18.906: Algebraic Topology II

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Fall 2025

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
∞-CATEGORIES

Language turns us all into jesters.

—Savannah Brown, [Bro24]

1.1 September 4

Here are some administrative notes.

- Office hours will be on Tuesday and Thursday immediately after class in 2-374.
- The syllabus will be posted to the course website later.
- The syllabus will contain some recommended textbooks, which are some free online texts that contain supersets of our class material.
- The grade will be 20% from a fifty-minute exam and 80% coming from problem sets. The exam will probably occur shortly before the drop deadline.

We hope to cover simplicial sets, ∞-categories, homotopy theory, Eilenberg–MacLane spaces, Postnikov towers, the Serre spectral sequence, and a little on vector bundles and characteristic classes. In particular, we see that the first part of the class is some purely formal nonsense, which we then use to set up the ∞-category of spaces, which is the natural setting for homotopy theory.

1.1.1 Category Theory

Let's recall some starting notions of category theory, though we may use more than we define here.

Warning 1.1. We will mostly ignore size issues. If it makes the reader feel better, we are willing to assume the existence of a countable ascending chain of inaccessible cardinals throughout the class.

Definition 1.2 (category)**.** A *category* $\mathcal{C}$ is a collection of objects, a collection of morphisms $\operatorname{Mor}_{\mathcal{C}}(A, B)$ for each pair of objects, a distinguished identity element id_A in $\operatorname{Mor}_{\mathcal{C}}(A, A)$, and a composition law

$$\circ\colon \operatorname{Mor}_{\mathcal{C}}(B, C) \times \operatorname{Mor}_{\mathcal{C}}(A, B) \to \operatorname{Mor}_{\mathcal{C}}(A, C).$$

We then require the composition law to be associative and unital with respect to the identity maps.

Remark 1.3. We will use the notation Hom for Mor whenever the category $\mathcal{C}$ is additive, meaning that these collections of morphisms are abelian groups, and the composition law is $\mathbb{Z}$-bilinear.

Definition 1.4 (functor)**.** A *functor* F between two categories $\mathcal{C}$ and $\mathcal{D}$ is a map which sends an object $A \in \mathcal{C}$ to an object $FA \in \mathcal{D}$ and a morphism $f\colon A \to B$ in $\mathcal{C}$ to a morphism $Ff\colon FA \to FB$. We further require F to respect identities and composition.

Definition 1.5 (isomorphism)**.** A morphism $f\colon A \to B$ in a category $\mathcal{C}$ is an *isomorphism* if and only if there is a morphism $g\colon B \to A$ for which $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Definition 1.6 (groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

Example 1.7. Any category $\mathcal{C}$ gives rise to a core groupoid $\operatorname{Core}\mathcal{C}$, which is the subcategory with the same objects but only taking the morphisms which are isomorphisms. One can check that this is in fact a subcategory.

In mathematics, one frequently encounters a category $\mathcal{C}$, and we are then interested in classifying the objects up to isomorphism.

Example 1.8. If $\mathcal{C} = \mathrm{Set}$, then isomorphisms are bijections, so sets "up to bijection" are simply given by their cardinalities.

Example 1.9. A commutative ring R gives rise to a category Mod_R of (left) R-modules. If R is a field, then this is a category of vector spaces, and objects up to isomorphism are given by their dimensions.

Example 1.10. One can consider the category Top of topological spaces, whose morphisms are continuous maps. (We will frequently restrict our category of topological spaces with some nicer subcategories, such as CW complexes or manifolds.) It is rather hard to classify objects up to isomorphism (here, isomorphisms are homeomorphisms), but there are some tools. For example, there are homology functors

$$\mathrm{H}_i(-; R)\colon \mathrm{Top} \to \mathrm{Mod}_R.$$

Because functors preserve isomorphisms, homeomorphic spaces must have isomorphic homology.

The definition of homology finds itself focused on continuous maps $|\Delta^n| \to X$, where $|\Delta^n|$ is the (topological) n-simplex.

Definition 1.11 (n-simplex)**.** The *(topological) n-simplex* $|\Delta^n|$ is the subspace

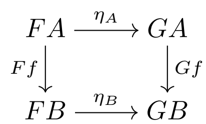
$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}_{\geq 0}^{n+1} : \sum_{i=0}^{n} t_i = 1 \right\}.$$

We will soon upgrade this topological n-simplex $|\Delta^n|$, which explains why we are writing $|\Delta^n|$ instead of Δ^n.

We are shortly going to get a lot of mileage out of the next example, so we spend some time to prove it in detail. We would like to define a category of functors between two given categories,[1] but this requires us to have a notion of morphism between functors.

Definition 1.12 (natural transformation)**.** Given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta\colon F \Rightarrow G$ is the data of a morphism $\eta_A\colon FA \to GA$ for each object $A \in \mathcal{C}$. We further require that $Gf \circ \eta_A = \eta_B \circ Ff$ for any morphism $f\colon A \to B$. A *natural isomorphism* is a natural transformation η in which each morphism η_A is an isomorphism.

Diagrammatically, the equation $Gf \circ \eta_A = \eta_B \circ Ff$ amounts to the commutativity of the following square.

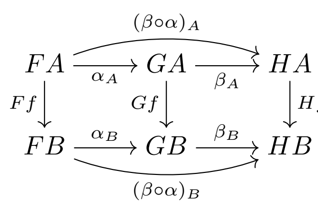


Anyway, here is our result.

Lemma 1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ where the objects are functors $\mathcal{C} \to \mathcal{D}$ and the morphisms are natural transformations.

Proof. We have explained our objects and morphisms, but we still have to provide identities and composition laws and check that everything works.

