all these claims later.

Remark 1.26. Here is an amusing way to view $L^2(\mathbb{R}^2, \chi_s)$: this amounts to sections of a line bundle on $\mathbb{RP}^1$, and after removing ∞, we see that we are looking at functions on $\mathbb{R}$. One can check that these are the functions which transform by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} f(z) = f\left(\frac{az+b}{cz+d}\right) |cz+d|^s.$$

Fix a Lie group G acting on an orientable manifold X. If G preserves a volume form dx on X, then $C_c^\infty(X)$ will have an invariant pairing

$$\langle \varphi, \psi \rangle := \int_X \varphi(x)\overline{\psi(x)}\,dx.$$

More generally, one can work with half-densities.

Definition 1.27 (density)**.** An s-density on a smooth manifold X is a section of a line bundle whose sections on an affine patch are just functions but which has transformations between coordinate charts $(x_n) \mapsto (x_n')$ given by

$$f(x_1, \ldots, x_n) \mapsto f(x_1', \ldots, x_n') \left| \det \left(\frac{\partial x_i'}{\partial x_j} \right) \right|^s.$$

The point of working with half-densities is that we can define the standard inner product between them in the usual way.

Example 1.28. Half-densities of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{RP}^1$ (in the obvious way) amounts to considering $L^2(\mathbb{R}^2, \chi_{-1})$.

1.2.2 Topological Vector Spaces

We spend a moment reviewing what we need about locally convex topological spaces; we refer to Appendix A for a more in-depth treatment.

Convention 1.29. All topological vector spaces are over $\mathbb{C}$ and are Hausdorff.

Definition 1.30 (locally convex)**.** A topological vector space V is *locally convex* if and only if 0 has an open neighborhood basis of convex sets.

Remark 1.31. Equivalently, by Corollary A.23 a topological vector space V if and only if its topology is generated by a collection of seminorms.

All representations in this class will actually be given by "Fréchet spaces."

Definition 1.32 (Fréchet)**.** A topological vector space V is *Fréchet* if and only if it is locally convex, has a countable basis of neighborhoods of 0, and is sequentially complete.

Remark 1.33. By Proposition A.25, having a countable basis of neighborhoods of 0 is equivalent to being metrizable. (In fact, one can choose the metric to be translation-invariant.) Once V is metrizable, being sequentially complete is equivalent to being complete.

We will also frequently take our vector spaces V to be separable.

Definition 1.34 (separable)**.** A topological space X is *separable* if and only if it admits a countable basis.

Convention 1.35. In this course, all Fréchet spaces are separable unless otherwise specified.

Non-Example 1.36. For $p < 1$, the space $L^p([0, 1])$ fails to be locally convex. In fact, the only open nonempty convex subspace is the whole space!

Example 1.37. For a topological space X, let $C(X)$ be the space of continuous functions, where the topology is given by uniform convergence. If X is (Hausdorff) compact, then $C(X)$ is a Banach space (given by $\|\cdot\|_\infty$). However, if X is merely a (Hausdorff, second countable) locally compact topological space, then $C(X)$ is merely a Fréchet space: write X as a countable union $\bigcup_i K_i$ of compact sets, and then we can use the seminorms $\|\cdot|_{K_i}\|_\infty$.

Example 1.38. If X is a manifold, then we can consider the topological space $C^k(X)$. (The topology is given by uniform convergence of the first k derivatives.) Similarly, we see that $C^k(X)$ is Banach when k is finite and X is compact; otherwise, it is merely Fréchet. The same sort of argument shows that the space $S(\mathbb{R}^n)$ of Schwartz functions is Fréchet.

Non-Example 1.39. If X fails to be compact, then the subspace $C_c^\infty(X)$ of $C^\infty(X)$ is not a Fréchet space because it fails to be complete.

Sometimes, we will find ourselves in a circumstance where we can restrict to a nice class of spaces.

Definition 1.40 (Banach)**.** A topological vector space V is a *Banach space* if and only if its topology is given by a norm, and it is complete with respect to that norm.

For example, Hilbert spaces are Banach spaces.

Here is one benefit of working with a Banach space.

Lemma 1.41. Fix a topological group G acting on a Banach space V. Then the action $G \times V \to V$ is continuous if and only if the induced map $\rho\colon G \to \operatorname{Aut}(V)$ is continuous in the strong topology, in which $\{E_i\} \to E$ if and only if $\{E_i v\} \to Ev$ converges for all $v \in V$.

Proof. The forward direction has little content: given a net $\{g_i\} \to g$, we know that $\rho(g_i)v \to \rho(g)v$ for each v by continuity, so it follows that $\rho(g_i) \to \rho(g)$.