- Identities: given a functor $F\colon \mathcal{C} \to \mathcal{D}$, there is an identity natural transformation $\mathrm{id}_F\colon F \to F$ given by $(\mathrm{id}_F)_A \coloneqq \mathrm{id}_{FA}$; checking that this is a natural transformation amounts to noting that $Ff \circ \mathrm{id}_{FA} = \mathrm{id}_{FB} \circ Ff$ for any morphism $f\colon A \to B$.
- Composition: given two natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, we define the composite natural transformation $(\beta \circ \alpha)\colon F \Rightarrow H$ by $(\beta \circ \alpha)_A \coloneqq \beta_A \circ \alpha_A$ for each $A \in \mathcal{A}$. Checking that this is a natural transformation amounts to checking the commutativity of the outer rectangle of



  which indeed commutes: the top and bottom triangles commute by definition of $\beta \circ \alpha$, and the two inner squares commute by naturality of α and β.
- Identities: given a natural transformation $\eta\colon F \Rightarrow G$, we need to check that $\mathrm{id}_G \circ \eta = \eta \circ \mathrm{id}_F = \eta$. Well, for any object A, we see that

$$(\mathrm{id}_G \circ \eta)_A = (\mathrm{id}_G \circ \eta)_A = \mathrm{id}_{G(A)} \circ \eta_A = \eta_A,$$

  and

$$(\eta \circ \mathrm{id}_F)_A = \eta_A \circ \mathrm{id}_{FA} = \eta_A.$$

- Associativity: given natural transformations α, β, and γ with appropriate domains and codomains, we must check that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$. Well, for any object A, we see that

$$((\alpha \circ \beta) \circ \gamma) = (\alpha_A \circ \beta_A) \circ \gamma_A = \alpha_A \circ (\beta_A \circ \gamma_A) = (\alpha \circ (\beta \circ \gamma))_A,$$

  as required. ■

[1] For those who are choosing to think about size issues, we remark that we will typically have one of $\mathcal{C}$ or $\mathcal{D}$ be locally small.

1.1.2 Homotopy Types, Intuitively

The homology functors $\mathrm{H}_i(-;R)\colon \mathrm{Top} \to \mathrm{Mod}_R$ factors through the homotopy category $\mathrm{Ho}(\mathrm{Top})$, which is obtained from Top by declaring homotopic maps to be equal.

We are going to be homotopy theorists for the most part, which means that we will be interested in understanding invariants of the category $\mathrm{Ho}(\mathrm{Top})$. One may complain that only studying spaces up to homotopy is an over-simplification. However, there are good reasons to only be interested in these "homotopy types" because such things also come up in other areas of mathematics.

Approximately speaking, a homotopy type is a collection of objects and morphisms between them. To ensure some level of homogeneity, one may require that any pair $f, g\colon A \to B$ of morphisms has a collection of "2-isomorphisms" $f \Rightarrow g$. Furthermore, there should be "3-isomorphisms" between these 2-isomorphisms, and this thinking continues inductively.

Example 1.14. Given two objects A and B with an isomorphism $f\colon A \to B$, one may think about these objects as being identified. Similarly, if we have a third isomorphism $g\colon B \to C$, then we can canonically identify all three objects. Here is diagram for this situation.

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \simeq \qquad A$$

Example 1.15. Given two objects A and B, there may be two isomorphisms $f, g\colon A \to B$. One may want to identify these two objects via either isomorphism, but then we don't want to forget about the other isomorphism, so perhaps we are thinking about an object with an automorphism. Here is a diagram for this situation.

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad \simeq \qquad \overset{g}{\circlearrowright} A$$

If one wanted to identify f and g, then there should be a "2-isomorphism" identifying f and g.

Example 1.16. We can think about a set as a homotopy type where all isomorphisms, 2-isomorphisms, and so on are all just the identity maps.

Example 1.17. We can think about a groupoid as a homotopy type where all 2-isomorphisms, 3-isomorphisms, and so on are all just the identity maps.

The above two examples will let us think about $\mathrm{Ho}(\mathrm{Top})$ as a category of ∞-groupoids. This is more or less why homotopy types are relevant to other areas of mathematics: one is frequently interested in not just isomorphisms between objects but also the uniqueness of those isomorphisms, and also the uniqueness of the isomorphisms identifying the isomorphisms, and so on.

Of course, we started with topological spaces, so let's explain how then make some ∞-groupoid.

Example 1.18. Given a topological space X, we can build a corresponding ∞-groupoid as follows.

- The points provide objects in our ∞-groupoids; these are functions $\left|\Delta^0\right| \to X$.
- The maps between points are given by paths; these are functions $\left|\Delta^1\right| \to X$.
- The maps between paths are given by homotopies of paths; these are functions $\left|\Delta^2\right| \to X$. Technically speaking, $\left|\Delta^2\right|$ gives two paths f and g whose composite should be homotopic to h. Thus, the structure of such a map tells us something about how composition should behave!

1.1.3 Simplices

After building up some intuition, we are now forced to do some combinatorics in order to get ourselves off of the ground.

Notation 1.19. For each integer $n \geq 0$, we define the category $[n]$ whose objects are the elements of $\{0, 1, \ldots, n\}$ and whose morphisms are given by

$$\operatorname{Hom}_{[n]}(i,j) := \begin{cases} \varnothing & \text{if } i < k, \\ * & \text{if } i \leq j, \end{cases}$$

where $*$ simply refers to some one-element set.

We remark that identities and the composition maps are then all uniquely defined (because everything is unique in the one-element set $*$); similarly, the coherence checks of identity and associativity have no content because everything is equal in $*$.

Remark 1.20. Combinatorially, $[n]$ is the poset category given by the totally ordered set

$$0 \leq 1 \leq 2 \leq \cdots \leq n.$$

Definition 1.21 (simplex)**.** The *simplex category* Δ has objects given by the categories $[n]$, and the morphisms are given by the collection of functors between any two such categories.