The reverse direction follows from the Uniform boundedness principle, which claims that $|E_i v|$ being bounded for every v implies that $\|E_i\|$ is bounded. Namely, to show that $G \times V \to V$ is continuous, we choose a net $\{(g_i, v_i)\} \to (g, v)$ in $G \times V$, and we would like to show that $g_i v_i \to gv$. By translation, we may assume that $g = 1$. We are given that $\rho(g_i) \to 1$ in the strong topology, which implies that $\rho(g_i)w \to w$ for all w, so we may also assume that $v = 0$ by translation. Because now $v_i \to 0$, it is enough to show that $\|\rho(g_i)\|$ is bounded, which is what follows from the Uniform boundedness principle. ■

Remark 1.42. A Banach space V can alternatively give $\operatorname{End} V$ the norm topology, but then the map $G \to \operatorname{Aut} V$ need not be continuous with the norm topology. For example, the action of $\mathbb{R}$ on $L^2(\mathbb{R})$ by translation $T_a f(x) \coloneqq f(x+a)$ fails to be continuous: as $a \to 0$, we see $T_a \to \mathrm{id}$, but $\|T_a - \mathrm{id}\| = 1$ for all $a \neq 0$.

1.2.3 Measures on a Space

Let's say a bit about measures.

Remark 1.43. Given a locally compact second countable topological space X, then $C(X)$ has topological dual $C(X)^*$, which is thought of as the (compact) measures on X. Given a measure μ, we can define its support $\operatorname{supp}\mu$ by having $x \notin \operatorname{supp}\mu$ if and only if there is an open neighborhood U of x for which $\mu(f) = 0$ for all f with $f|_U = 0$. It follows that $\operatorname{supp}\mu$ is closed, and one can even check that it is compact by construction of $C(X)^*$. If X is an orientable manifold, then we remark that the Poincaré pairing $f \mapsto \int_X f\varphi\,\omega$ defines an embedding $C_c(X) \to C(X)^*$.

Non-Example 1.44. Continuing with Remark 1.43, we can take the discrete space $X = \mathbb{N}$. Then $C(X)$ is given by sequences in $\mathbb{C}$, but $C(X)^*$ is given by finite sequences in $\mathbb{C}$, and we have some convergence $\sigma^i \to \sigma$ if and only if there is a finite subset $S \subseteq \mathbb{N}$ containing all the supports, and we have pointwise convergence in S. One can check that this is separable, sequentially complete, but it is not complete and thus not a Fréchet space!

Remark 1.45. One can replace the topology on $C(X)^*$ with the weak-$*$ topology, in which $\{\mu_i\} \to \mu$ if and only if $\{\mu_i(f)\} \to \mu(f)$ for all f. However, $C(\mathbb{N})^*$ continues to not be complete: it embeds into the space of all linear maps $C(\mathbb{N}) \to \mathbb{C}$, but it is not a closed subset of this space.

Next class, we will take G to be a Lie group, and we will find that compact measures on G is an algebra, and it acts continuously on our continuous representations. The point is that it more or less plays the role of the group algebra.

Appendix A

Functional Analysis

In this appendix, we introduce the small amount of functional analysis we will need in order to get going with infinite-dimensional vector spaces. In other words, we need to set up the theory of Fréchet spaces. Throughout this appendix, $\mathbb{F}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$. Our exposition is largely stolen from [Con90].

A.1 Locally Convex Spaces

We begin with the following definition.

Definition A.1 (topological vector space)**.** Fix a topological field k. Then a *topological vector space* is a vector space V over k equipped with a topology so that addition map $+\colon V \times V \to V$ and scalar multiplication map $\cdot\colon k \times V \to V$ are both continuous. In these notes, all topological vector spaces will be assumed to be Hausdorff.

A Fréchet space will be a complete topological vector space admitting two notable definitions: having its topology is defined by a countable family of seminorms, or being locally convex and metrizable. As such, let's quickly recall the definition of a seminorm.

Definition A.2 (seminorm)**.** Fix a vector space V over $\mathbb{F}$. Then a *seminorm* is a function $p\colon V \to \mathbb{R}$ satisfying the following.

- Subadditive: we have $p(x+y) \le p(x) + p(y)$ for any $x, y \in V$.
- Homogeneous: we have $p(\lambda x) = |\lambda|\, p(x)$ for any $x \in V$ and $\lambda \in \mathbb{F}$.

Remark A.3. The homogeneity implies that $p(0) = 0$, which is sometimes included in the definition. Similarly, the subadditivity now implies that $2p(x) = p(x) + p(-x)$ is at least $p(0) = 0$, so p is automatically nonnegative; this is also sometimes included in the definition.

It is worthwhile to have a few ways to check continuity.

Lemma A.4. Fix a topological vector space V over $\mathbb{F}$, and let $p\colon V \to \mathbb{R}$ be a seminorm. Then the following are equivalent.

(i) p is continuous.

(ii) $\{v : p(v) < 1\}$ is open.

(iii) p is continuous at 0.

Proof. This is [Con90, Proposition 1.3]. Of course (i) implies (ii). We show the remaining implications independently.

- We show (ii) implies (iii). For any net $\{x_i\}$ converging to 0, we want to show that $\{p(x_i)\} \to 0$, which is true because any x_i in the open neighborhood εU of 0 has $p(x_i) < \varepsilon$.
- We show (iii) implies (i). Note that any net $\{x_i\}$ converging to some x has

$$|p(x_i) - p(x)| \le p(x_i - x),$$

and $p(x_i - x) \to 0$ because $x_i - x \to 0$. ■

We now start talking about convex sets, but we will relate our definitions back to seminorms.

Definition A.5 (convex)**.** Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *convex* if and only if any two $a, b \in A$ has

$$ta + (1-t)b \in A$$

for any $t \in [0, 1]$.

Example A.6. Let $p\colon V \to \mathbb{R}$ be a seminorm. Then we claim that $A := \{v \in V : p(v) < 1\}$ is convex. Indeed, for $a, b \in A$ and $t \in [0, 1]$, we see that

$$p(ta + (1-t)b) = tp(a) + (1-t)p(b),$$

which is still less than 1, so $ta + (1-t)b \in A$.

Example A.7 (convex hull)**.** For any subset $A \subseteq V$, we may define the convex hull

$$\operatorname{conv}(A) := \left\{ \sum_{i=1}^{n} t_i a_i : \{a_i\}_i \subseteq A, \{t_i\}_i \in [0,1], t_1 + \cdots + t_n = 1 \right\}.$$

Note that $\operatorname{conv}(A)$ is convex: for two points $\sum_i t_i a_i$ and $\sum_j s_j b_j$ and $t \in [0, 1]$, the sum $\sum_i t t_i a_i + \sum_j (1 - t) s_j b_j$ still has $\sum_i t t_i + \sum_j (1-t)s_j = t + (1-t) = 1$. In fact, if B is convex and contains A, then $\operatorname{conv}(A) \subseteq B$ because the sums $\sum_i t_i a_i$ can be checked to be in B by induction.

Convex sets on their own turn out to not be good enough for our purposes, so we will need extra adjectives.

Definition A.8 (balanced)**.** Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *balanced* if and only if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{F}$ such that $|\lambda| \le 1$.

Example A.9. Let $p\colon V \to \mathbb{R}$ be a seminorm. Then we claim that $A := \{v \in V : p(v) < 1\}$ is balanced. Indeed, for $a \in A$ and λ with $|\lambda| \le 1$, we see that $p(\lambda a) = |\lambda|\, p(a)$, which is still less than 1, so $\lambda a \in A$.

Example A.10. For any subset $A \subseteq V$, the subset

$$\operatorname{bal}(A) := \bigcup_{|\lambda| \leq 1} \lambda A$$

is balanced. Indeed, for any μ with $|\mu| \leq 1$, we see that $\mu \operatorname{bal}(A) = \bigcup_\lambda \mu\lambda A$ is contained in $\operatorname{bal}(A)$ because $|\lambda\mu| \leq 1$ whenever $|\lambda| \leq 1$. Of course, we always have $A \subseteq \operatorname{bal}(A)$, and one can see that any balanced set containing A must contain each λA and hence contain $\operatorname{bal}(A)$.

It turns out that passing to balanced convex sets is not too big of a burden.

Lemma A.11. Fix a topological vector space V over $\mathbb{F}$. Any convex open neighborhood of 0 contains a balanced convex open neighborhood of 0.

Proof. Let U be a convex open neighborhood of 0. The point is to use the continuity of scalar multiplication: the continuity of

$$\cdot \colon \mathbb{F} \times V \to V$$

provides a basic open neighborhood $B(0,\varepsilon) \times U'$ of $(0,0)$ of $\mathbb{F} \times V$ such that $B(0,\varepsilon)U' \subseteq U$. We claim that $\operatorname{conv}(B(0,\varepsilon)U')$ is the desired open neighborhood. Here are our checks; set $U'' := B(0,\varepsilon)U'$ for brevity.

- Because U is already convex, Example A.7 explains that $U'' \subseteq U$ implies that $\operatorname{conv}(U'') \subseteq U$.
- Convex: note $\operatorname{conv}(U'')$ is convex by Example A.7.
- Open: because scalar multiplication by a nonzero number is a homeomorphism, we see that $U'' := B(0,\varepsilon)U'$, which is

  $$\bigcup_{0<|\lambda|<\varepsilon} \lambda U',$$

  is open. Then once U' is open, we see that $\operatorname{conv}(U')$ can be written as a union

  $$\bigcup_{n\geq 1} \left(\bigcup_{t_1+\cdots+t_n=1} t_1U'' + \cdots + t_nU'' \right),$$

  which is open because the sum of two open subsets is open (indeed, the sum of open sets is a union of translates of just one of the open sets).
- Balanced: the previous step realized U'' as a union $\bigcup_{0<|\lambda|<\varepsilon} \lambda U'$, which can be shown to be balanced exactly as in Example A.10. ■

Remark A.12. Here is a sample application: suppose that 0 has a neighborhood basis of convex sets. Then Lemma A.11 implies that 0 also admits a neighborhood basis of balanced convex sets: for each U in the neighborhood basis, the lemma produces a smaller open subset which is still convex but now also balanced. Similarly, admitting a countable neighborhood basis of convex sets can be upgraded to admitting a countable neighborhood basis of balanced convex sets.