Remark 1.22. Combinatorially, we see that a functor $F\colon [n] \to [m]$ amounts to the data of an increasing map. Indeed, whenever $i \leq j$ in $[n]$, which is equivalent to having a morphism $i \to j$, we see that there is a morphism $Fi \to Fj$, which is equivalent to the requirement $Fi \leq Fj$.

Example 1.23. There are six morphisms $[1] \to [2]$, as follows.

- If $0 \mapsto 0$, then $1 \in [1]$ can go anywhere.
- If $0 \mapsto 1$, then 1 maps to 1 or 2 in $[2]$.
- If $0 \mapsto 2$, then 1 maps to 2.

Example 1.24. For each nonnegative integer n, there is a unique map $[n] \to [0]$ for each n. Indeed, everything must go to 0.

Remark 1.25. There is an important functor $F\colon \Delta \to \mathrm{Top}$ given by sending $[n] \mapsto |\Delta^n|$. Let's explain what this functor is on morphisms: given an increasing map $f\colon [n] \to [m]$, then we need to provide a continuous map $Ff\colon |\Delta^n| \to |\Delta^m|$. Well, we may identify $[n]$ and $[m]$ with bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, so f is now a function on bases, so it upgrades uniquely to a linear map $\mathbb{R}^n \to \mathbb{R}^m$ given by

$$Ff\left(\sum_{i=0}^n t_i e_i\right) := \sum_{i=0}^n t_i e_{f(i)}.$$

Thus, we see that Ff does restrict to a map $|\Delta^n| \to |\Delta^m|$. Functoriality follows by the uniqueness of the construction of Ff: given two increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we see $Fg \circ Ff$ and $F(g \circ f)$ definitionally are both defined as $g \circ f$ on the basis of $\mathbb{R}^n$. (Of course, we should mention $\mathrm{id}_{[n]}\colon [n] \to [n]$ defines the identity on $\mathbb{R}^n$.)

1.1.4 Simplicial Sets

The following is the first important definition of this course.

Definition 1.26 (simplicial set)**.** A *simplicial set* is a functor $\Delta^{\text{op}} \to \text{Set}$. We let sSet denote the category of such functors.

Note that this "functor category" is in fact a category by Lemma 1.13. Here are some examples of simplicial sets.

Example 1.27 ($\text{Sing}(X)$)**.** There is a functor $\text{Sing}\colon \text{Top} \to \text{sSet}$ such that

$$\text{Sing}(X)\colon [n] \mapsto \text{Mor}_{\text{Top}}(|\Delta^n|, X).$$

Proof. We have many checks to do, which we handle in sequence.

- We define $\text{Sing}(X)$ on morphisms. Well, given an increasing map $f\colon [n] \to [m]$, the functor F of Remark 1.25 provides a continuous map $Ff\colon |\Delta^m| \to |\Delta^n|$, so there is a map

$$(- \circ Ff)\colon \text{Mor}_{\text{Top}}(|\Delta^n|, X) \to \text{Mor}_{\text{Top}}(|\Delta^m|, X).$$

- We check that $\text{Sing}(X)$ is a functor. First, the identity morphism $\text{id}_{[n]}\colon [n] \to [n]$ goes to the map

$$(- \circ F\text{id}_{[n]})\colon \text{Mor}_{\text{Top}}(|\Delta^n|, X) \to \text{Mor}_{\text{Top}}(|\Delta^n|, X),$$

which is the identity because $F\text{id}_{[n]} = \text{id}_{|\Delta^n|}$. Second, given increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we need to check that

$$(- \circ F(g \circ f)) = (- \circ Ff) \circ (- \circ Fg),$$

which is true because $F(g \circ f) = Fg \circ Ff$.

- We define Sing on morphisms. Well, given a continuous map $f\colon X \to Y$, we use the map

$$(f \circ -)\colon \text{Mor}_{\text{Top}}(|\Delta^n|, X) \to \text{Mor}_{\text{Top}}(|\Delta^n|, Y).$$

- We check that Sing is a functor. First, the identity $\text{id}_X\colon X \to X$ goes to the map $(\text{id}_X \circ -)$, which is just the identity composition. Second, given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$, we note that

$$(g \circ -) \circ (f \circ -) = ((g \circ f) \circ -)$$

by the associativity of composition. ■

Remark 1.28. It turns out that not all simplicial sets arise from this construction. In particular, it turns out that the image of Sing has many nice properties.

Remark 1.29. It will turn out that the homotopy type of X is uniquely determined by $\text{Sing}(X)$. This is remarkable because one expects Top to be a difficult category, even taken up to homotopy, but sSet just looks like some combinatorial data.

Example 1.30 (nerve)**.** Fix a category $\mathcal{C}$. Then there is a "nerve" functor $N\colon \text{Cat} \to \text{sSet}$ such that

$$N(\mathcal{C})\colon [n] \mapsto \text{Fun}([n], \mathcal{C}).$$

The proof of this claim is exactly the same as in Example 1.27 (note that $\text{Fun}([n], \mathcal{C}) = \text{Mor}_{\text{Cat}}([n], \mathcal{C})$), except now there is now need for the auxiliary functor $F\colon \Delta \to \text{Top}$ because Δ is already a category. (Being brazen, one can copy the same proof but erasing all Fs, replacing Top with Cat throughout, and replacing $|\Delta^\bullet|$s with $[\bullet]$s throughout.)

Check: fully faithful.

Remark 1.31. As in Remark 1.28, nerves of categories have some nice properties which prevent them from producing all simplicial sets. It turns out that ∞-categories will be some kind of simultaneous generalization of Sings and nerves.

1.1.5 Simplicial Sets by Combinatorics

Even though we will avoid doing so as much as possible in the sequel, it can be worthwhile to have a purely combinatorial description of a simplicial set. Let's begin by classifying increasing maps. We will get some utility out of the following lemma, which allows us to think about increasing maps f in terms of the multi-set $\operatorname{im} f$.

Lemma 1.32. Let $f, g\colon [n] \to [m]$ be increasing maps. Suppose that

$$\#f^{-1}(\{k\}) = \#g^{-1}(\{k\})$$

for all $k \in [m]$. Then $f = g$.

Proof. We proceed by induction on n. If $n = 0$, then $[n]$ is a singleton, so there is a unique $k \in [m]$ for which $f^{-1}(\{k\})$ and $g^{-1}(\{k\})$ are nonempty, namely $f(0)$ and $g(0)$ respectively, so the result follows.

For the induction, we are given two increasing maps $f, g\colon [n+1] \to m$. There are two steps.

1. The main claim is that $f(n+1) = g(n+1)$. To show this, note that $\operatorname{im} f = \operatorname{im} g$ because these sets are just the $k \in [m]$ with nonempty fibers. Thus, because $n+1$ is the maximum of $[n+1]$, we see that $f(n+1)$ and $g(n+1)$ are maximal elements of $\operatorname{im} f$ and $\operatorname{im} g$, respectively, so $f(n+1) = g(n+1)$ follows.

2. We now complete the proof. Note that

$$\#f|_{[n]}^{-1}(\{k\}) = \begin{cases} \#f^{-1}(\{k\}) & \text{if } f(n+1) \neq k, \\ \#f^{-1}(\{k\}) - 1 & \text{if } f(n+1) = k, \end{cases}$$

and similar for g, so $f|_{[n]}$ and $g|_{[n]}$ have fibers of the same cardinality, so $f|_{[n]} = g|_{[n]}$ by the induction, so $f = g$ follows because they are already equal on $n+1$. ■

Let's now classify injective maps.

Definition 1.33 (face maps)**.** Given some $i \in [n]$, we define the *face map* $\delta^i\colon [n-1] \to [n]$ to be the embedding which omits i by sending the set $\{0, \ldots, i-1\}$ to itself and sending the set $\{i, \ldots, n\}$ to one more than each element.

Lemma 1.34. Every injective increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = \delta^{i_1} \circ \cdots \circ \delta^{i_r},$$

where $i_1 > i_2 > \cdots > i_r$.

Proof. We proceed in steps.

1. Given a decreasing sequence $i_1 > i_2 > \cdots > i_r$, we claim that the map

$$(\delta^{i_1} \circ \cdots \circ \delta^{i_r})\colon [n] \to [n+r]$$

avoids the set $\{i_1, \ldots, i_r\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $i_1 > i_2 > \cdots > i_r > i_{r+1}$. By the induction, the map

$$(\delta^{i_2} \circ \cdots \circ \delta^{i_{r+1}}) \colon [n] \to [n+r]$$

already avoids the set $\{i_2, \ldots, i_{r+1}\}$. Then δ^{i_1} preserves the set $\{0, \ldots, i_1 - 1\}$ (and in particular preserves the omitted set $\{i_2, \ldots, i_{r+1}\}$) while going on to omit i_1, so the total composite $\delta^{i_1} \circ \cdots \circ \delta^{i_{r+1}}$ successfully omits $\{i_1, \ldots, i_{r+1}\}$.

2. We show that any injective f is a composite of $\delta^\bullet$s as given. Well, given an injective increasing map $f \colon [n] \to [m]$, set $I := [m] \setminus \operatorname{im} f$, and arrange the elements of I as $\{i_1, \ldots, i_r\}$ in decreasing order. Then $\delta^{i_1} \circ \cdots \circ \delta^{i_r}$ is another injective increasing map which omits I by the previous step, so it equals f by Lemma 1.32.

3. We show that two composites of decreasing $\delta^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that
$$\delta^{i_1} \circ \cdots \delta^{i_r} = \delta^{i'_1} \circ \cdots \delta^{i'_{r'}}$$
as maps $[n] \to [m]$, and the sequence of indices are both strictly decreasing; denote this map by f for brevity. By the first step, the size of the fibers of f can be read off of the indices $i_\bullet$ or $i'_\bullet$ (an index is present exactly when not in $\operatorname{im} f$), so these sequences must be equal. ■

Remark 1.35. It follows that δ^i is the unique injection $[n] \to [n+1]$ omitting a given element of $[n+1]$.

Remark 1.36. The requirement that the indices are strictly decreasing is necessary for the uniqueness. Indeed, if $i \le j$, then $\delta^i \circ \delta^j$ avoids i and $j+1$, so it equals $\delta^{j+1} \circ \delta^i$.

Analogously, we have should handle surjective increasing maps.

Definition 1.37 (degeneracy maps)**.** Given some $j \in [n+1]$, we define the *degeneracy map* $\sigma^j \colon [n+1] \to [n]$ to be the surjection which hits j twice by sending the set $\{0, \ldots, j\}$ to itself and sending $\{j+1, \ldots, n+1\}$ to one less than each element.

Lemma 1.38. Every surjective increasing map $f \colon [n] \to [m]$ can be written uniquely as a composite

$$f = \sigma^{j_1} \circ \cdots \circ \sigma^{j_r},$$

where $j_1 \ge j_2 \ge \cdots \ge j_r$.

Proof. The structure of this proof is similar to Lemma 1.34, but the technical core requires a couple modifications.

1. Given a decreasing sequence $j_1 \ge j_2 \ge \cdots \ge j_r$, we claim that the map

$$(\sigma^{j_1} \circ \cdots \circ \sigma^{j_r}) \colon [n] \to [n-r]$$

has fiber over $k \in [n-r]$ of size equal to $1 + \#\{t : j_t = k\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $j_1 \ge \cdots \ge j_{r+1}$. By induction, we know that the fiber of $\sigma := \sigma^{j_2} \circ \cdots \circ \sigma^{j_{r+1}}$ over k is $1 + \{2 \le t \le r+1 : j_t = k\}$.