The previous remark allows us to make the following definition.

Definition A.13 (locally convex)**.** A topological vector space V is *locally convex* if and only if 0 admits a neighborhood basis of convex sets.

Remark A.14. By translation, it is equivalent to require V to have a basis of convex sets. (Namely, if $\mathcal{U}$ is the neighborhood basis of 0, then $\bigcup_{v\in V} v+\mathcal{U}$ is the basis of V.) By Remark A.12, we can also upgrade these notions to having bases of balanced convex sets.

It is notable that the previous two definitions avoid the mention of any topology. In order to continue not doing any topology, we pick up the following definition, which provides a linear algebraic stand-in for "contains an open neighborhood of the origin."

Definition A.15 (absorbing). Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *absorbing* if and only if any $v \in V$ admits some $\varepsilon > 0$ such that $tv \in A$ for all $t \in [0, \varepsilon)$.

For example, we see that 0 is contained in any absorbing subset.

Remark A.16. Of course, if A is absorbing, and $A \subseteq B$, then B is absorbing: for each v, the ε which worked for A continues to work for B.

Example A.17. Let's explain the remark given before the definition. If V is a topological vector space over $\mathbb{F}$, then we claim that any open neighborhood U of 0 is absorbing. This will follow by the continuity of scalar multiplication: for any $v \in V$, the map $\mathbb{R} \to V$ given by $t \mapsto tv$ is continuous. Thus, because $0 \in U$, there must be $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)v \subseteq U$.

Example A.18. Let $p\colon V \to \mathbb{R}$ be a seminorm, and set $A := \{v \in V : p(v) < 1\}$. Then we claim that $(-a) + A$ is absorbing for all $a \in A$; for example, setting $a = 0$ will imply that A is absorbing. Now, for any $v \in V$, we need to show that $a + tv \in A$ for small t. Well, $p(a + tv) \le p(a) + |t|\, p(v)$, so taking any t with $p(v)\,|t| < (1 - p(a))$ will do. (In particular, any t will work if $p(v) = 0$.)

We are now ready to construct some seminorms.

Notation A.19. Fix a vector space V over $\mathbb{F}$. For any absorbing subset $A \subseteq V$, we define $\|\cdot\|_A : V \to \mathbb{R}_{\ge 0}$ by

$$\|v\|_A = \inf\{t \ge 0 : v \in tA\}.$$

Here are some basic facts about this construction.

Remark A.20. Because A is absorbing, we see that any $v \in V$ does in fact have some $t > 0$ for which $(1/t)v \in A$ and hence $v \in tA$, so the infimum is a real number.

Remark A.21. Suppose further that A is convex. Then we claim that $v \in tA$ whenever $t > \|v\|_A$; note if $v = 0$, then $\|0\|_A = 0$, so there is nothing to do. The main point is to note that $sv \in A$ implies that $s'v \in A$ for any $s' \in [0, s]$ by convexity. Thus, if $t > \|v\|_A$, then we know there is $s < t$ such that $v \in sA$, so $(1/s)v \in A$ while $1/t < 1/s$, so $(1/t)v \in A$, so $v \in tA$.

Let's put all our adjectives together, finally explaining the relationship between seminorms and convex sets.

Proposition A.22. Fix a vector space V over $\mathbb{F}$ and a subset $A \subseteq V$.

(a) There is a seminorm $p\colon V \to \mathbb{R}$ such that $A = \{v \in V : p(v) < 1\}$ if and only if A is nonempty, convex, balanced, and $(-a) + A$ is absorbing for all $a \in A$.

(b) If A is merely convex, balanced, and absorbing, then there is a seminorm $p\colon V \to \mathbb{R}$ such that $\{v : p(v) < 1\} \subseteq A$.

Proof. This is [Con90, Proposition 1.14]. The forward direction of (a) follows from combining Examples A.6, A.9 and A.18. It remains to show the reverse direction of (a) and (b). For both of these, we will take $p := \|\cdot\|_A$, which is a well-defined function by Remark A.20 (once we know that A in (a) is absorbing). We will run our checks in a few pieces.