Now, $\sigma^{j_1} \circ \sigma$ has the same-size fibers over any $k < j_1$ as σ because σ^{j_1} preserves $\{0, \ldots, j_1\}$. For $k > j_1$, we note that the fibers of σ over each such k is 1 because $k > j_i$ for each i (by the induction), so $\sigma^{j_1} \circ \sigma$ also has fiber of size 1 over this k. Lastly, for $k = j_1$, we see that the fiber increases in size by 1 because σ^{j_1} sends $j_1 + 1$ (whose fiber has size 1 for σ) to j_1. This casework completes the proof.

2. We show that any surjective f is a composite of $\sigma^\bullet$s as given. Well, let J be the multi-subset of $[m]$ hit multiple times by f, counted with multiplicity, and we may arrange the elements of J as $\{j_1, \ldots, j_r\}$ in decreasing order. Then $\sigma^{j_1} \circ \cdots \circ \sigma^{j_r}$ and f have fibers of the same size by the previous step, so they are equal functions by Lemma 1.32.

3. We show that two composites of decreasing $\sigma^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that

$$\sigma^{j_1} \circ \cdots \circ \sigma^{j_r} = \sigma^{j'_1} \circ \cdots \circ \sigma^{j'_{r'}}$$

as maps $[n] \to [m]$, and the sequences of indices are both decreasing; denote this map by f. Well, the fibers of f can be read off the indices $\{j_\bullet\}$ or $\{j'_\bullet\}$ by the first step, so these sequences must be equal. ■

Remark 1.39. As in Remark 1.35, we note that the requirement that the indices are decreasing is necessary for the uniqueness. Indeed, if $i < j$, then $\sigma^i \circ \sigma^j$ hits $j-1$ twice and i twice (counted with multiplicity), so $\sigma^i \circ \sigma^j = \sigma^{j-1} \circ \sigma^i$.

We are now ready to classify general maps.

Lemma 1.40. Every increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \sigma^{j_s}),$$

where $i_1 > \cdots > i_r$ and $j_1 \geq \cdots \geq j_s$.

Proof. The main point is to show that any increasing map f admits a unique decomposition as $\delta \circ \sigma$ where $\delta\colon [k] \to [m]$ is injective and $\sigma\colon [n] \to [k]$ is surjective. The existence and uniqueness of the required decomposition now follows by the existence and uniqueness of the decomposition $f = \delta \circ \sigma$ with Lemmas 1.34 and 1.38. For example, to get the uniqueness, if

$$\delta^{i_1} \circ \cdots \circ \delta^{i_r} \circ \sigma^{j_1} \circ \cdots \sigma^{j_s} = \delta^{i'_1} \circ \cdots \circ \delta^{i'_{r'}} \circ \sigma^{j'_1} \circ \cdots \sigma^{j'_{s'}},$$

then the composites of the $\delta^\bullet$s and of the $\sigma^\bullet$s must each be equal (because those are injections and surjections, respectively), and then the equalities of the indices follows from using Lemmas 1.34 and 1.38, respectively.

It remains to show the main claim. We show existence and uniqueness separately.

- Existence: note $\operatorname{im} f \subseteq [m]$ is some totally ordered subset, so we let its cardinality be $k+1$. By suitably ordering the elements of $\operatorname{im} f$, we receive a totally ordered bijection $[k] \to \operatorname{im} f$. Then we see that f decomposes into

$$\underbrace{[n] \xrightarrow{f} \operatorname{im} f \leftarrow [k]}_{\sigma} = \underbrace{[k] \to \operatorname{im} f \subseteq [m]}_{\delta},$$

as required.

- Uniqueness: suppose we have two equal decompositions $f = \delta \circ \sigma = \delta' \circ \sigma'$ where $\sigma\colon [n] \to [k]$ and $\sigma'\colon [n] \to [k']$ and $\delta\colon [k] \to [m]$ and $\delta\colon [k'] \to [m]$. To begin, note that the injectivity of δ and δ' implies that $k+1$ and $k'+1$ are both the cardinality of $\operatorname{im} f$, so $k = k'$ follows. Now, because δ and δ' have the same image, and both are injective, it follows that all their fibers from $[m]$ have the same size (as either 0 or 1)! Thus, $\delta = \delta'$ follows from Lemma 1.32. The injectivity of δ now shows that $\delta \circ \sigma = \delta \circ \sigma'$ implies $\sigma = \sigma'$. ■

Remark 1.41. As in Remarks 1.35 and 1.39, we note that putting δs before σs is important for the uniqueness. Suppose we have some $\sigma^j \circ \delta^i$, and then we have the following cases.

- If $j > i$, then σ^j fixes $\{0, \ldots, i+1\}$, so $\sigma^j \circ \delta^i$ avoids i and hits j twice. This is the same as $\delta^i \circ \sigma^{j-1}$.
- If $j = i$ or $j = i-1$, then $\sigma^j \circ \delta^i$ fixes $\{0, \ldots, i-1\}$ throughout, and the elements at least i get $+1$ from δ^i and -1 from σ^j. Thus, $\sigma^j \circ \delta^i = \mathrm{id}$.
- If $j < i-1$, then $\sigma^j \circ \delta^i$ avoids $i-1$ (-1 from σ^j) and hits j twice. This is the same as $\delta^{i-1} \circ \sigma^j$.

Having access to generators of these maps and some relations between them allows us to provide a combinatorial definition of a simplicial set.

Definition 1.42. A *combinatorial simplicial set* is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with face maps $d_0, \ldots, d_n \colon X_n \to X_{n-1}$ and degeneracy maps $s_0, \ldots, s_n \colon X_n \to X_{n+1}$ (for each n) satisfying the following simplicial identities

$$\begin{cases} d_j d_i = d_i d_{j+1} & \text{if } i \le j, \\ s_j s_i = s_i s_{j-1} & \text{if } i < j, \end{cases} \qquad \text{and} \qquad \begin{cases} d_i s_j = s_{j-1} d_i & \text{if } i < j, \\ d_i s_j = \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets is a function $f_n \colon X_n \to Y_n$ for each n commuting with the face and degeneracy maps; i.e., $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$.

Remark 1.43. One can check that there is a category of combinatorial simplicial sets. In particular, the identity is given by $(\mathrm{id}_X)_n \coloneqq \mathrm{id}_{X_n}$, and composition is defined by $(g \circ f)_n \coloneqq g_n \circ f_n$ (which commutes with the face and degeneracy maps because g and f do).

Proposition 1.44. There is an isomorphism of categories from the category of simplicial sets to the category of combinatorial simplicial sets by sending $X \in \mathrm{sSet}$ to a combinatorial simplicial set given by

$$\begin{cases} X_n \coloneqq X([n]), \\ d_\bullet \coloneqq X(\delta^\bullet) & \text{for each } n \in \mathbb{N}, \\ s_\bullet \coloneqq X(\sigma^\bullet) & \text{for each } n \in \mathbb{N}. \end{cases}$$

Proof. We run our many checks in sequence.

- To check that $X \in \mathrm{sSet}$ is sent to a combinatorial simplicial set $\{X_n\}$, we just need to check that the $d_\bullet$s and $s_\bullet$s satisfy the simplicial identities. This follows from the functoriality of X and Remarks 1.35, 1.39 and 1.41.
- We define $X \mapsto \{X_n\}$ on morphisms. Well, a functor $f \colon X \Rightarrow Y$ of simplicial sets defines maps $f_{[n]} \colon X([n]) \to Y([n])$, which we claim assembles into a morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets by $f_n \coloneqq f_{[n]}$. To check this, we need to check compatibility with the face and degeneracy maps. Well, $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$ follow by naturality of f because these amount to requiring
$$f_{[n-1]} \circ X(\delta^n) = Y(\delta^n) \circ f_{[n]} \qquad \text{and} \qquad f_{[n+1]} \circ X(\sigma^n) = Y(\sigma^n) \circ f_{[n]}.$$
- We show that $X \mapsto \{X_n\}$ is functorial. To begin, note $\mathrm{id} \colon X \Rightarrow X$ goes to the identity maps $\mathrm{id}_n \colon X_n \to X_n$. Then given $f \colon X \Rightarrow Y$ and $g \colon Y \Rightarrow Z$, we see that the composite $(g \circ f) \colon \{X_n\} \to \{Z_n\}$ is given by $(g \circ f)_n = (g \circ f)_{[n]} = g_{[n]} \circ f_{[n]} = g_n \circ f_n$, as required.

- We define a map from combinatorial simplicial sets back to simplicial sets. Well, given a combinatorial simplicial set $\{X_n\}$, we begin defining our functor $X\colon \Delta^{\text{op}} \to \text{Set}$ by $X([n]) := X_n$. On morphisms $f\colon [n] \to [m]$, we need to define some map $Xf\colon X_m \to X_n$. For this, we note that Lemma 1.40 allows us to write f uniquely as a composite

$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \circ \sigma^{j_s}),$$

  where $i_\bullet$ is strictly decreasing and $j_\bullet$ is decreasing. Thus, we define

$$Xf := (s_{j_s} \circ \cdots \circ s_{j_1}) \circ (d_{i_r} \circ \cdots \circ d_{i_1}).$$

  For example, $f = \text{id}_{[n]}$ is equal to the empty composite everywhere, so $X\text{id}_{[n]} = \text{id}_{X_n}$.

  To complete our functoriality check, because any morphisms can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s, it is enough to check functoriality for such morphisms. Namely, we have to check that

$$\begin{cases} X(\delta_i\delta_j) = X(\delta_j)X(\delta_i), \\ X(\sigma_i\sigma_j) = X(\sigma_j)X(\sigma_i), \\ X(\delta_i\sigma_j) = X(\sigma_j)X(\delta_i), \\ X(\sigma_j\delta_i) = X(\delta_i)X(\sigma_j). \end{cases}$$

  For the first, this is by definition when $i > j$ and follows from the simplicial identities otherwise; the second is similar. The third is also automatic, and the last follows from the simplicial identities again.

- We define our map on morphisms. Well, given a morphism $F\colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets, we already have our component morphisms $F_n\colon X_n \to Y_n$ which will become our morphisms $F_{[n]}\colon X([n]) \to Y([n])$. It remains to check the naturality of $F\colon X \Rightarrow Y$. Well, let $f\colon [n] \to [m]$ be an increasing map, we should check that $Xf \circ F_m = F_n \circ Yf$. Because f can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s (by Lemma 1.40), it is enough to check this for $f \in \{\delta^\bullet, \sigma^\bullet\}$, which now follows because F started its life as a morphism of combinatorial simplicial sets.

- We show that $\{X_n\} \mapsto X$ is functorial. To begin, note $\text{id}\colon \{X_n\} \to \{X_n\}$ goes to the identity maps $\text{id}_{[n]}\colon X([n]) \to X([n])$. Then given $f\colon \{X_n\} \to \{Y_n\}$ and $g\colon \{Y_n\} \to \{Z_n\}$, we see that the composite $(g \circ f)\colon X \Rightarrow Z$ is given by $(g \circ f)_{[n]} = (g \circ f)_n = g_n \circ f_n = g_{[n]} \circ f_{[n]}$.

- We complete the check that we have defined inverse equivalences. For concreteness, let $A\colon \{X_n\} \mapsto X$ and $B\colon X \mapsto \{X_n\}$ be our functors.