- In (a), we show that A is absorbing. It is enough to check that $0 \in A$. Well, being nonempty, there is some $a \in A$. Because A is balanced, we see $-a \in A$, and because A is convex, it follows that $0 \in A$.
- If A is absorbing and balanced, we check that $p(\lambda v) = |\lambda|\, p(v)$ for $v \in V$ and $\lambda \in \mathbb{F}$. Well, $\lambda v \in tA$ if and only if $v \in \frac{t}{\lambda}A$, which is equivalent to $v \in \frac{t}{|\lambda|}A$ because A is balanced! It follows that

$$\{t \geq 0 : \lambda v \in tA\} = |\lambda| \left\{ \frac{t}{|\lambda|} : v \in \frac{t}{|\lambda|} A \right\},$$

so the check follows.
- If A is convex, balanced, and absorbing, then we check that $p(v+w) \leq p(v) + p(w)$ for $v, w \in V$. The geometric input is that $tA + sA \subseteq (t+s)A$ for any $t, s > 0$; this follows by convexity because

$$tA + sA = (t+s)\left(\frac{t}{t+s}A + \frac{s}{t+s}A \right)$$

is contained in $(t+s)A$ by convexity. Now, for the check, we note that having $p(v) < t$ and $p(w) < s$ implies that $v \in tA$ and $w \in sA$ by Remark A.21, so $v + w \in (t+s)A$, so $p(v+w) \leq t+s$. Sending $t \to p(v)$ and $s \to p(w)$ completes the check.
- We complete the proof of (b). The above checks show that p is a seminorm, so it remains to check that $\{v : p(v) < 1\} \subseteq A$. This follows from A being balanced: if $p(v) < 1$, then there is $t < 1$ such that $v \in tA$, and $tA \subseteq A$ because A is balanced.
- We complete the proof of (a). The previous check shows that $\{v : p(v) < 1\} \subseteq A$, so it remains to check the other inclusion. Well, for any $a \in A$, we see that $(-a) + A$ is absorbing, so $a + ta \in A$ for small $t > 0$. It follows that $a \in (1+t)^{-1}A$, so $p(a) < (1+t)^{-1} < 1$ follows. ■

Corollary A.23. Fix a topological vector space V over $\mathbb{F}$. The following are equivalent.

(i) V is locally convex.

(ii) The topology on V is induced by a family of seminorms.

Proof. We show the implications separately.

- Suppose that V is locally convex, so 0 admits a neighborhood basis $\mathcal{U}$ of balanced convex sets by Remark A.14. By Example A.17, we see that each $U \in \mathcal{U}$ has $(-a) + U$ absorbing for all $a \in U$, so Proposition A.22 provides a seminorm $p_U \colon V \to \mathbb{R}$ such that $U = \{v : p_U(v) < 1\}$. Note that p_U is continuous by Lemma A.4.

  Lastly, we should check that the topology given by the seminorms $\{p_U\}$ is the correct one. Well, this topology has basis given by finite intersections of sets of the form

$$\{v \in V : p_U(v) \in (a,b)\},$$

  where $(a,b) \subseteq \mathbb{R}$. The continuity of the p_Us implies that any such subset is open in V. Conversely, any open neighborhood of 0 in V contains some $U \in \mathcal{U}$ and therefore contains $\{v \in V : p(v) \in (-1,1)\}$, so a comparison of the neighborhood bases (via translation) implies that the open neighborhood of 0 is still open.

- Suppose that V has its topology generated by a family of seminorms $\{p_i\}$. Well, because $p_i(0) = 0$ for each i, an open neighborhood basis of 0 can be given by finite intersections of sets of the form $p_i^{-1}((-\varepsilon, \varepsilon))$. Of course, this is just
$$\varepsilon\{v \in V : p_i(v) < 1\},$$
which we note is convex by Example A.6. Thus, 0 admits a neighborhood basis of convex sets. ■

Now that we have an understanding of locally convex spaces, we may define Fréchet spaces.

Definition A.24 (Fréchet)**.** A topological vector space X is *Fréchet* if and only if it is locally convex, metrizable, and complete.

The bizarre addition here is metrizable. This condition fits in with the other ones as follows.

Proposition A.25. Fix a locally convex topological vector space V over $\mathbb{F}$. Then the following are equivalent.

(i) V has its topology induced by a translation-invariant metric.

(ii) V is metrizable.

(iii) V has a countable neighborhood basis of 0.

(iv) The topology on V is induced by a countable family of seminorms.

Proof. The implication (i) to (ii) has no content, and (ii) to (iii) follows by taking the neighborhood basis of open subsets given by $\{v : d(v, 0) < 1/n\}$ for positive integers n. Next, (iii) implies (iv) by the proof of the forward direction of Corollary A.23, which built one seminorm for each balanced convex subset in the neighborhood basis of 0.

Lastly, we have to show that (iv) implies (i). Well, given the countable family of seminorms $\{p_i\}_{i\geq 1}$, we define the function $d\colon V \times V \to \mathbb{R}$ by
$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x-y)}{1 + p_i(x-y)}.$$
Here are our checks on d.

- We check that d is a metric. The summation always converges because $\frac{p_i(x-y)}{1+p_i(x-y)} \leq 1$ always. Continuing, $d(x, x) = 0$ follows because $p_i(0) = 0$ for all i, and the triangle inequality follows from the subadditivity of each of the p_is.