  Let's check $BA = \text{id}$. On an object $\{X_n\}$, we see that $BA\{X_n\}$ has $(BA\{X_n\})_n = A\{X_n\}([n]) = X_n$ and simplicial maps d_i and s_j given by $A(\{X_n\})(\delta^i)$ and $A(\{X_n\})(\sigma^j)$ which are d_i and s_j, respectively. On morphisms, we see $BAf = f$ because $(BAf)_n = Af_{[n]} = f_n$ for each n.

  Lastly, let's check $AB = \text{id}$. On an object X, we analogously see that $ABX([n]) = BX_n = X([n])$; further, to check that $ABX(f) = X(f)$ for an increasing map f, we note that Lemma 1.40 reduces this check to $\delta^\bullet$ and $\sigma^\bullet$ by functoriality, which similarly follows by construction of A and B (which turns $\delta^\bullet$s and $\sigma^\bullet$s to $d_\bullet$s and $s_\bullet$s and vice versa). Lastly, on morphisms, we see $(ABf)_{[n]} = Bf_n = f_{[n]}$ for each n. ■

Remark 1.45. In light of Proposition 1.44, we will occasionally identify simplicial sets and combinatorial simplicial sets. In particular, the term "combinatorial simplicial set" will not appear again.

Remark 1.46. There is also a notion of "semi-simplicial set" where we remove all the data associated to the $s_\bullet$s. This notion is sufficient to work with homology, but because we are now homotopy theorists, we work with simplicial sets.

1.2 September 4

The first problem set will be posted in about a day.

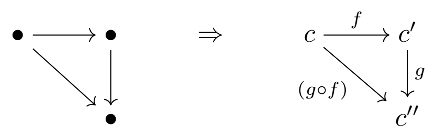
1.2.1 More on Simplicial Sets

It is worthwhile to explain nerves a little more.

Exercise 1.47. Fix a category $\mathcal{C}$. We work our $N(\mathcal{C})_i$ for $i \in \{0, 1, 2\}$.

Proof. Here we go.

- We see $N(\mathcal{C})_0$ consists of functors from the category $\{\bullet\}$, which are just objects of $\mathcal{C}$.
- Similarly, $N(\mathcal{C})_1$ consists of functors from the category $\{\bullet \to \bullet\}$ to $\mathcal{C}$, which are just morphisms of $\mathcal{C}$.
- Lastly, we note $N(\mathcal{C})_2$ consists of functors from the category $\{\bullet \to \bullet \to \bullet\}$ to $\mathcal{C}$, which amounts to the data of a diagram



  so that the nerve is required to know something about composition!

Let's work out N on some morphisms. For example, the canonical map $\sigma_0 \colon [n] \to [0]$ picks out the identity diagram, and the maps δ^i pick out some sub-diagrams. ■

It turns out that sSet is a presheaf category.

Definition 1.48 (presheaf)**.** Fix a category $\mathcal{C}$. Then a *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Accordingly, the presheaf category $\mathrm{PSh}(\mathcal{C})$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

Example 1.49. We see that $\mathrm{sSet} = \mathrm{PSh}(\Delta)$.

These categories are nice because they admit Yoneda embeddings.

Lemma 1.50 (Yoneda)**.** Fix a category $\mathcal{C}$. Then there is a functor $\text{よ} \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ which is defined on objects by

$$\text{よ}(c) \coloneqq \mathrm{Mor}_{\mathcal{C}}(-, c).$$

Furthermore, よ is fully faithful.

Remark 1.51. Another way to state the last conclusion is that there is a canonical bijection between $\mathrm{Mor}_{\mathcal{C}}(c_1, c_2)$ and natural transformations $\text{よ}(c_1) \Rightarrow \text{よ}(c_2)$.

Definition 1.52 (representable)**.** A presheaf $\mathcal{F}$ on a category $\mathcal{C}$ is *representable* if and only if there is an object $c \in \mathcal{C}$ for which $\mathcal{F}$ is isomorphic to $\text{よ}(c)$.

We are now allowed to remove the absolute value bars from our Δ^n.

Notation 1.53. We define Δ^n as the simplicial set $\text{よ}([n])$.

Example 1.54. We see that $(\Delta^n)_m$ consists of the order-preserving maps $[m] \to [n]$. For example, $(\Delta^2)_0$ has three elements, and $(\Delta^2)_1$ has six elements. Here are three of the elements of $(\Delta^2)_1$.

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ & \searrow & \downarrow \\ & & 2 \end{array}$$

Here is another important simplicial set.

Definition 1.55 (boundary)**.** For each $n \geq 0$, we define the *boundary* $\partial\Delta^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with $\partial\Delta^n(m)$ given by the non-surjective maps $[m] \to [n]$.

Definition 1.56 (horn)**.** For each $i \in [n]$, we define the *ith horn* $\Lambda_i^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with Λ_n^i given by the maps $[m] \to [n]$. We say that Λ_i^n is an *inner horn* if and only if $0 < i < n$; otherwise, Λ_i^n is an outer horn.

Remark 1.57. There are canonical inclusions $\Lambda_n^i \subseteq \partial\Delta^n \subseteq \Delta^n$ for each relevant i and n.

Example 1.58. Intuitively, Λ_i^n deletes the face opposite i. For example, here is Λ_0^2.

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ & \searrow & \\ & & 2 \end{array}$$

One can similarly draw Λ_1^2 (which omits $0 \to 2$) and Λ_2^2 (which omits $0 \to 1$).

1.2.2 Lifting Horns

These horns allow us to state a special property of nerves.

Proposition 1.59. Fix a category $\mathcal{C}$. Then any map $\Lambda_i^n \to N(\mathcal{C})$ from an inner horn Λ_i^n extends uniquely to a map $\Delta^n \to N(\mathcal{C})$.