  It remains to check the positivity of d. Well, if $x \neq y$, then because V is Hausdorff, we see that $p_i(x - y) > 0$ for some p_i. (Otherwise, the constant net $\{x - y\}$ would converge to both 0 and $x - y$.) Thus, $d(x, y) > 0$.

- We check that d is translation-invariant. Well, for any $a \in V$, we see that $d(x + a, y + a)$ is a function of $(x + a) - (y + a) = (x - y)$ and will equal $d(x, y)$.

- Lastly, we check that d induces the topology on V. It is enough to check that these two topologies have the same convergent nets. Well, a net $\{x_i\}$ converges to some $x \in V$ if and only if $p_\bullet(x_i - x) \to 0$ for all seminorms $p_\bullet$. This surely implies that $d(x_i, x) \to 0$, and conversely, $d(x_i, x) \to 0$ will require that $p_\bullet(x_i - x) \to 0$ for each $p_\bullet$. ■

A.2 The Open Mapping Theorem

In this section, we review the proof of the Open mapping theorem in order to extend the usual proof (for Banach spaces) to the setting of Fréchet spaces.

As usual, our proof will have to rely on the Baire category theorem. Before introducing any strange terminology, let's start with a statement on just metric spaces.

Lemma A.26. Let X be a nonempty complete metric space. Then a countable intersection of dense open subsets is dense.

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be our collection of dense open subsets. We would like to show that their intersection $\bigcap_{i\in\mathbb{N}} U_i$ intersects any open subset V of X. The idea is to recursively choose nearby elements in $U_i \cap V$ for each i, and then use completeness of X to finish the proof. We proceed in steps.

1. We build a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ recursively, as follows. To start us off, we note $V \cap U_0$ is nonempty and open (by density of U_0), so we are granted a point $x_0 \in U_0$ and ε_0 such that $B(x_0, \varepsilon_0) \subseteq V \cap U_0$. For the recursion, we suppose that we are given such an open neighborhood $B(x_n, \varepsilon_n)$, and then because U_{n+1} is open and dense, we are provided a point x_{n+1} in the intersection and some $\varepsilon_{n+1} < \varepsilon_n/3$ such that
$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}.$$
2. We claim that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is (rapidly) Cauchy. Indeed, note $\varepsilon_{n+1} < \varepsilon_n/2$ for each n, so $\varepsilon_n < 2^{-n}$ follows by an induction. Thus, $d(x_n, x_{n+1}) < 2^{-n}$ for each n, so our sequence is rapidly Cauchy. To finish checking that it is Cauchy, we note that whenever $i < j$, we have
$$d(x_i, x_j) \le \sum_{k=i}^{j-1} \underbrace{d(x_k, x_{k+1})}_{<2^{-k}},$$
which is upper-bounded by 2^{-i+1}.
3. Now, we let x be a limit point of $\{x_n\}_{n\in\mathbb{N}}$. (This is where we used completeness!) Because our sequence is eventually in $B(x_n, \varepsilon_n)$ for any given n, we see that $x \in B(x_n, \varepsilon_n)$ for each n. Thus, $x \in V \cap U_0$ by the first step of the construction, and $x \in U_n$ for each $n \ge 1$ by the recursive step of the construction. ■

The previous lemma now upgrades to the Baire category theorem.

Theorem A.27 (Baire category)**.** Let X be a nonempty complete metric space. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable collection of dense open subsets. Then the intersection $\bigcap_{i\in\mathbb{N}} U_i$ is not contained in a countable union of nowhere dense subsets.

Proof. Suppose for the sake of contradiction that we have
$$\bigcap_{i\in\mathbb{N}} U_i \subseteq \bigcup_{j\in\mathbb{N}} A_j,$$
where each A_j is nowhere dense. It thus follows that
$$\bigcap_{i\in\mathbb{N}} U_i \cap \bigcap_{j\in\mathbb{N}} X \setminus \overline{A_j}$$
is empty. We claim that $X \setminus \overline{A_j}$ is open and dense, which yields the desired contradiction by Lemma A.26. Certainly $X\setminus\overline{A_j}$ is open; for density, note that $\overline{A_j}$ contains no open subset, which means that the complement intersects any open subset. ■

Corollary A.28. Let X be a nonempty complete metric space. Then X is not the countable union of nowhere dense subsets.

Proof. This follows from taking $U_i = X$ for each i in Theorem A.27. ■

We now proceed with the Open mapping theorem. We will isolate the application of the Baire category theorem to the following lemma: Corollary A.28 shows that the hypothesis is satisfied.

Lemma A.29. Let $f\colon X \to Y$ be a linear map of locally convex topological vector spaces. Suppose that $\operatorname{im} f$ is not the union of nowhere dense subsets. Then $\overline{f(U)}$ contains an open neighborhood of 0 for each open neighborhood U of 0.