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & N(\mathcal{C}) \\ \big\downarrow & \overset{!}{\nearrow} & \\ \Delta^n & & \end{array}$$

Remark 1.60. In fact, a simplicial set is the nerve of a category if and only if it satisfies the conclusion of Proposition 1.59. Thus, we have a characterization of the image of the fully faithful nerve functor! Amusingly, this allows one to give an alternate definition of a category in terms of simplicial sets; this is not circular because one can define simplicial sets as combinatorial simplicial sets.

Example 1.61. We show that any map $\Lambda_2^1 \to N(\mathcal{C})$ admits a unique extension to Δ^2. Well, Λ_2^1 specifies two maps $f\colon c_0 \to c_1$ and $g\colon c_1 \to c_2$, which we complete to a map from Δ^2 by defining the map $c_0 \to c_2$ to be the composite.

Example 1.62. It turns out that extending maps $\Lambda_1^3 \to N(\mathcal{C})$ and $\Lambda_2^3 \to N(\mathcal{C})$ to Δ^3 encodes associativity of composition.

Non-Example 1.63. One does not expect any map $\Lambda_0^2 \to N(\mathcal{C})$ to always extend to Δ^2. Indeed, Λ_0^2 only has the maps $0 \to 1$ and $0 \to 2$, but there is no obvious way to then produce a map $1 \to 2$ in the nerve!

Remark 1.64. One can check that a category is a groupoid if and only if the outer horns also admit horn fillings. The point is that being a groupoid allows one to reverse all the arrows, so coherence of composition allows one to do the filling.

We now turn to Sing.

Proposition 1.65. The functor $\operatorname{Sing}: \operatorname{sSet} \to \operatorname{Top}$ admits a left adjoint $|\cdot| : \operatorname{Top} \to \operatorname{sSet}$. In fact, $|\Delta^n|$ is defined to be the topological n-simplex.

It is worthwhile to know how to construct adjoints.

Theorem 1.66. Fix a category $\mathcal{C}$. Then $\operatorname{PSh}(\mathcal{C})$ has all limits and colimits.

Theorem 1.67. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories, where $\mathcal{D}$ admits colimits. For any functor $F: \mathcal{C} \to \mathcal{D}$, there is a unique functor $G: \operatorname{PSh}(\mathcal{C}) \to \mathcal{D}$ preserving colimits for which the composite

$$\mathcal{C} \xrightarrow{\sharp} \operatorname{PSh}(\mathcal{C}) \xrightarrow{G} \mathcal{D}.$$

In fact, G is a left adjoint.

Remark 1.68. This property characterizes Sing: indeed, for any topological space Y, we need to have $\operatorname{Sing}(Y)(n)$ to be

$$\operatorname{Mor}_{\operatorname{sSet}}(\Delta^n, \operatorname{Sing}(Y)) = \operatorname{Mor}_{\operatorname{Top}}(|\Delta^n|, Y).$$

We are now able to characterize the image of Sing.

Proposition 1.69. Fix a topological space Y. Then any map $\Lambda_i^n \to \operatorname{Sing} Y$ admits a lift to a map $\Delta^n \to \operatorname{Sing} Y$.

Proof. By the adjunction, it is enough to lift a map $|\Lambda_i^n| \to Y$ to a map $|\Delta^n| \to Y$. But this is not hard because there are projection maps $|\Delta^n| \to |\Lambda_i^n|$. ■

1.2.3 Kan Complexes

Proposition 1.69 motivates the following definition.

Definition 1.70 (Kan complex)**.** A *Kan complex* is a simplicial set X in which every $\Lambda_i^n \to X$ admits a lift to a map $\Delta^n \to X$.

Example 1.71. By Proposition 1.69, we see that $\operatorname{Sing} Y$ is always a Kan complex.

Example 1.72. By Remark 1.64, we see that $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$

At long last, we may define ∞-categories, which is intended to simultaneously generalize nerves and Kan complexes.

Definition 1.73 (∞-category, quasicategory)**.** An ∞-*category quasicategory* is a simplicial set X where every inner horn $\Lambda_i^n \to X$ admits a lift to $\Delta^n \to X$. We may call X_0 the *objects*, call X_1 the *morphisms*, and call X_n the n-morphisms for $n \geq 1$. More concretely, for any $E \in \mathcal{C}_2$, we may say that $d_1 E$ exhibits a 2-*isomorphism* between $d_0 E$ and $d_2 E$.

Definition 1.74 (homotopic)**.** Two maps $f, g \colon X \to Y$ are *homotopic* if and only if there is a map $h \colon X \times \Delta^1 \to Y$ such that the composites with $d_0 \colon X \times \Delta^0 \to X \times \Delta^1$ and $d_1 \colon X \times \Delta^0 \to X \times \Delta^1$ are g and f, respectively.

Remark 1.75. It turns out that being homotopic is an equivalence relation; the symmetry check uses the fact that Y is a Kan complex.

Definition 1.76 (homotopy equivalent)**.** Two Kan complexes X and Y are homotopy equivalent if and only if there are maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identities.

We will make use of the following hard(!) theorem.

Theorem 1.77 (Quillen)**.** If X is a CW complex, then $|\operatorname{Sing} X|$ is homotopy equivalent to X. Similarly, if X is a Kan complex, then $\operatorname{Sing}|X|$ is homotopy equivalent to X.

Corollary 1.78. The homotopy category of topological spaces is equivalent to the homotopy category of Kan complexes.

This theorem is a purely motivational statement: it allows us to pass from topological spaces to just Kan complexes.

Bibliography

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Bro24] Savannah Brown. *healed my shame! became an orb*. 2024. URL: https://www.youtube.com/watch?v=rygQ2s9MwUc.

List of Definitions