Proof. This more or less follows from unwinding the hypothesis. Because V is locally convex, we may shrink U to make it convex and balanced (by Lemma A.11); we will only use this at the end of the proof. The hypothesis is applied as follows: because U is open, it is absorbing (by Example A.17), so $V = \bigcup_{i\in\mathbb{N}} iU$, so

$$\operatorname{im} f = \bigcup_{i>0} iU.$$

Thus, the hypothesis implies that one of the iU fails to be nowhere dense; because multiplication by $i > 0$ is a homeomorphism, we see that U is also fails to be nowhere dense, so it contains an open subset V.

We now upgrade V into an open neighborhood of 0. Well, simply set $V' := \frac{1}{2}(V - V)$. Then $f(V')$ is contained in $\frac{1}{2}(U - U)$ by linearity, which is $\frac{1}{2}(U + U)$ because U is balanced, which is contained in U because U is convex. ■

It remains to use the hypothesis that X is complete, which is done in the following lemma.

Lemma A.30. Let $f\colon X \to Y$ be a continuous linear map of metrizable locally convex topological spaces. Suppose that X is complete and that $\overline{f(U)}$ contains an open neighborhood of 0 for each open neighborhood U of 0. Then f is open.

Proof. The idea is to use the completeness of X to construct points of U which go to a required open neighborhood. We proceed in steps.

1. We are going to show that $f(U)$ contains an open neighborhood of 0 for each open neighborhood U of 0, so let's spend a moment to explain why this is enough. For each open subset $U' \subseteq X$ and $x \in U'$, we note that $f(U' - x)$ contains an open neighborhood V_x of the origin. Thus, $f(U')$ contains the open neighborhood $f(x) + V_x$, so $f(U')$ equals

$$\bigcup_{x\in U'} f(x) + V_x,$$

which is open because it is a union of open subsets.

2. We unwind the hypothesis on f. By shrinking our open neighborhood U of 0, we may assume that U is convex and balanced (by Lemma A.11), so there is a seminorm p on X for which U is $B(0,1)$ for some translation-invariant metric d on X, chosen via Proposition A.25. Similarly, by hypothesis on f, we know that $\overline{f(U)}$ contains some open neighborhood V of 0, which we may again shrink until it is $B(0,2)$ for some translation-invariant metric d on Y. It will be worthwhile to remove the closure from this statement. Well, for any $y \in Y$, we see that $y \in d(y,0)V$ and so $y \in \overline{q(y) \cdot f(U)}$, so any $\varepsilon > 0$ has some $x \in X$ for which $d(x,0) < d(y,0)$ and $d(y, f(x)) < \varepsilon$.

3. We will actually show that $V \subseteq f(U)$, so choose some $y \in V$. The completeness of X will be used via a limiting process to produce an element of U mapping to y. To start us off, fix some $\varepsilon > 0$ (to be fixed at the end of the proof), and we take $x_0 := 0$. Now, if we are given $x_0 + \cdots + x_n$, we may select x_{n+1} so that $d(x_{n+1}, 0) < d(y - f(x_0 + \cdots + x_n), 0)$ and

$$d(y - f(x_0 + \cdots + x_n), f(x_{n+1})) < \varepsilon/2^{n+1}$$

by the above paragraph.

4. We complete the proof. Now, by construction, $d(x_n, 0) < \varepsilon/2^{n-1}$ for all $n \geq 2$, so the sequence of partial sums is rapidly Cauchy. As in the proof of Lemma A.26, it follows that these partial sums converge to some $x \in X$.

   We claim that this x is the desired element. To start, we see that $f(x_0 + \cdots + x_n) \to y$ as $n \to \infty$ by construction, so $f(x) = y$ by continuity of f!

   It remains to check that $x \in U$. Well, $d(x, 0)$ is bounded by

$$\sum_{n=0}^{\infty} d(x_n, 0) < d(x_0, 0) + d(x_1, 0) + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}},$$

   and $d(x_1, 0) < d(y, 0) < 2$ and $\sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}} = \varepsilon$, so $x \in U$ for ε small enough. ■

Theorem A.31 (Open mapping). Let $f \colon X \to Y$ be a continuous linear map of Fréchet spaces. If f is surjective, then f is open.

Proof. By Corollary A.28, we see that X is not a countable union of nowhere dense subsets. The result now follows from combining Lemmas A.29 and A.30. ■

Corollary A.32. Let $f \colon X \to Y$ be a bijective continuous linear map of Fréchet spaces. Then f has continuous inverse.

Proof. Let g be the inverse map. Checking that g is continuous is equivalent to checking that f is open, which follows from Theorem A.31. ■

A.3 The Hahn–Banach Theorem

In this section, we review the proof of the Hahn–Banach theorem. This section will be filled with plenty of nonsense. Ultimately, we are interested in extending continuous linear functionals on Fréchet spaces, but along the way, we will show that linear functionals separate convex sets.

As with Banach spaces, we check if a linear functional is continuous by checking if it is bounded, but the definition of bounded needs to be adjusted.

Definition A.33 (bounded). Fix a topological vector space V over $\mathbb{F}$. A linear functional $\ell \colon V \to \mathbb{F}$ is *bounded* if and only if there is an open neighborhood U of 0 and a constant $c > 0$ such that $|\ell(x)| \leq c$ for all $x \in U$.

Lemma A.34. Fix a topological vector space V over $\mathbb{F}$ and a linear functional ℓ on V. Then the following are equivalent.

(i) ℓ is continuous.

(ii) ℓ is bounded.

(iii) ℓ is continuous at 0.

Proof. We use Lemma A.4. The implication from (i) to (ii) is direct; the proof that (iii) implies (i) is identical to the proof in Lemma A.4. To show (ii) implies (iii), we note $|\ell|$ is a seminorm, and by considering nets, we see that it is enough to check that $|\ell|$ is continuous at 0, which follows from Lemma A.4(ii) and the fact that ℓ is bounded. ■

Corollary A.35. Fix a topological vector space V over $\mathbb{F}$, and let $p\colon V\to\mathbb{R}$ be a continuous seminorm. Given a linear functional $\ell\colon V\to\mathbb{F}$, if $\ell\le p$ pointwise, then ℓ is continuous.

Proof. Because $\ell\le p$, we see that ■

Thus, it will be important to be able to extend linear functionals along with an upper bound against a seminorm. By considerations with Zorn's lemma, we will find that the hard part is extending the linear functional one step, which is the content of the next lemma.

Lemma A.36. Fix a vector space V over $\mathbb{R}$, a seminorm p on V, and a linear functional ℓ on a subspace $W\subseteq V$ such that $\ell\le p$ pointwise. Given any $v'\in V$, there is an extension ℓ' to a linear functional on a subspace W' containing v' such that $\ell'\le p$ pointwise.

Proof. If $v'\in W$ already, then there is nothing to do. Otherwise, for any real number c, we see that we may extend ℓ to a linear functional ℓ' on $W':=W+\mathbb{R}v'$ by setting $\ell'(v'):=c$. Namely, we have

$$\ell'(w+tv')=\ell(w)+tc$$

for any $w\in W$ and $t\in\mathbb{R}$.

We would like to show that we can choose c so that $\ell'\le p$ pointwise. This requires a little trickery. By scaling, it is enough to only check with $t\in\{\pm1\}$ (because $t=0$ follows by hypothesis). Thus, we need both $\ell(w)+c\le p(w+v')$ and $\ell(w)-c\le p(w-v')$ for all $w\in W$. Now, such a c exists if and only if

$$\sup_{w\in W}\left(\ell(w)-p(w-v')\right)\overset{?}{\le}\inf_{w\in W}\left(p(w+v')-\ell(w)\right).$$

For this, we should check that $\ell(w)-p(w-v')\le p(w'+v')-\ell(w')$ for any $w,w'\in W$, which is equivalent to $\ell(w+w')\le p(w-v')+p(w'+v')$. This last inequality follows because $\ell\le p$ and the subadditivity of p. ■

Theorem A.37 (Hahn–Banach)**.** Fix a vector space V over $\mathbb{R}$, a seminorm p on V, and a linear functional ℓ on a subspace $W\subseteq V$ such that $\ell\le p$ pointwise. Then ℓ extends to a linear functional ℓ' on V such that $\ell'\le p$.

Proof. After Lemma A.36, the rest of this proof is largely formal nonsense. We use Zorn's lemma on the partially ordered set $\mathcal{P}$ of pairs (V',ℓ'), where V' is an intermediate subspace, and ℓ' is a functional on V' bounded above by p; the ordering is given by $(V',\ell')\le(V'',\ell'')$ if and only if $V'\subseteq V''$ and $\ell''|_{V'}=\ell'$. Our application of Zorn's lemma is in two steps.

- We claim that $\mathcal{P}$ has a maximal element, for which we use Zorn's lemma. First, note $\mathcal{P}$ is nonempty because it has (W,ℓ). Secondly, any ascending chain $\{(W_i,\ell_i)\}_i$ in $\mathcal{P}$ has upper bound given by setting $V':=\bigcup_i W_i$ and defining ℓ' as the union of the ℓ_is. We can see that V' is still a vector space, and the nature of the partial ordering verifies that ℓ' is a well-defined functional extending ℓ. Thus, so (V',ℓ') is indeed an upper bound for our chain.
- Let (V',ℓ') be a maximal element of $\mathcal{P}$. We claim that $V'=V$, which will complete the proof. We already have $V'\subseteq V$, so it remains to show the other inclusion. Well, for any $v\in V$, we see that (V',ℓ') can be extended up to $V'+\mathbb{R}v$ by Lemma A.36, so the maximality of (V',ℓ') requires $V'+\mathbb{R}v=V'$. Thus, $v\in V'$, so $V\subseteq V'$ follows. ■

BIBLIOGRAPHY

[Con90] John B. Conway. *A course in functional analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions